CIME-EMS Summer School on Splines and PDEs

Error estimation for Adaptive IsoGeometric Methods

Design & Analysis of AIGMs

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joint work with Annalisa Buffa

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July 7, 2017 — Cetraro

⋆ standard Adaptive Finite Element Method (AFEM) of the form:

$$\boxed{\text{SOLVE}} \rightarrow \boxed{\text{ESTIMATE}} \rightarrow \boxed{\text{MARK}} \rightarrow \boxed{\text{REFINE}}$$

$\mathcal{Q}_k$: mesh (conforming or with a fixed level of nonconformity) at step k of the adaptive loop

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- refine the marked subset $\mathcal{M}_k$ (and *some* others) to obtain $\mathcal{Q}_{k+1}$ and go to step SOLVE

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... [Dörfler — SIAM JNA, 1996] [Morin, Nochetto, Siebert — SIAM JNA, 2000]

:: convergence & optimality [Binev, Dahmen, Devore — NM, 2004] [Stevenson — MC, 2007]

[Bonito, Nochetto — SIAM JNA, 2010] ...

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⋆ *Primer of adaptive finite element methods*
Multiscale and Adaptivity: Modeling, Numerics and Applications
Cetraro - July 6 - 11, 2009

[Nochetto, Veeser — LNM, 2012]

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⋆ *A Posteriori Error Estimation Techniques for Finite Element Methods*
Oxford Univ. Press
[Verfürt — 2013]

⋆ extensions of tensor–product splines

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- **(T)HB–splines: (truncated) hierarchical B–splines** [Forsey & Bartels — CG, 1988] . . .

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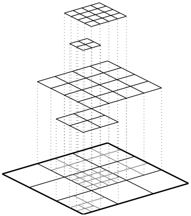

- (analysis suitable) T–splines [Sederberg, Zheng, Bakenov, Nasri – ACMTG, 2003] . . .

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- polynomial splines over (hierarchical) T–meshes

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- locally refined (LR) B-splines [Dokken, Lyche, Pettersen – CAGD, 2013] [Bressan – CAGD, 2013]

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:: adaptive spline models

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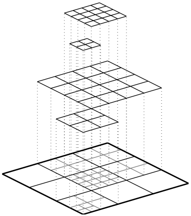

. . . many key properties are directly preserved by construction
. . . the refinement rules are simple and straightforward

⇒ *facilitates the design of adaptive isogeometric methods (AIGM)*

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   - SOLVE: hierarchical setting and scheme
   - ESTIMATE: a posteriori error analysis
   - [MARK]
   - REFINE: properties & grading
2. AIGM: complexity & optimality
   - [complexity of REFINE]
   - [total error & approximation classes]
   - Local upper bound
   - Optimal marking and convergence rates
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3. AIGM: numerical examples

$\Omega^0 \supseteq \Omega^1 \supseteq \ldots \supseteq \Omega^{N-1}$
domain hierarchy

- each domain Ω^ℓ is the union of the closure of elements of the tensor–product grid $G^{\ell-1}$

$$\Omega^\ell = \bigcup \left\{ \overline{Q} : Q \in G^{\ell-1} \right\}.$$

- active elements of level ℓ

$$\mathcal{G}^\ell = \left\{ Q \in G^\ell : Q \subset \Omega^\ell \wedge Q \not\subset \Omega^{\ell+1} \right\}$$

- hierarchical mesh

$$\mathcal{Q} = \left\{ Q \in \mathcal{G}^\ell,\ \ell = 0, 1, \ldots, N-1 \right\}$$

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$$\mathcal{Q}^* \succeq \mathcal{Q} \quad \Leftrightarrow \quad \mathcal{Q}^* \text{ is a refinement of } \mathcal{Q}$$

each element $Q^* \in \mathcal{Q}^*$ either also belongs to $\mathcal{Q}$ or it is obtained via refinement of $Q \in \mathcal{Q}$

:: model problem

$$-\operatorname{div}(\mathbf{A}\nabla u) = f \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = 0, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded domain and f is any square integrable function

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at step k of the adaptive loop:

$\mathcal{Q}_k$: (strictly) admissible mesh

$\mathbb{S}(\mathcal{Q}_k)$: THB–spline space over $\mathcal{Q}_k$

U_k: discrete solution

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- $U_k = \text{SOLVE}(\mathcal{Q}_k)$

- function space: $\mathbb{V} := H_0^1(\Omega)$
- bilinear form:

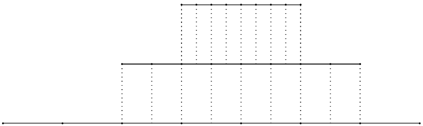
$$a(u, v) := \int_\Omega \mathbf{A}\, \nabla u \nabla v \qquad \forall\, u, v \in \mathbb{V}$$

- weak solution: find $u \in \mathbb{V}$ s.t.

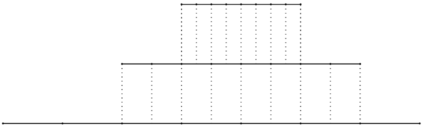
$$a(u, v) \; = < f, v > \qquad \forall\, v \in \mathbb{V}$$

- Galerkin solution: find $U_k \in \mathbb{S}_D(\mathcal{Q}_k)$ s.t.

$$a(U_k, V) \; = < f, V > \qquad \forall\, V \in \mathbb{S}_D(\mathcal{Q}_k)$$



subdomain hierarchy: $\Omega^0 \supseteq \Omega^1 \supseteq \ldots \supseteq \Omega^{N-1}$



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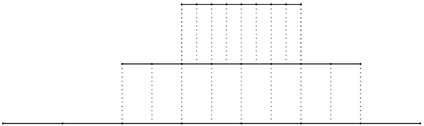
a B–spline $\beta \in B^\ell$ is active iff $\operatorname{supp}\beta \subseteq \Omega^\ell$ and $\operatorname{supp}\beta \not\subseteq \Omega^{\ell+1}$

HB–splines

$\{\beta \in B^\ell :$ β is active, $\ell = 0, 1, \ldots\}$

B–splines of level ℓ: B^ℓ



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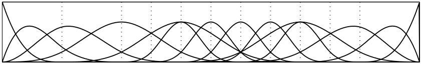
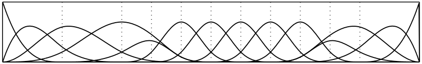


hierarchical B–splines on 2 levels



truncated hierarchical B–splines on 2 levels

$\tau \in \operatorname{span} B^\ell$

$$\operatorname{trunc}^{\ell+1}\tau = \sum_{\substack{\beta \in B^{\ell+1} \\ \operatorname{supp}\beta \not\subseteq \Omega^{\ell+1}}} c_\beta^{\ell+1}(\tau)\,\beta$$

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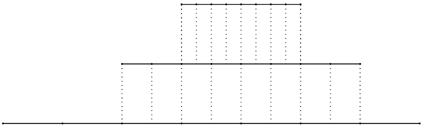
$\{\operatorname{trunc} \beta \in B^\ell : \beta \text{ is active}, \ \ell = 0, \ldots, N-1\}$

where $\operatorname{trunc}\beta = \operatorname{trunc}^{N-1}(\ldots(\operatorname{trunc}^{\ell+1}\beta))$

[Giannelli, Jüttler, Speleers — CAGD, 2012 & ACOM 2014]



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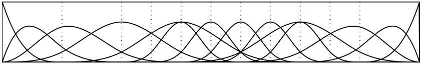

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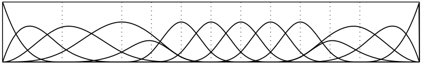

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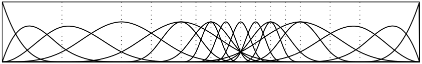
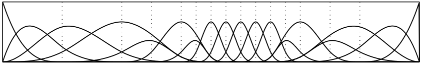


hierarchical B–splines on 3 levels



truncated hierarchical B–splines on 3 levels

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where $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded domain and f is any square integrable function

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at step k of the adaptive loop:

$\mathcal{Q}_k$: (strictly) admissible mesh

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U_k: discrete solution

- $U_k = \text{SOLVE}(\mathcal{Q}_k)$
- $\varepsilon_k = \text{ESTIMATE}(\mathcal{Q}_k, U_k, f)$

- residual representation:

$$< R, v > = \sum_{Q \in \mathcal{Q}_k} \int_Q r\, v$$

$$r = f + \operatorname{div}(A\nabla U_k) \text{ in any } Q \in \mathcal{Q}_k$$

- error indicators:

$$\varepsilon^2_{\mathcal{Q}_k}(U_k, \mathcal{Q}_k) = \sum_{Q \in \mathcal{Q}_k} h_Q^2 ||r||^2_{L^2(Q)}$$

- ingredients for upper bound:
  partition of unity + # basis functions

Let u be the exact weak solution of the model problem, the error estimator

$$\varepsilon^2_{\mathcal{Q}_{\boldsymbol{k}}}(U_k, \mathcal{Q}_k) = \sum_{Q \in \mathcal{Q}_{\boldsymbol{k}}} \varepsilon^2_{\mathcal{Q}_{\boldsymbol{k}}}(U_k, Q) \qquad \text{with} \qquad \varepsilon^2_{\mathcal{Q}_{\boldsymbol{k}}}(U_k, Q) = h^2_{\boldsymbol{Q}} ||r||^2_{\boldsymbol{L^2}(Q)}$$

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- is *reliable* $\Rightarrow$ it is an **upper bound** of the error of the Galerkin approximation U_k

$$|||u - U_k|||^2_\Omega \leq C_{up}\, \varepsilon^2_{\mathcal{Q}_k}(U_k, \mathcal{Q}_k)$$

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- is *reliable* $\Rightarrow$ it is an **upper bound** of the error of the Galerkin approximation U_k

$$|||u - U_k|||^2_\Omega \leq C_{up}\, \varepsilon^2_{\mathcal{Q}_{\boldsymbol{k}}}(U_k, \mathcal{Q}_k)$$

- is *efficient* $\Rightarrow$ it is a **lower bound** for the error of U_k up to oscillations

$$\varepsilon^2_{\mathcal{Q}_{\boldsymbol{k}}}(U_k, \mathcal{Q}_k) \leq C_{lb} \left(|||u - U_k|||^2_\Omega + \operatorname{osc}_{\mathcal{Q}}(U_k, \mathcal{Q})^2 \right)$$

  where

$$\operatorname{osc}^2_{\mathcal{Q}_{\boldsymbol{k}}}(U_k, \mathcal{Q}_k) = \sum_{Q \in \mathcal{Q}_{\boldsymbol{k}}} \operatorname{osc}^2(U_k, Q) \quad \text{with} \quad \operatorname{osc}(U_k, Q) = h_Q \|r - \Pi_{\mathbf{n}} r\|_{L^2(Q)}$$

  and $\Pi_{\mathbf{n}} : L^2(Q) \to \mathbb{Q}_{\mathbf{n}}$, $\mathbf{n} = (n_1, \ldots, n_d)$, denotes the L^2 projector onto the space of polynomials of degree n_j in the space direction j.

:: model problem

$$-\operatorname{div}(\mathbf{A}\nabla u) = f \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = 0, \tag{2}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded domain and f is any square integrable function

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- $\varepsilon_k = \text{ESTIMATE}(\mathcal{Q}_k, U_k, f)$
- $\mathcal{M}_k = \text{MARK}(\mathcal{Q}_k, \varepsilon_k)$

- selects $\mathcal{M}_k$ using *Dörfler marking* (bulk chasing)

$$\varepsilon_{\mathcal{Q}_k}(U_k, \mathcal{M}_k) \geq \theta \varepsilon_{\mathcal{Q}_k}(U_k, \mathcal{Q}_k)$$

for $0 < \theta < 1$

[Dörfler — SIAM JNA, 1996]

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- $\mathcal{M}_k = \text{MARK}(\mathcal{Q}_k, \varepsilon_k)$
- $\mathcal{Q}_{k+1} = \text{REFINE}(\mathcal{Q}_k, \mathcal{M}_k)$

- refines not only the marked elements but also a suitable set of elements in their neighborhood so that

$\mathcal{Q}_{k+1}$ is a (strictly) admissible mesh

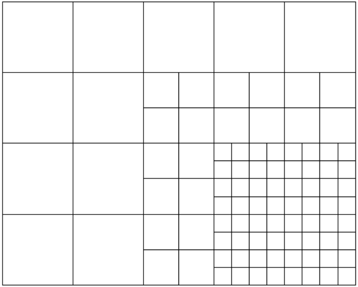
$\Rightarrow$ the # of basis functions acting on any given point is bounded

$\Rightarrow$ the support of any basis function can be compared with the element size

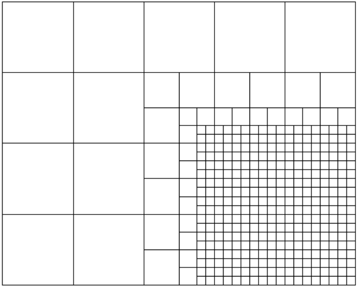
a mesh is admissible of class m if each active element $Q \in \mathcal{G}^\ell$
belongs to the support of basis functions of at most levels $\ell - m + 1, \dots, \ell$

$m = 2 \Rightarrow$ basis functions of at most 2 levels $(\ell - 1, \ell)$ on any element $Q \in \mathcal{G}^\ell$
$m = 3 \Rightarrow$ basis functions of at most 3 levels $(\ell - 2, \ell - 1, \ell)$ on any element $Q \in \mathcal{G}^\ell$

... ...



- admissible of class 2 for $\mathbf{p} \leq (2, 2)$
- admissible of class 3 for any $\mathbf{p}$



- admissible of class 2 for $\mathbf{p} = (1, 1)$
- admissible of class 4 for any $\mathbf{p}$

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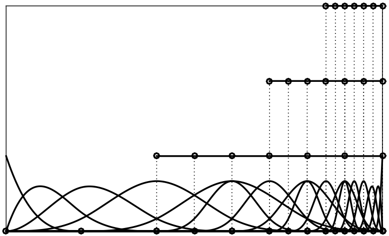
to generate admissible meshes with a certain structure . . .



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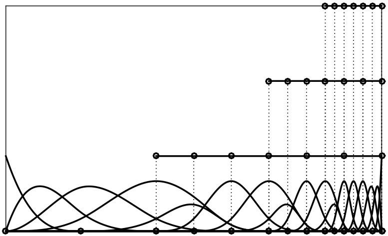

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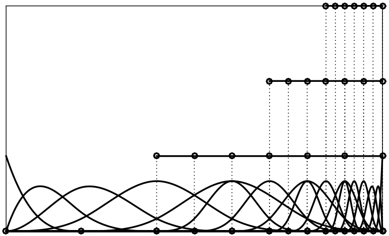
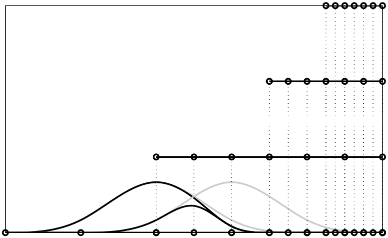


HB–splines

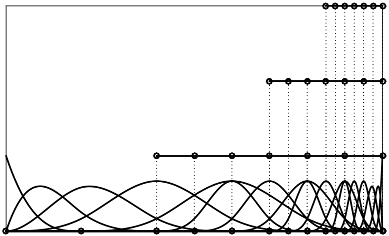


THB–splines

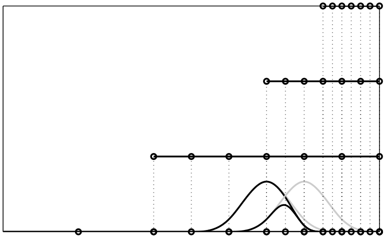
given an element $Q \in \mathcal{G}^\ell$ marked for refinement, its neighborhood collects all active elements of level $\ell - m + 1$ whose intersection with the support extension of Q w.r.t. level $\ell - m + 2$ is not empty

$\Rightarrow$ when Q is refined all elements in $\mathcal{N}(\mathcal{Q}, Q, m)$ have also to be recursively refined in order to generate a (strictly) admissible mesh

... by considering THB–splines we can exploit the truncation
to generate admissible meshes with a certain structure ...



HB–splines



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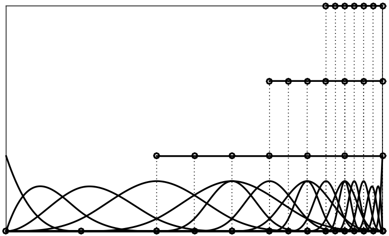
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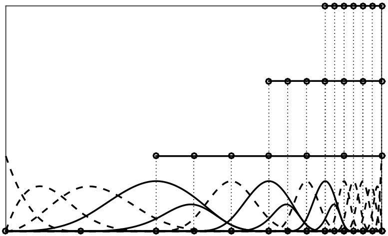
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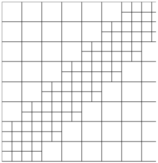
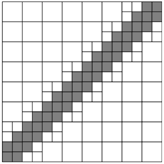


$\mathcal{Q}_k$



$\mathcal{M}_k$

$\mathcal{Q} =$ **REFINE(**$\mathcal{Q}, \mathcal{M}, m$**)**

for all $Q_{\mathcal{M}} \in \mathcal{Q} \cap \mathcal{M}$

$\mathcal{Q} =$ REFINE_RECURSIVE($\mathcal{Q}, Q_{\mathcal{M}}, m$)

end

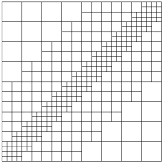
$\mathcal{Q} =$ **REFINE_RECURSIVE(**$\mathcal{Q}, Q_{\mathcal{M}}, m$**)**

for all Q in the *neighborhood* of $Q_{\mathcal{M}}$

$\mathcal{Q} =$ REFINE_RECURSIVE($\mathcal{Q}, Q, m$)

end

subdivide $Q_{\mathcal{M}}$ and update $\mathcal{Q}$



$\mathcal{Q}_{k+1}$ for $\mathbf{p} = (2, 2)$

SOLVE → ESTIMATE → MARK → REFINE

- $\varepsilon_k = \text{ESTIMATE}(\mathcal{Q}_k, U_k, f) \rightarrow$ a posteriori upper bound in energy norm

$$|||u - U_k|||^2_\Omega \leq C_{up}\varepsilon^2_k(U_k, \mathcal{Q}_k)$$

- $\mathcal{M}_k = \text{MARK}(\mathcal{Q}_k, \varepsilon_k) \rightarrow$ Dörfler marking (for estimator)

$$\varepsilon_k(U_k, \mathcal{M}_k) \geq \theta\varepsilon_k(U_k, \mathcal{Q}_k)$$

- $\mathcal{Q}_{k+1} = \text{REFINE}(\mathcal{Q}_k, \mathcal{M}_k) \rightarrow$ reduction of the estimator

$\Rightarrow$ towards the *contraction* of the quasi–error

$$|||u - U_{k+1}|||^2_\Omega + \gamma\varepsilon^2_{k+1}(U_{k+1}) \leq \alpha^2\left[|||u - U_k|||^2_\Omega + \gamma\varepsilon^2_k(U_k)\right]$$

for any $k \geq 0$, $\gamma > 0$ and $0 < \alpha < 1$

SOLVE → ESTIMATE → MARK → REFINE

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$\Rightarrow$ and the *convergence* of the adaptive scheme

$$|||u - U_{k+1}|||_\Omega + \gamma\varepsilon_{k+1}(U_{k+1}) \leq M\alpha^k$$

for any $k \geq 0$, $\gamma > 0$ and $0 < \alpha < 1$

the overlay $\mathcal{Q}_*$ of the two meshes $\mathcal{Q}_1$ and $\mathcal{Q}_2$
$$\mathcal{Q}_* = \mathcal{Q}_1 \otimes \mathcal{Q}_2.$$
is the coarsest common refinement of $\mathcal{Q}_1$ and $\mathcal{Q}_2$

- if $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are strictly admissible $\Rightarrow$ $\mathcal{Q}_*$ is strictly admissible
- $\mathcal{Q}_*$ has a bounded cardinality:

$$\#\mathcal{Q}_* = \#(\mathcal{Q}_1 \otimes \mathcal{Q}_2) \leq \#\mathcal{Q}_1 + \#\mathcal{Q}_2 - \#\mathcal{Q}_0,$$

where $\mathcal{Q}_0$ is the initial mesh configuration.

$\mathcal{Q}^* = \mathsf{REFINE}(\mathcal{Q}, \mathcal{M}, m)$

...an estimate of the form:

$$\boxed{\#\mathcal{Q}^* - \#\mathcal{Q} \leq \Lambda \# \mathcal{M}}$$

...is not valid by considering a constant independent of the refinement step...

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...an estimate of the form:

$$\#\mathcal{Q}^* - \#\mathcal{Q} \leq \Lambda \, \# \mathcal{M}$$

...is not valid by considering a constant independent of the refinement step...

$\Rightarrow$ the cumulative effect for a sequence of admissible meshes must be considered:

$$\#\mathcal{Q}_J - \#\mathcal{Q}_0 \leq \Lambda \sum_{j=0}^{J-1} \#\mathcal{M}_j$$

AFEM [Binev, Dahmen, Devore — NM, 2004] [Stevenson — MC, 2007]

AST–splines [Morgenstern, Peterseim — CAGD, 2015] [Morgenstern — SIAMJNA, 2016]

$\mathcal{M} := \bigcup_{j=0}^{J-1} \mathcal{M}_j$

$\Rightarrow$ set of marked elements used to generate the sequence of strictly admissible meshes $\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_J$ starting from $\mathcal{Q}_0 = G^0$

$$\mathcal{Q}_j = \mathrm{REFINE}(\mathcal{Q}_{j-1}, \mathcal{M}_{j-1}, m), \quad \mathcal{M}_{j-1} \subseteq \mathcal{Q}_{j-1} \quad \text{for } j \in \{1, \ldots, J\}$$

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$\Rightarrow$ there exists a constant $\Lambda > 0$ so that

$$\#\mathcal{Q}_J - \#\mathcal{Q}_0 \leq \Lambda \sum_{j=0}^{J-1} \#\mathcal{M}_j\,,$$

$$\Lambda = \Lambda(d, p, m) := 4(4\tilde{C} + 1)^d$$

$$\tilde{C} := \left(2^{-1} + \frac{2}{1 - 2^{1-m}} C_s\right), \qquad C_s = C_s(p, m) := 2^{m-2}(2\,p + 1), \qquad p := \max_{i=1,\ldots,d} p_i.$$

$$|||u - U_k|||_\Omega^2 + \gamma\, \varepsilon_k^2(U_k, \mathcal{Q}_k) \quad \text{vs.} \quad |||u - U_k|||_\Omega^2 + \mathrm{osc}_k^2(U_k, \mathcal{Q}_k)$$

quasi–error | total error

- estimators dominate oscillations + global lower and upper bounds imply:

$$\boxed{\text{total error} \approx \text{estimator} \approx \text{quasi-error}}$$

$\swarrow$ $\searrow$

the marking strategy is governed by the error indicators
$\Downarrow$
the decay rate of the AIGM must be characterized by the total error

the AIGM controls the quasi–error (contraction property)

- quasi-optimality of total error:

$$|||u - U_k|||_\Omega^2 + \mathrm{osc}_k^2(U_k, \mathcal{Q}_k) \lesssim \inf_{V \in \mathbb{S}_D} \left(|||u - V|||_\Omega^2 + \mathrm{osc}_k^2(V, \mathcal{Q}_k) \right)$$

$$\mathbb{Q}_M := \{\text{admissible mesh } \mathcal{Q} : \#\mathcal{Q} - \#\mathcal{Q}_0 \leq M\}$$

...the notion of total error is introduced to define:

$$\mathbb{A}_s := \left\{(v, f, \mathbf{A}) : |v, f, \mathbf{A}|_s := \sup_{M>0}(M^s\, \sigma(M; v, f, \mathbf{A})) < \infty\right\}, \quad s > 0$$

approximation class

where

$$\sigma(M; v, f, \mathbf{A}) := \inf_{\mathcal{Q}\in\mathbb{Q}_{\mathcal{M}}} \sigma_{\boldsymbol{e}}(u, f, \mathbf{A})^{1/2}$$

characterizes the quality of the best approximation in $\mathbb{Q}_M$ w.r.t.

$$\sigma_{\boldsymbol{e}}(u, f, \mathbf{A}) = \inf_{V\in\mathbb{S}_{\boldsymbol{D}}(\mathcal{Q})} \left(|||u - V|||_\Omega^2 + \operatorname{osc}^2_{\mathcal{Q}}(V, \mathcal{Q})\right)$$

best total error

:: local upper bound

if $\mathcal{Q}^* \succsim \mathcal{Q}$ then

$$|||U - U^*|||_\Omega^2 \leq C_{lub}\, \varepsilon_{\mathcal{Q}}^2(U, \mathcal{R})$$

where $\mathcal{R}_{\mathcal{Q} \to \mathcal{Q}^*}$ is the refined set of elements

derived in terms of quasi–interpolation (QI) operators

- quasi–interpolant of level ℓ

  [Buffa, Garau, Giannelli, Sangalli — LNCSE, 2016]

  :: spline spaces on tensor–product meshes

  :: class of L^2 stable QI operators

- hierarchical quasi–interpolant based on THB-splines

  [Speleers, Manni — NM, 2016]

  :: hierarchical spline space

  :: direct multi–level construction in terms of the truncated basis

QI of level ℓ

- $\mathcal{I}^\ell : L^2(\Omega) \to V^\ell$

$$\mathcal{I}^\ell v = \sum_{\boldsymbol{i} \in I^\ell} \lambda^\ell_{\beta_{\boldsymbol{i}}}(v)\beta_{\boldsymbol{i}} \quad \text{with} \quad I^\ell := \left\{ \boldsymbol{i} : \beta_{\boldsymbol{i}} \in \mathcal{B}^\ell \right\}, \qquad \ell = 0, \ldots, N-1$$

$\{\lambda^\ell_{\beta_{\boldsymbol{i}}}\}_{\beta_{\boldsymbol{i}} \in \mathcal{B}^\ell}$: local projection onto one element $Q_{\beta_{\boldsymbol{i}}}$ in the support of $\beta_{\boldsymbol{i}}$ so that

$$(\star) \quad \frac{|\operatorname{supp} \beta_{\boldsymbol{i}}|}{|Q_{\beta_{\boldsymbol{i}}}|} \leq C \qquad \text{(for some constant } C \text{ that depends on the degree } \mathbf{p}\text{)}$$

:: QI setting for LUB

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---

hierarchical QI

- $\mathcal{I}_\mathcal{Q} : L^2(\Omega) \to \operatorname{span} \mathcal{T}(\mathcal{Q})$

$$\mathcal{I}_\mathcal{Q} v := \sum_{\ell=0}^{N-1} \sum_{i \in I^\ell_\mathcal{Q}} \lambda^\ell_{\beta_i}(v)\tau_i \quad \text{with } I^\ell_\mathcal{Q} := \text{index set of active (T)HB-splines at level } \ell$$

and $\beta_i = \operatorname{mot} \tau_i$ ...

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and $\beta_i = \operatorname{mot} \tau_i$ ...

... when considering an admissible hierarchical mesh $\mathcal{Q}$,
an element Q_{β_i} that satisfies $(\star)$ may be chosen between
the active elements of level ℓ that belongs to the support of β_i.

...sketch of the proof: $E^* = U - U^*$

$$\mathcal{I}_{\mathcal{Q}} : \mathrm{span}\,\mathcal{T}(\mathcal{Q}) \to \mathrm{span}\,\mathcal{T}(\mathcal{Q}^*), \quad \text{if } w \in \mathbb{S}_D(\mathcal{Q}^*) \quad \Rightarrow \quad \mathcal{I}_{\mathcal{Q}} w = w \quad \text{in} \quad \Omega_{\mathcal{Q}} := \Omega \setminus \Omega_{\mathcal{R}}$$

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- we can consider the approximation $V \in \mathbb{S}_D(\mathcal{Q})$ defined as

$$V = \begin{cases} \mathcal{I}_{\mathcal{Q}} E^* & \text{in } \Omega_{\mathcal{R}}, \\ E^* & \text{in } \Omega_{\mathcal{Q}}, \end{cases} \qquad \text{so that} \qquad E^* - V = \begin{cases} E^* - \mathcal{I}_{\mathcal{Q}} E^* & \text{in } \Omega_{\mathcal{R}}, \\ 0 & \text{in } \Omega_{\mathcal{Q}}. \end{cases} \qquad (\star\star)$$

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- by combining

$$a(E^*, E^*) = a(U, E^*) - a(U^*, E^*) \quad \text{with} \quad a(E^*, E^*) = a(E^*, E^* - V) \quad \text{and} \quad (\star\star)$$

we have

$$a(E^*, E^*) \leq \sum_{Q \in \mathcal{R}} ||r(U)||_{L^2(Q)} ||E^* - \mathcal{I}_{\mathcal{Q}} E^*||_{L^2(Q)},$$

...sketch of the proof: $E^* = U - U^*$

$$\mathcal{I}_{\mathcal{Q}} : \operatorname{span}\mathcal{T}(\mathcal{Q}) \to \operatorname{span}\mathcal{T}(\mathcal{Q}^*), \quad \text{if } w \in \mathbb{S}_D(\mathcal{Q}^*) \quad \Rightarrow \quad \mathcal{I}_{\mathcal{Q}} w = w \quad \text{in} \quad \Omega_{\mathcal{Q}} := \Omega \setminus \Omega_{\mathcal{R}}$$

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$$a(E^*, E^*) \leq \sum_{Q \in \mathcal{R}} ||r(U)||_{L^2(Q)} ||E^* - \mathcal{I}_{\mathcal{Q}} E^*||_{L^2(Q)},$$

- the definition of $\varepsilon_{\mathcal{Q}}(U, Q)$ & the approximation properties of $\mathcal{I}_{\mathcal{Q}}$ lead to

$$|||E^*|||_\Omega^2 = a(E^*, E^*) \lesssim \sum_{Q \in \mathcal{R}} \varepsilon_{\mathcal{Q}}(U, Q) ||E^*||_{H^1(S(Q, \ell(Q) - m + 1))}$$

$$\lesssim \left(\sum_{Q \in \mathcal{R}} \varepsilon_{\mathcal{Q}}^2(U, Q) \right)^{1/2} \left(\sum_{Q \in \mathcal{R}} ||E^*||^2_{H^1(S(Q, \ell(Q) - m + 1))} \right)^{1/2} \lesssim \varepsilon_{\mathcal{Q}}(U, \mathcal{R}) \, |||E^*|||_\Omega$$

:: optimal marking and $\#\mathcal{M}_k$

$$\mathcal{Q}^* \gtrsim \mathcal{Q}, \qquad \mu := \frac{1}{2}\left(1 - \frac{\theta^2}{\theta_*^2}\right), \qquad \theta_* := \sqrt{\frac{C_{glb}}{1 + C_{lub}(1 + \Lambda_{osc})}}, \qquad \theta \in (0, \theta_*)$$

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- optimal marking: if

$$|||u - U^*|||_\Omega^2 + \mathrm{osc}^2_{\mathcal{Q}^*}(U^*, \mathcal{Q}^*) \le \mu \left[|||u - U|||_\Omega^2 + \mathrm{osc}^2_{\mathcal{Q}}(U, \mathcal{Q})\right]$$

then the refined set of elem. $\mathcal{R} = \mathcal{R}_{\mathcal{Q}\to\mathcal{Q}^*}$ satisfies the Dörfler property

$$\varepsilon_{\mathcal{Q}}(U, \mathcal{R}) \ge \theta\, \varepsilon_{\mathcal{Q}}(U, \mathcal{Q})$$

main ingredients: reduction of total error, perturbation of oscillations
global lower bound (C_{glb}), local upper bound (C_{lub})...

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main ingredients: reduction of total error, perturbation of oscillations
global lower bound (C_{glb}), local upper bound (C_{lub})...

▶ cardinality of $\mathcal{M}_k$: if $(u, f, \mathbf{A}) \in \mathbb{A}_s$, the AIGM generates a sequence

$$\{\mathcal{Q}_k, \mathbb{S}_D(\mathcal{Q}_k), U_k\}_{k\ge 0}$$

so that for any $k \ge 0$

$$\#\mathcal{M}_k \lesssim |u, f, \mathbf{A}|_s^{1/s} \left[|||u - U_k|||_\Omega^2 + \mathrm{osc}_k^2(U_k, \mathcal{Q}_{||})\right]^{-\frac{1}{2s}}$$

main ingredients: minimal cardinality of $\mathcal{M}_k$, overlay with bounded cardinality, quasi-optimality of total error, optimal marking...

by combining the pieces together...

- cardinality of $\mathcal{M}_k$ (ass. on Dörfler's parameter, min. card. of $\mathcal{M}_k$)
- complexity estimate
- property: the element residual dominates the oscillations
- global lower bound
- contraction of the quasi–error
- ...

we obtain

if $(u, f, \mathbf{A}) \in \mathbb{A}_s$, the AIGM generates a sequence

$$\{\mathcal{Q}_k, \mathbb{S}_D(\mathcal{Q}_k), U_k\}_{k \geq 0}$$

so that

$$\left[|||u - U_k|||_\Omega^2 + \operatorname{osc}_k^2(U_k, \mathcal{Q}_k)\right]^{\frac{1}{2}} \lesssim |u, f, \mathbf{A}|_s \left(\#\mathcal{Q}_k - \#\mathcal{Q}_0\right)^{-s}$$

for any $k \geq 0$

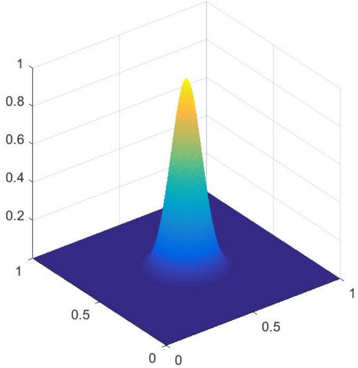
1. AIGM: design & convergence
   - SOLVE: hierarchical setting and scheme
   - ESTIMATE: a posteriori error analysis
   - [MARK]
   - REFINE: properties & grading
2. AIGM: complexity & optimality
   - [complexity of REFINE]
   - [total error & approximation classes]
   - Local upper bound
   - Optimal marking and convergence rates
3. AIGM: numerical examples

GeoPDEs implementation → joint work with C. Bracco and R. Vázquez

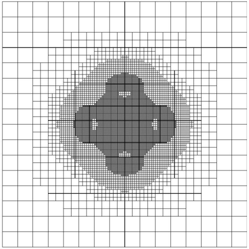
GeoPDEs implementation → joint work with C. Bracco and R. Vázquez

$$-\Delta u = f \quad \text{in } \Omega = [0,1] \times [0,1]$$

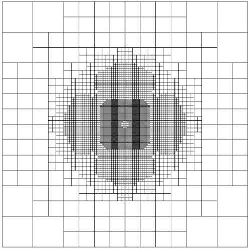
with exact solution:



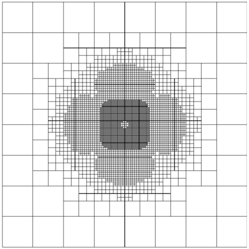
#1: $\mathbf{p} = (2,2)$, 14 iterations



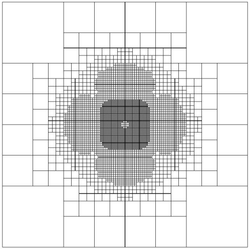
$m = 2$, DOF $= 8740$



$m = 3$, DOF $= 4936$

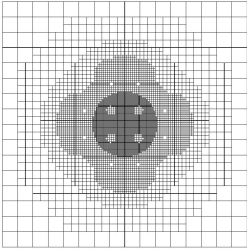


$m = 4$, DOF $= 4804$

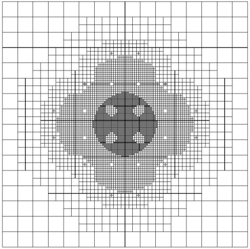


$m = \infty$, DOF $= 4776$

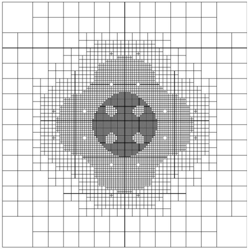
#1: $\mathbf{p} = (3, 3)$, 14 iterations



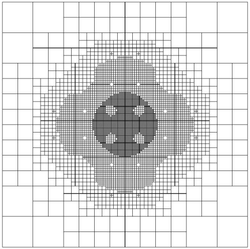
$m = 2$, DOF $= 6253$



$m = 3$, DOF $= 5729$

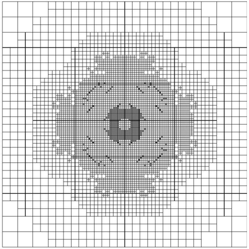


$m = 4$, DOF $= 5645$

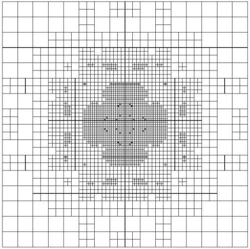


$m = \infty$, DOF $= 5705$

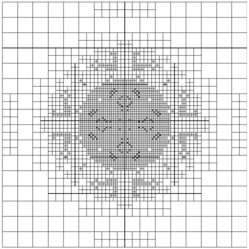
#1: $\mathbf{p} = (4, 4)$, 14 iterations



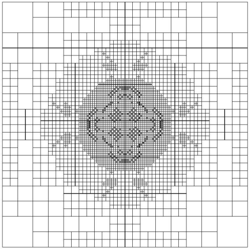
$m = 2$, DOF $= 5092$



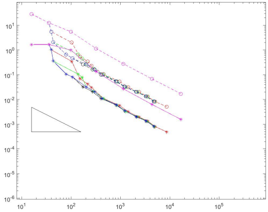
$m = 3$, DOF $= 2332$



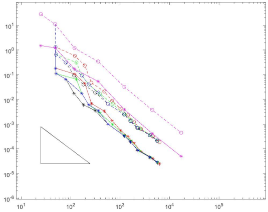
$m = 4$, DOF $= 2568$



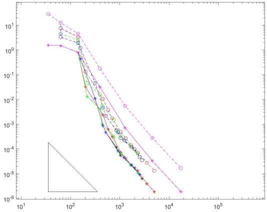
$m = \infty$, DOF $= 2864$



$\mathbf{p} = (2, 2)$



$\mathbf{p} = (3, 3)$

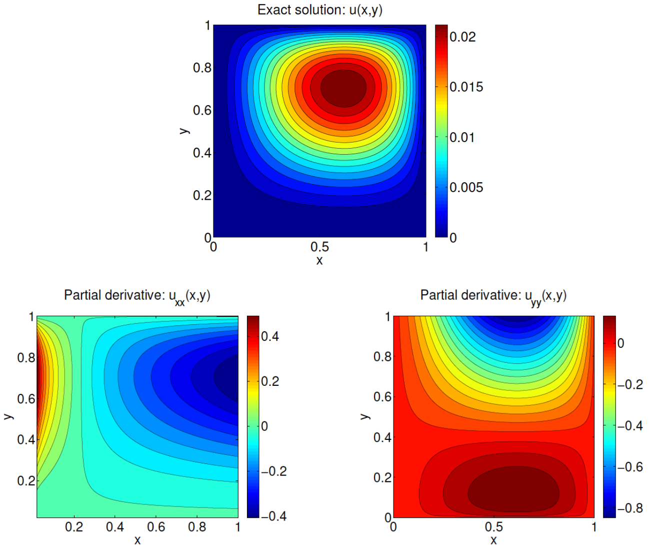


$\mathbf{p} = (4, 4)$

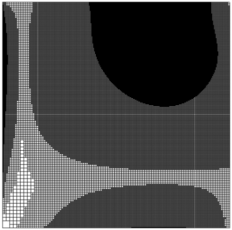
	$m = 2$	$m = 3$	$m = 4$	$m = \infty$	global refinement
estimator	o- -	o- -	o- -	o- -	o- -
error	*—	*—	*—	*—	*—

:: example #2: singular solution

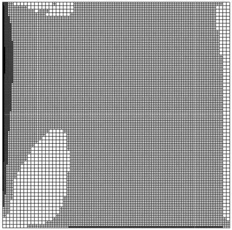
$-\Delta u = f \quad \text{in } \Omega, \qquad u|_{\partial\Omega=0}, \qquad$ with



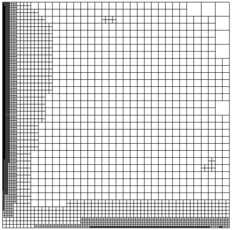
#2: $m = 2$, 23 iterations, 9 levels



$\mathbf{p} = (2, 2)$
DOF = 100,968

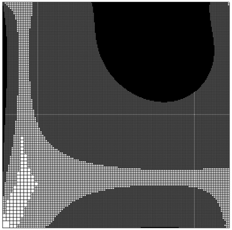


$\mathbf{p} = (3, 3)$
DOF = 19,521

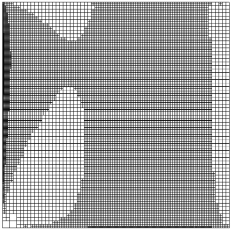


$\mathbf{p} = (4, 4)$
DOF = 7,386

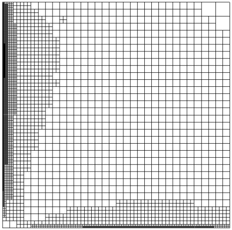
#2: $m = 3$, 23 iterations, 9 levels



$\mathbf{p} = (2,2)$
DOF = 96,301

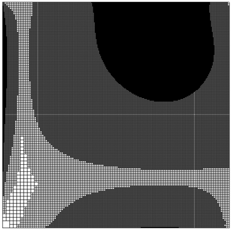


$\mathbf{p} = (3,3)$
DOF = 16,402

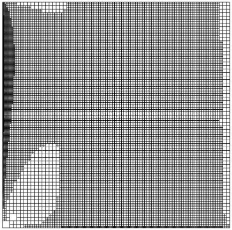


$\mathbf{p} = (4,4)$
DOF = 6,192

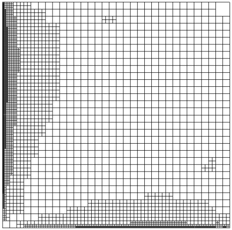
#2: $m = 4$, 23 iterations, 9 levels



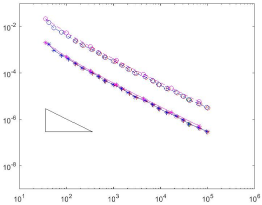
$\mathbf{p} = (2,2)$
DOF = 96,228



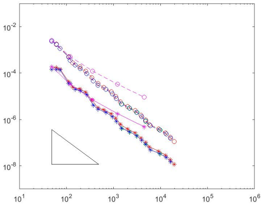
$\mathbf{p} = (3,3)$
DOF = 16,318



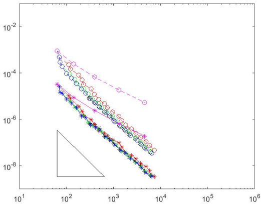
$\mathbf{p} = (4,4)$
DOF = 6,492



$\mathbf{p} = (2, 2)$



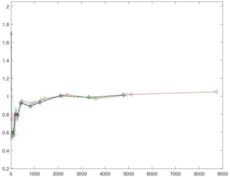
$\mathbf{p} = (3, 3)$



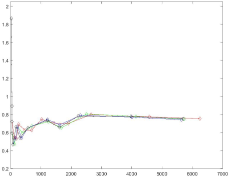
$\mathbf{p} = (4, 4)$

	$m = 2$	$m = 3$	$m = 4$	$m = \infty$	global refinement
estimator	○- -	○- -	○- -	○- -	○- -
error	*—	*—	*—	*—	*—

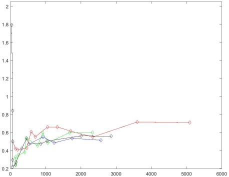
$$\text{effectivity index} = \frac{\text{estimator}}{\text{error}}$$



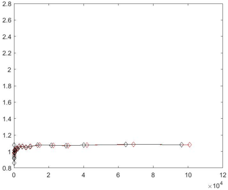
$\mathbf{p} = (2, 2)$



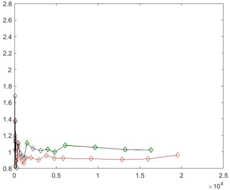
$\mathbf{p} = (3, 3)$



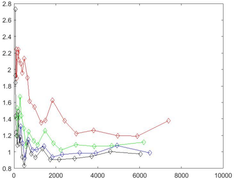
$\mathbf{p} = (4, 4)$



$\mathbf{p} = (2, 2)$



$\mathbf{p} = (3, 3)$



$\mathbf{p} = (4, 4)$

	$m = 2$	$m = 3$	$m = 4$	$m = \infty$
effectivity index	◇—	◇—	◇—	◇—

- Error estimation & convergence
  - hierarchical setting & schemes
  - a posteriori error analysis
  - contraction property

[Buffa, Giannelli — M3AS, 2016]

- Linear complexity of hierarchical refinement
  - mesh overlay with bounded cardinality
  - complexity estimate

[Buffa, Giannelli, Morgenstern, Peterseim — CAGD, 2016]

- Optimality
  - total error and approximation classes
  - local upper bound
  - additional assumptions on mark

[Buffa, Giannelli — MATHICSE report (EPFL), 2017]