Fast Assembly of Bernstein-Bézier FEM

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Splines and PDEs

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Outline

1. Bernstein Polynomials
2. FEM Assembly
3. Computation of Bernstein-Bézier Moments
4. Assembly of Element Matrices
5. Further Applications of BB Moments

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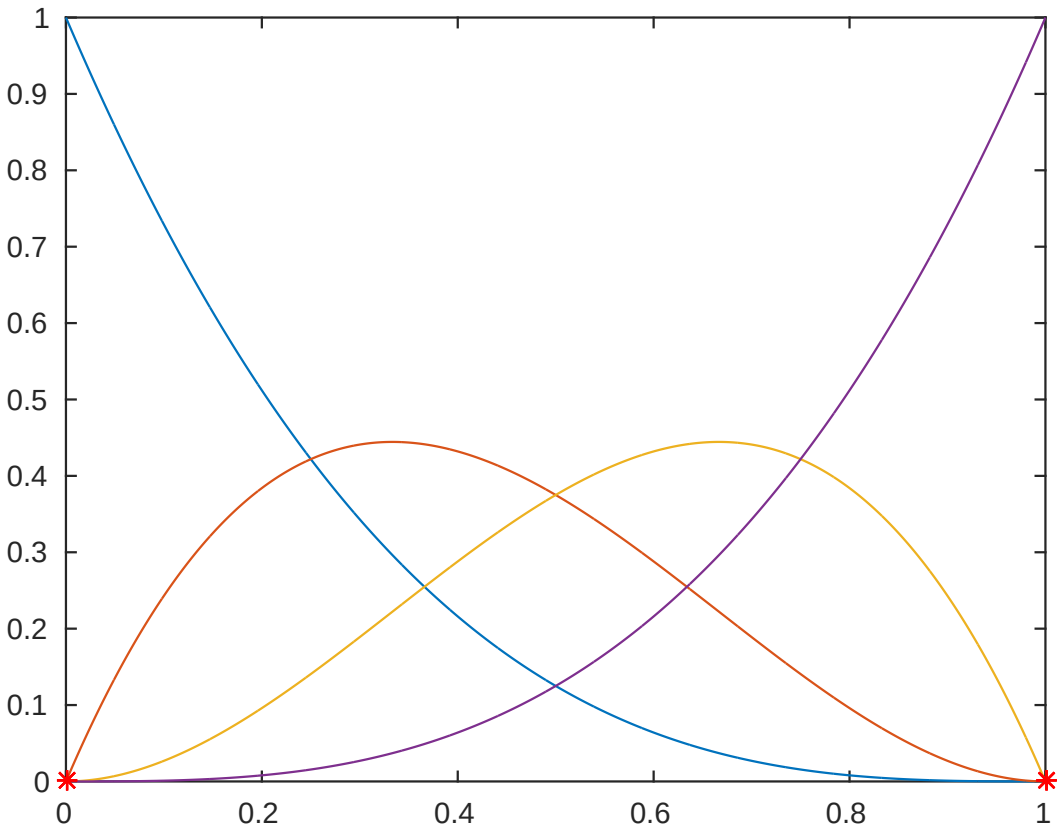
2. FEM Assembly
3. Computation of Bernstein-Bézier Moments
4. Assembly of Element Matrices
5. Further Applications of BB Moments

Bernstein Polynomials

- Definition in 1D: $B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, \ldots, n,$ a basis for the space of polynomials of degree n.



Bernstein Polynomials

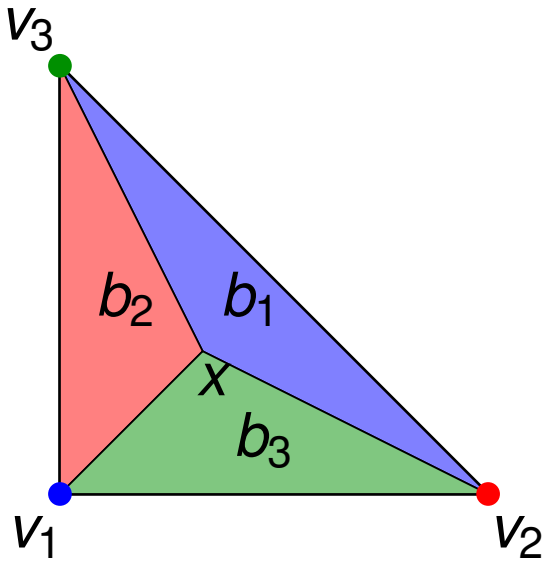
- A standard tool in curve and surface modelling:
  - Good algorithms for evaluation, differentiation, degree raising.
  - Optimal condition numbers among all non-negative bases.
  - Bernstein-Bézier techniques for smooth piecewise polynomials in 2D and 3D.
- Long ago suggested as shape functions in the finite element method (de Boor, Schumaker), alternative to Lagrange interpolation basis.
- Turn out to be optimal for the assembly of high order FEM system matrices on simplices[1]

[1] M. Ainsworth, G. Andriamaro and O. Davydov, Bernstein-Bézier finite elements of arbitrary order and optimal assembly procedures, *SIAM J. Sci. Comp.*, **33** (2011), 3087–3109.

Bernstein Polynomials

Bernstein basis polynomials on a d-simplex T

- $B_\xi^{n,d}(x) := \frac{n!}{\xi_1!\cdots\xi_{d+1}!} b_1^{\xi_1}(x)\cdots b_{d+1}^{\xi_{d+1}}(x),\ \xi \in \mathcal{I}_{n,d},\ x \in \mathbb{R}^d$
- $\mathcal{I}_{n,d} := \left\{\xi = (\xi_1,\ldots,\xi_{d+1}) \in \mathbb{Z}_+^{d+1} : |\xi| = n\right\},\ |\xi| = \sum_{i=1}^{d+1}\xi_i$
- $b_1(x),\ldots,b_{d+1}(x)$ – barycentric coordinates of x w.r.t. a simplex $T = [v_1,\ldots,v_{d+1}] \subset \mathbb{R}^d$



$b_i(x)$ are linear polynomials satisfying

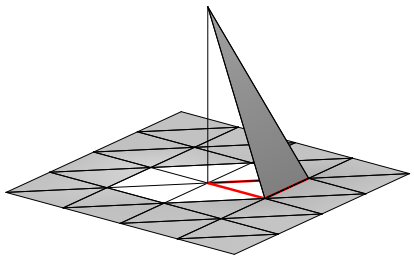
$$b_i(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

- $b_i(x)$ are linear $\Longrightarrow$ $B_\xi^{n,d}(x)$ is a polynomial of degree n

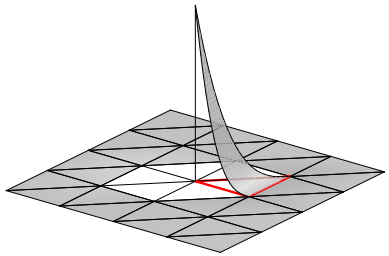
Bernstein Polynomials

Example: Bernstein polynomial on a triangle ($n = 3$, $d = 2$)

$b_1(x)$



$B^{3,2}_{(3,0,0)} = b^3_1(x)$



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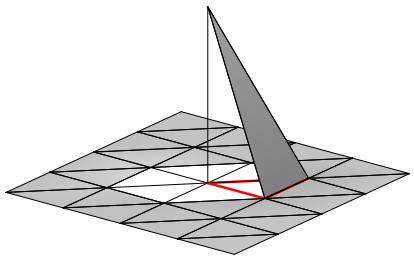
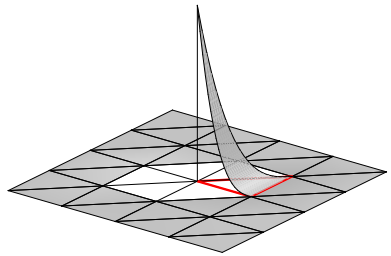
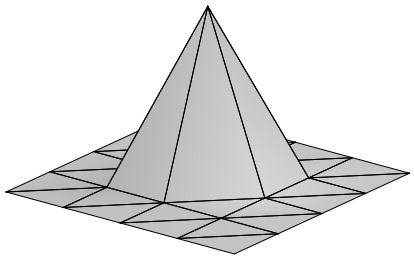


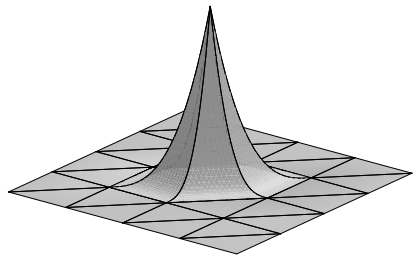
Courant hat function

$B^{3,2}_{(3,0,0)} = b^3_1(x)$



Bernstein-Bezier basis function





Bernstein Polynomials

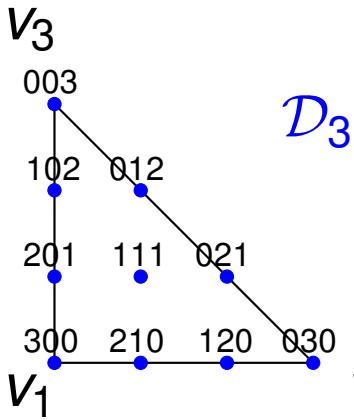
Domain points (Bernstein-Bézier techniques)

- $\mathcal{I}_{n,d} = \left\{\xi = (\xi_1, \ldots, \xi_{d+1}) \in \mathbb{Z}_+^{d+1} : \ |\xi| = n\right\}$ can be identified with the set of domain points

  $\mathcal{D}_{n,T} = \left\{p_\xi := \frac{1}{n}\sum_{i=1}^{d+1} \xi_i v_i\right\}_{\xi \in \mathcal{I}_{n,d}} \subset T = [v_1, \ldots, v_{d+1}].$

- Example: in 2D

  $\mathcal{D}_{n,T} = \left\{p_{(i,j,k)} := \frac{iv_1 + jv_2 + kv_3}{n}\right\}_{i+j+k=n} \subset T = [v_1, v_2, v_3].$



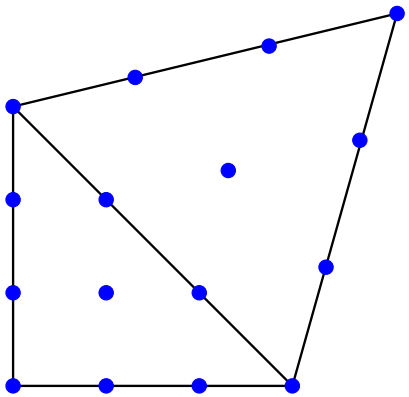
$\mathcal{D}_{3,T}$

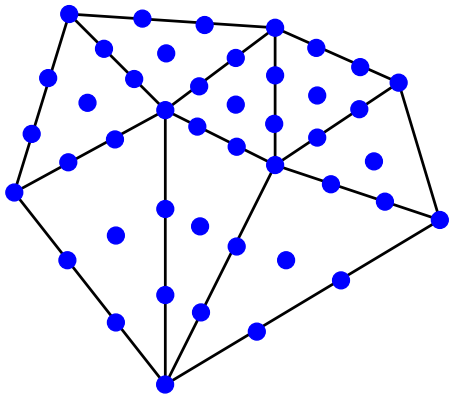
$$\mathcal{I}_{3,2} = \{(3,0,0),(0,3,0),(0,0,3),(2,1,0),(1,2,0),(0,2,1),(0,1,2),(2,0,1),(1,0,2),(1,1,1)\}$$

- Geometric splitting: “Degrees of freedom" located in p_ξ correspond to faces, edges, vertices, etc., which is important for the assembly of global system matrices.

Domain points

- We can identify domain points on faces shared by two or more simplices because the restriction of a d-dimensional Bernstein polynomial to a k-dimensional face is the k-dimensional Bernstein polynomial for the same domain point in this face

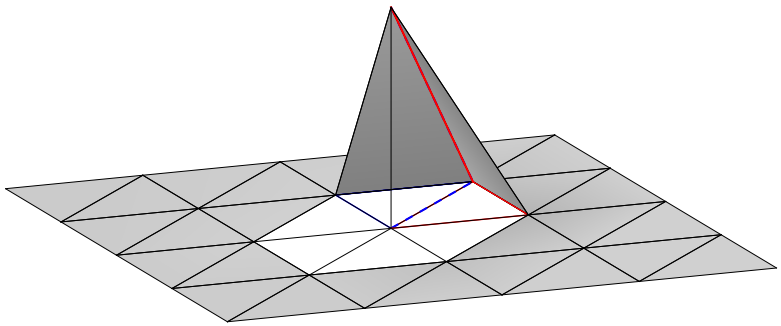




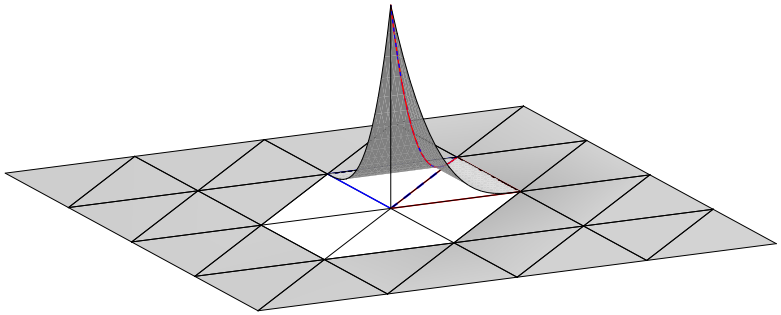
- Domain points coincide with interpolation points for Lagrange basis functions

Continuity across common faces

Linear shape functions



BB vertex shape functions



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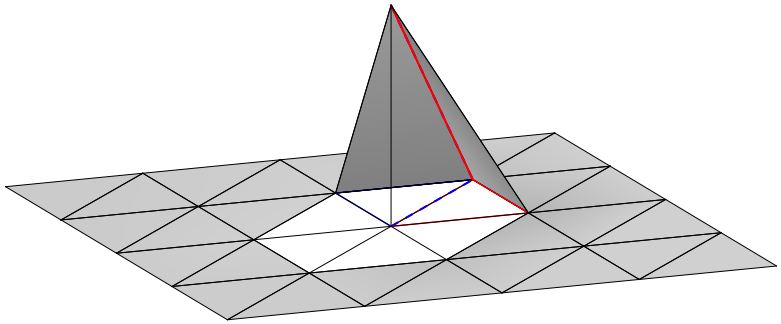

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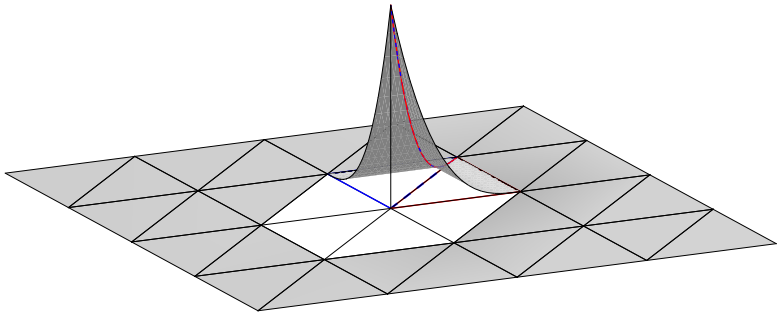
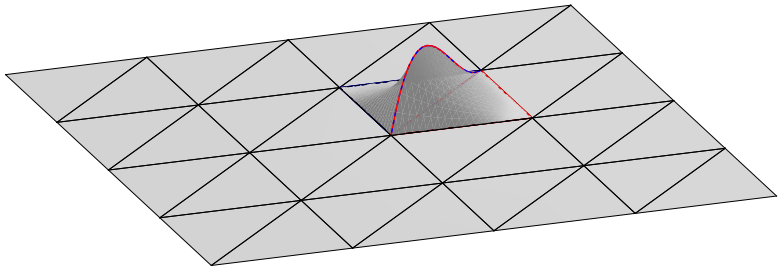
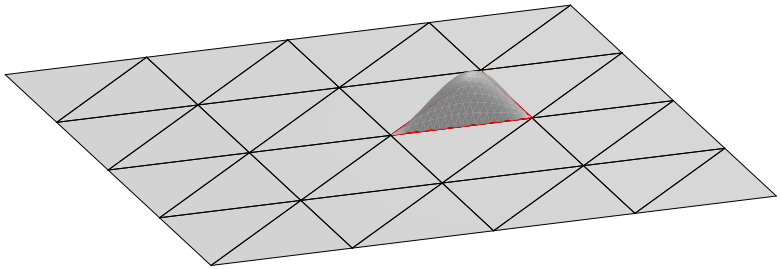


BB edge shape functions

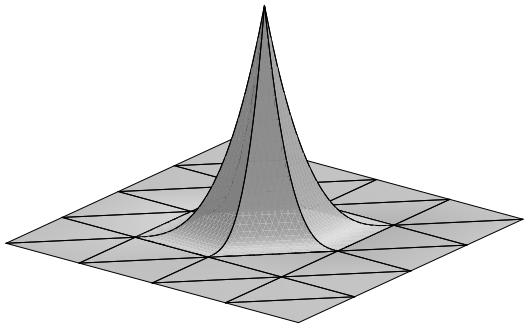


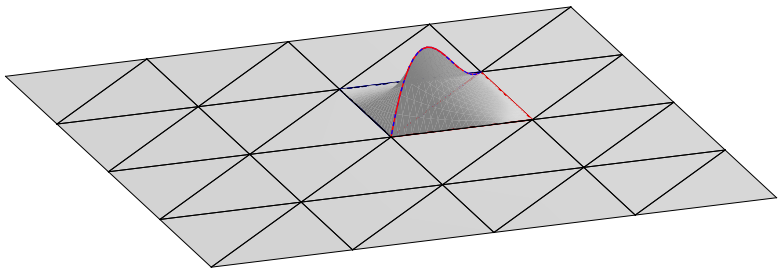
BB interior shape function

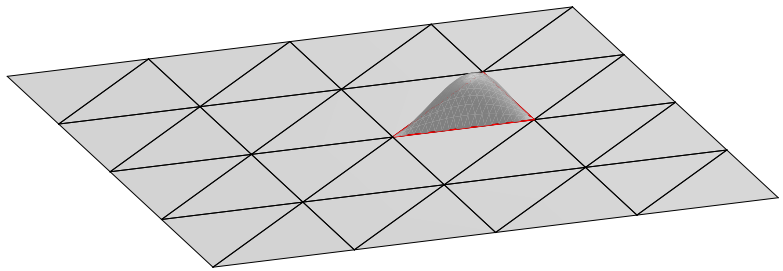


Bernstein Polynomials

Bernstein basis functions are C^0 B-splines on triangulations:

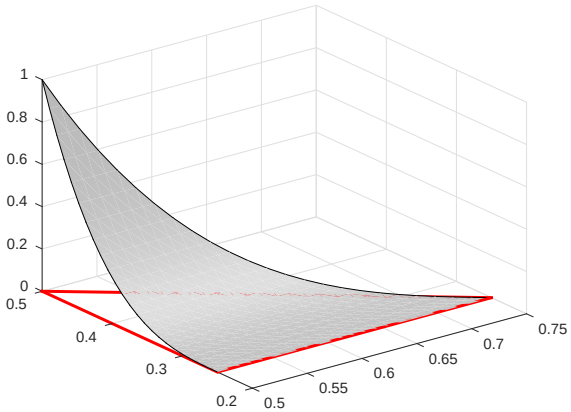




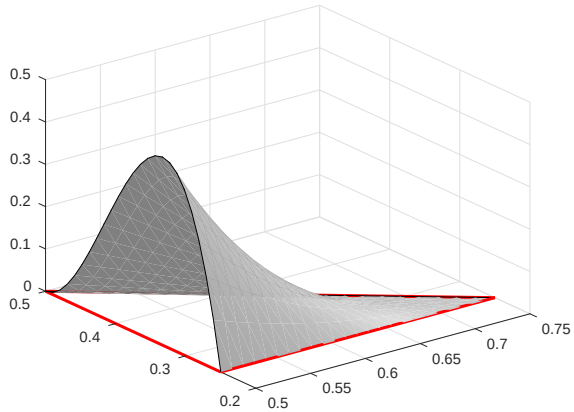


Bernstein versus Lagrange shape functions: degree 3

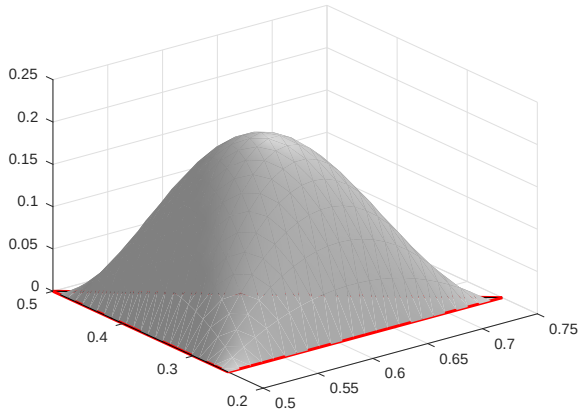
Vertex

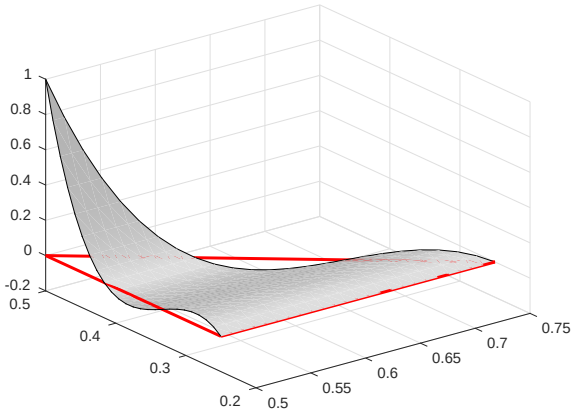


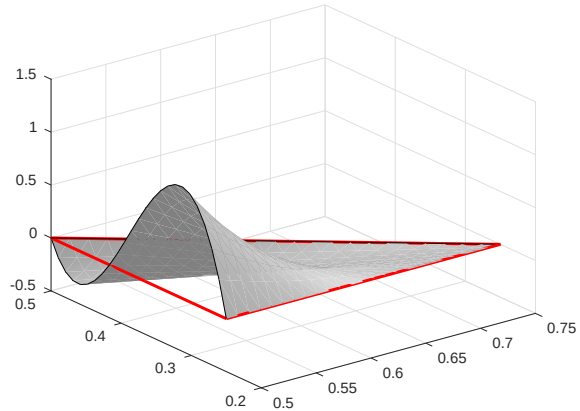
Edge

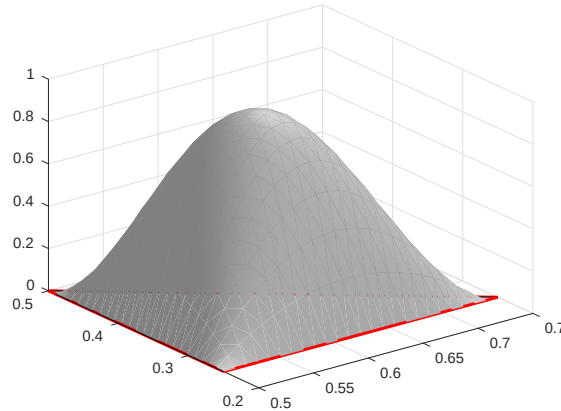


Internal



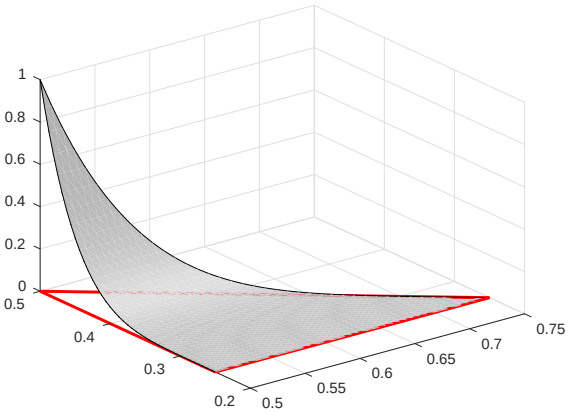


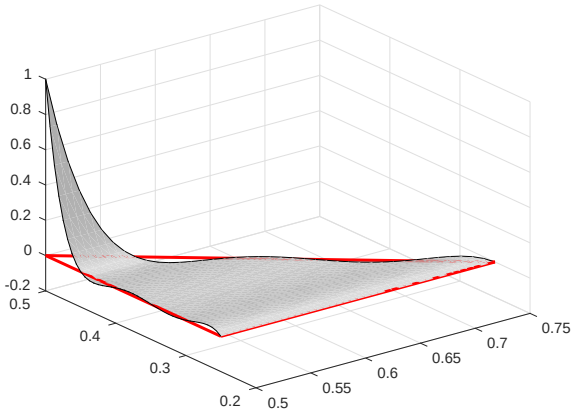




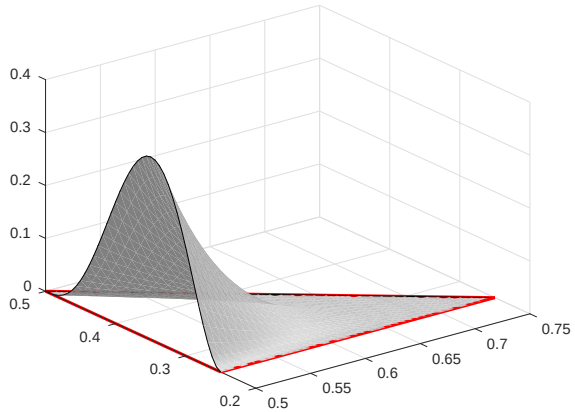
Bernstein versus Lagrange shape functions: degree 5

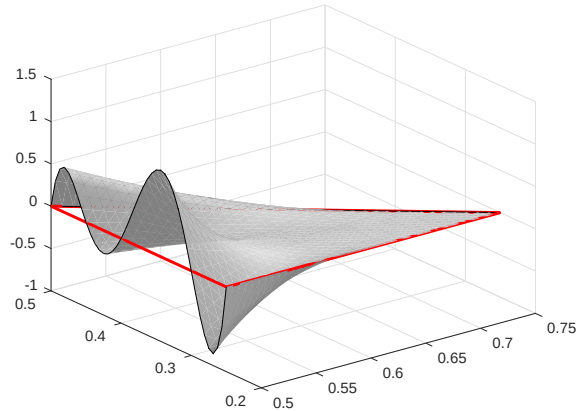
Vertex





Edge





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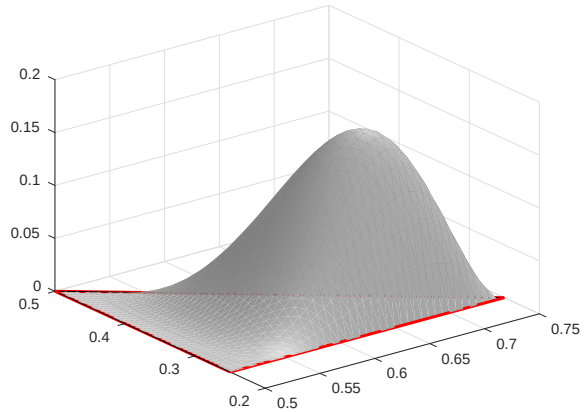
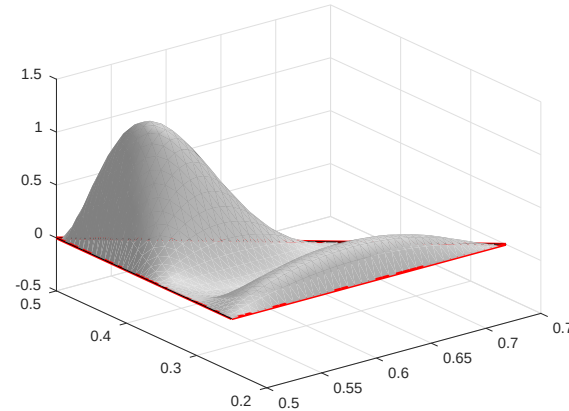
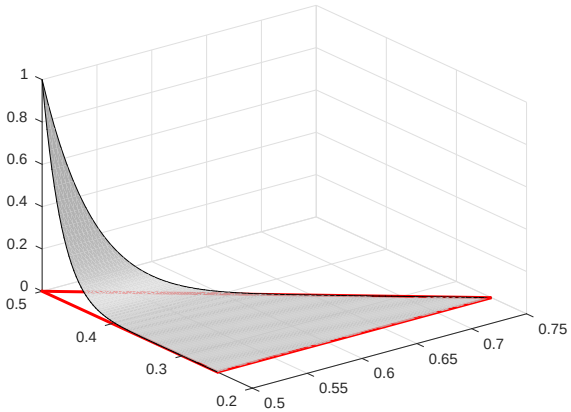
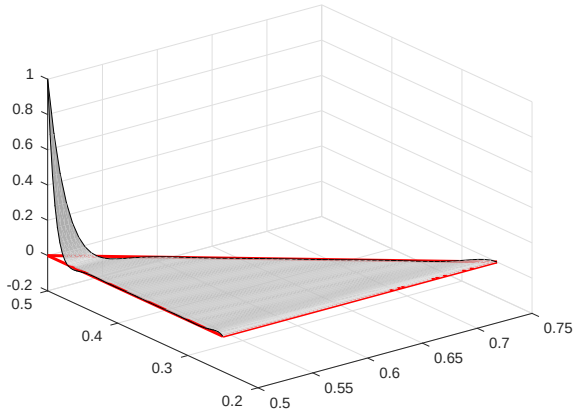




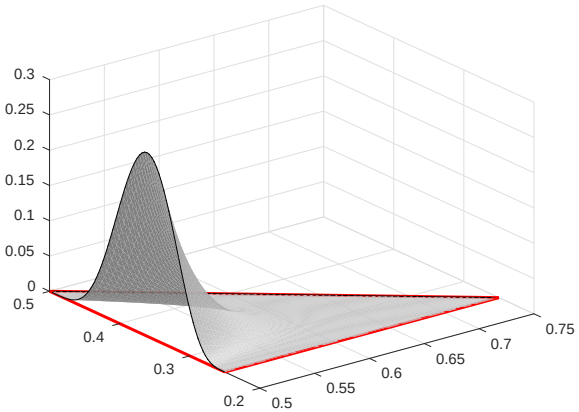
Bernstein versus Lagrange shape functions: degree 9

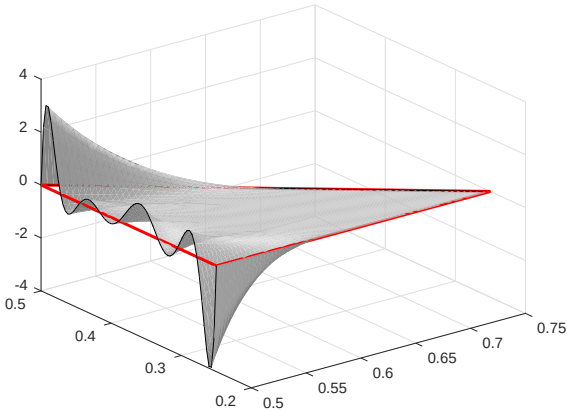
Vertex





Edge





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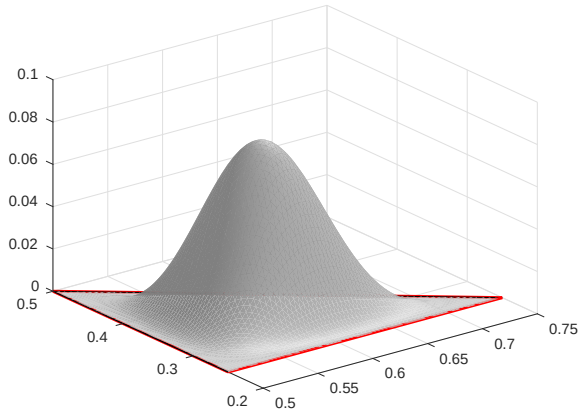
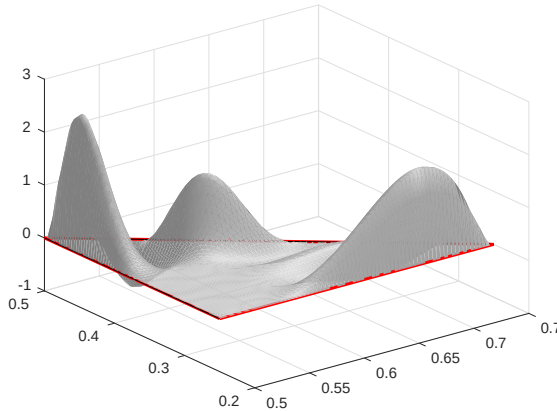
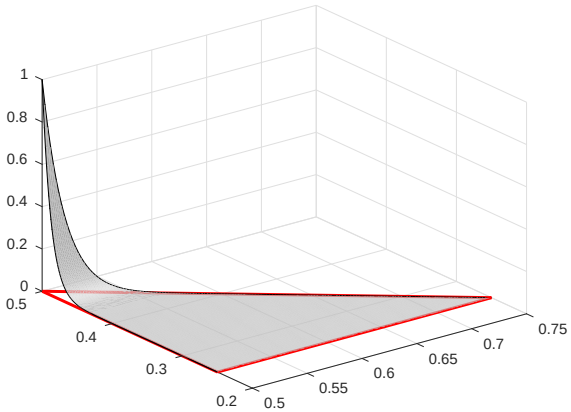
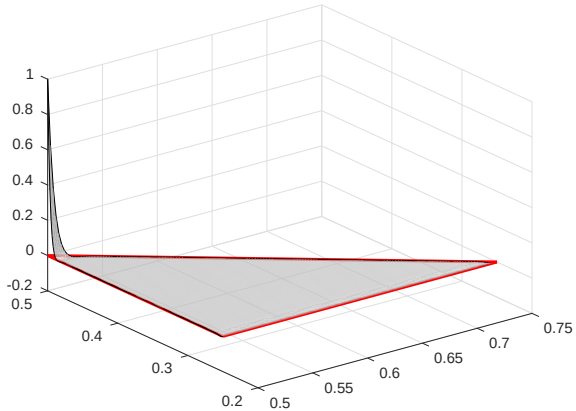




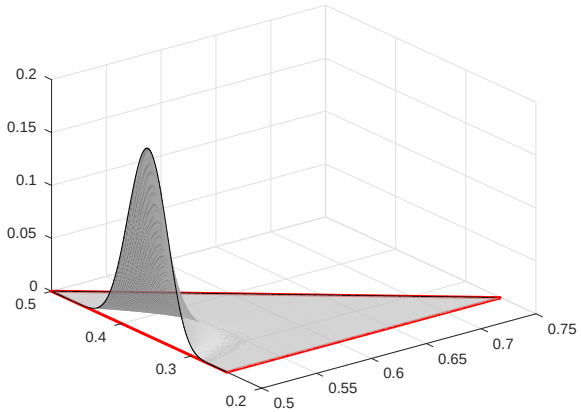
Bernstein versus Lagrange shape functions: degree 20

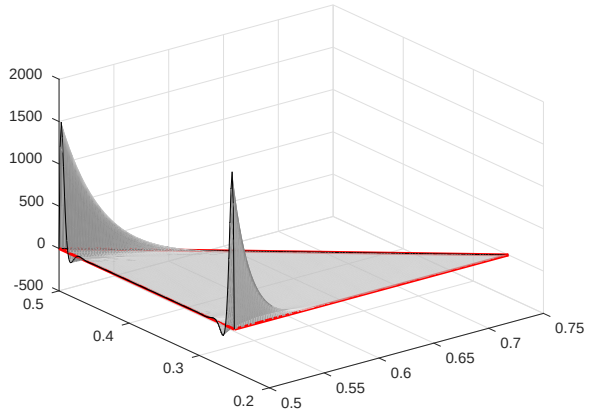
Vertex





Edge





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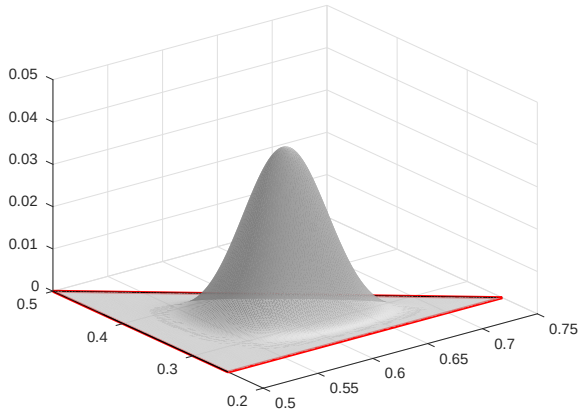

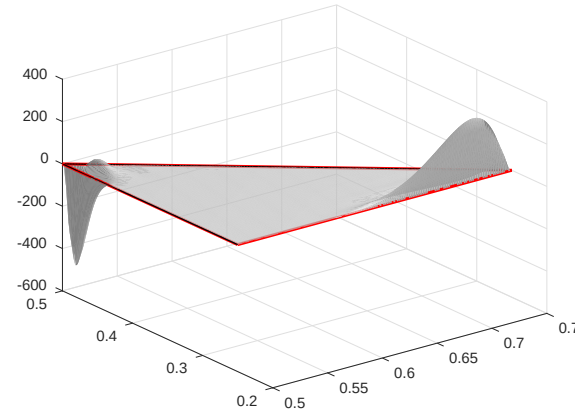

Bernstein Polynomials

Key properies for the fast computation of system matrices

- Product is a Bernstein polynomial:

$$B_{\xi}^{n,d} B_{\zeta}^{m,d} = \frac{\binom{n}{\xi}\binom{m}{\zeta}}{\binom{2n}{\xi+\zeta}} B_{\xi+\zeta}^{n+m,d}$$

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- Gradient is a sparse linear combination of Bernstein polynomials: $\nabla B_\xi^{n,d} = n \sum_{\alpha \in \mathcal{I}_{1,d}} B_{\xi-\alpha}^{n-1,d} \nabla b_\alpha$,

  sparse: the number of terms is independent of degree;
  ∇b_α are constant vectors depending on T;
  b_α are barycentric coordinates,
  numbered as Bernstein polynomials of degree 1:
  $b_\alpha := B_\alpha^{1,d}$, $\alpha \in \mathcal{I}_{1,d} = \{(1,0,\ldots,0),\ldots,(0,0,\ldots,1)\}$.

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- Hence other differential operators such as Hessian, curl, div also produce sparse linear combinations of Bernstein polynomials.

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Dirichlet problem for linear second order elliptic equation:

$$\begin{cases} -\text{div}\,(A\nabla u) + \mathbf{b}\cdot\nabla u + cu &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{cases}$$

Weak formulation of the homogenuous problem ($g = 0$):
Find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$

$$\int_\Omega \Big[\nabla u \cdot A\nabla v + (\mathbf{b}\cdot\nabla u)v + cuv\Big]\mathrm{d}x = \int_\Omega fv\mathrm{d}x. \quad (1)$$

FEM Assembly

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$$\int_\Omega \Big[\nabla u \cdot A \nabla v + (\mathbf{b} \cdot \nabla u) v + cuv\Big] \mathrm{d}x = \int_\Omega fv \mathrm{d}x. \quad (1)$$

Galerkin method:

- Choose a finite dimensional subspace $\mathcal{P}$ of $H_0^1(\Omega)$ with basis $\{\phi_1, \ldots, \phi_N\}$.
- Replace u by a linear combination $u = \sum_{i=1}^N u_i \phi_i$ and require (1) to hold for each $v = \phi_j$, $j = 1, \ldots, N$.
- This results in a linear system w.r.t. $\{u_i\}_{i=1}^N$, with system matrices in LHS and load vector in RHS, with entries:

$$\int_\Omega \nabla \phi_i \cdot A \nabla \phi_j \, \mathrm{d}x, \quad \int_\Omega (\mathbf{b} \cdot \nabla \phi_i) \phi_j \mathrm{d}x, \quad \int_\Omega c \phi_i \phi_j \, \mathrm{d}x, \quad \int_\Omega f \phi_j \, \mathrm{d}x$$

FEM Assembly

Simplicial finite elements

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- $\Omega \subset \mathbb{R}^d$ is triangulated into simplices ('elements').
- $\{\phi_i\}_{i=1}^N$ is a set of basis functions for the space of continuous piecewise polynomials of total degree n.
- Assume that for each simplex T, nonzero restrictions $\phi_{T,\xi}$, $\xi \in I_T$, of the basis functions to T ('shape functions') form a basis for the space $\mathbb{P}_n$ of polynomials of degree n.

Assembling the linear system

- Compute for each element T:
  - (a) load vector $\int_T f\phi_{T,\xi}\,\mathrm{d}x, \quad \xi \in I_T,$
  - (b) mass matrix $\int_T c\phi_{T,\xi}\phi_{T,\zeta}\,\mathrm{d}x, \quad \xi,\zeta \in I_T,$
  - (c) stiffness matrix $\int_T \nabla\phi_{T,\xi}\cdot A\nabla\phi_{T,\zeta}\,\mathrm{d}x, \quad \xi,\zeta \in I_T,$
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$$\int_\Omega \nabla\phi_i\cdot A\nabla\phi_j\,\mathrm{d}x = \sum_{T\subset \operatorname{supp}\phi_i\cap\operatorname{supp}\phi_j} \int_T \nabla\phi_{T,\xi_i}\cdot A\nabla\phi_{T,\xi_j}\,\mathrm{d}x,$$

where $\phi_{T,\xi_i} = \phi_i|_T, \quad \phi_{T,\xi_j} = \phi_j|_T.$

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- For Bernstein basis functions the load vector consists of Bernstein-Bézier moments $\int_T f(x)\,B_\xi^{n,d}(x)\,\mathrm{d}x$, $\xi \in \mathcal{I}_{n,d}$.

Outline

1. Bernstein Polynomials
2. FEM Assembly
3. Computation of Bernstein-Bézier Moments
4. Assembly of Element Matrices
5. Further Applications of BB Moments

Computation of Bernstein-Bézier Moments

Consider Bernstein-Bézier moments (of degree n):

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- Hence, look for a quadrature $\int_T g(x)\, \mathrm{d}x \approx \sum_{i=1}^{N} w_i g(x_i)$ with $N \sim n^d$ centers, to be used with $g(x) := f(x)\, B_\xi^{n,d}(x)$.

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- Stroud conical product rule is especially well suited because it allows fast evaluation of the quadrature by sum factorization.

Sum factorization

- Assume we need to compute double sums

$$\Sigma_{k,\ell} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j}(k,\ell)\, f_{i,j} \quad \text{for all} \quad k,\ell = 1,\ldots,n.$$

  This requires in general $\sim n^4$ multiplications.

- If $c_{i,j}(k,\ell)$ is factorized as $c_{i,j}(k,\ell) = c_i(k)c_j(\ell)$, then

$$\Sigma_{k,\ell} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j}(k,\ell) f_{i,j} = \sum_{i=1}^{n} c_i(k) \sum_{j=1}^{n} c_j(\ell) f_{i,j}.$$

- Hence, the cost can be reduced to $2n^3$ by computing

1. $\sigma(i,\ell) = \sum_{j=1}^{n} c_j(\ell) f_{i,j}, \quad i,\ell = 1,\ldots,n.$ (n^3 multipl.)
2. $\Sigma_{k,\ell} = \sum_{i=1}^{n} c_i(k)\sigma(i,\ell), \quad k,\ell = 1,\ldots,n.$ (n^3 multipl.)

Sum factorization

- For multiple sums

$$\Sigma_{k_1,\ldots,k_d} = \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} c_{i_1,\ldots,i_d}(k_1,\ldots,k_d)\, f_{i_1,\ldots,i_d}, \quad k_j = 1,\ldots,n,$$

  such that

$$c_{i_1,\ldots,i_d}(k_1,\ldots,k_d) = c_{i_1}(k_1)\cdots c_{i_d}(k_d),$$

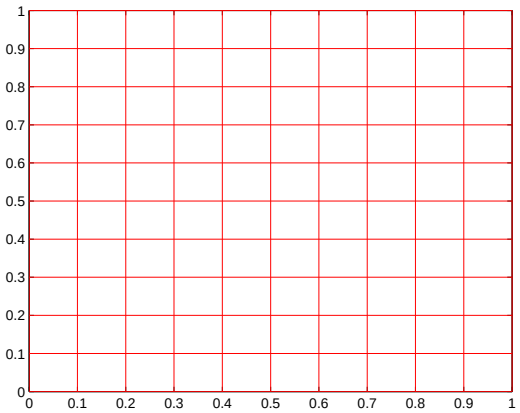
  the cost reduces from n^{2d} to dn^{d+1}.

- The method of sum factorization in FEM goes back to [Orszag, 1980] who used it with tensor-product shape functions on quadrilateral partitions.

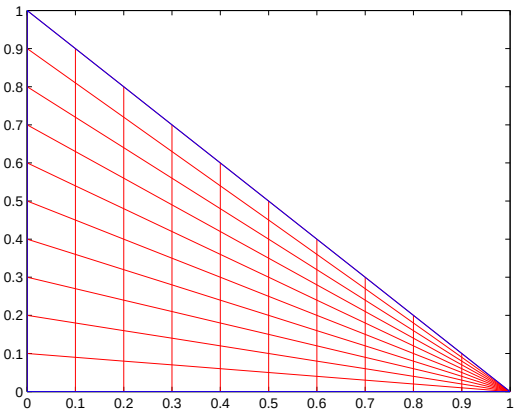
Duffy transformation

$$x(t) = \sum_{k=1}^{d+1} b_k x_k : [0,1]^d \to T = \langle x_1, \ldots, x_{d+1} \rangle$$

$$b_1 = t_1, \quad b_2 = t_2(1 - b_1), \quad b_3 = t_3(1 - b_1 - b_2), \quad \cdots,$$
$$b_d = t_d(1 - b_1 - \cdots - b_{d-1}), \quad b_{d+1} = (1 - b_1 - \cdots - b_d)$$



$\Longrightarrow$



Computation of Bernstein-Bézier Moments

Integral over simplex T reduced to box $[0,1]^d$

$$\int_T f(x)\mathrm{d}x = d!|T| \int_0^1 dt_1(1-t_1)^{d-1} \int_0^1 dt_2(1-t_2)^{d-2} \cdots \int_0^1 dt_d f(x(t))$$

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Gauss-Jacobi quadrature (exact for poly. of degree $2q-1$)

$$\int_0^1 (1-s)^\alpha s^\beta g(s)ds \approx \sum_{i=1}^{q} w_i^{(\alpha,\beta)} g(\xi_i^{(\alpha,\beta)})$$

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Stroud conical product rule:

$$\int_T f(x)\mathrm{d}x \approx d!|T| \sum_{i_1=1}^{q} w_{i_1}^{(d-1,0)} \sum_{i_2=1}^{q} w_{i_2}^{(d-2,0)} \cdots \sum_{i_d=1}^{q} w_{i_d}^{(0,0)} f(x_{i_1,i_2,\ldots,i_d}),$$

$$\text{where } x_{i_1,i_2,\ldots,i_d} := x(\xi_{i_1}^{(d-1,0)}, \xi_{i_2}^{(d-2,0)}, \ldots, \xi_{i_d}^{(0,0)}).$$

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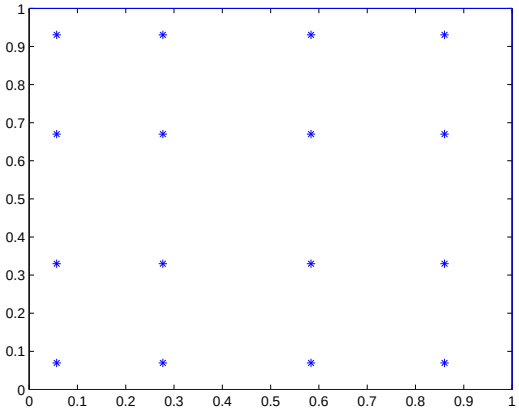
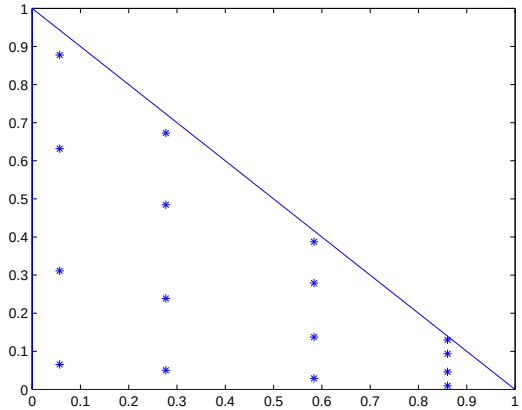


$(\xi_{i_1}^{(1,0)}, \xi_{i_2}^{(0,0)})$



x_{i_1,i_2}

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Stroud rule for the computation of the moments ($q \sim n$):

$$\rho_\eta^{n,d}(f) = \int_T f(x)\, B_\eta^{n,d}(x)\, \mathrm{d}x$$

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- Positive weights with respect to the data $f(x_{i_1,\ldots,i_d})$.
- A naive way of computing these integrals could go through precomputing the weights $w_{i_1}^{(d-1,0)} \cdots w_{i_d}^{(0,0)} B_\eta^{n,d}(x_{i_1,\ldots,i_d})$ for all $\eta \in \mathcal{I}_{n,d}$ and all $(i_1, \ldots, i_d) \in \{1, \ldots, q\}^d$, which already results in excessive storage requirement $(nq)^d \sim n^{2d}$ (square of the size n^d of the moment vector), and then computing all integrals with $\sim n^{2d}$ operations.

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- We reduce this cost to $\sim n^{d+1}$ operations by using sum factorization thanks to another nice property of Bernstein polynomials: factorization w.r.t. Duffy transformation.

Computation of Bernstein-Bézier Moments

Factorization of Bernstein polynomials w.r.t. Duffy transform

$$b_1 = t_1, \quad b_2 = t_2(1-b_1), \quad b_3 = t_3(1-b_1-b_2), \quad \cdots,$$
$$b_d = t_d(1-b_1-\cdots-b_{d-1}), \quad b_{d+1} = (1-b_1-\cdots-b_d).$$

Hence

$$b_1 = t_1, \quad b_2 = (1-t_1)t_2, \quad b_3 = (1-t_1)(1-t_2)t_3, \quad \cdots,$$
$$b_d = (1-t_1)\cdots(1-t_{d-1})t_d, \quad b_{d+1} = (1-t_1)\cdots(1-t_d).$$

$$\begin{aligned} B_\eta^{n,d}(x(t)) &= \frac{n!}{\eta_1!\cdots\eta_{d+1}!} b_1^{\eta_1} b_2^{\eta_2} \cdots b_{d+1}^{\eta_{d+1}} \\ &= \frac{n!}{\eta_1!\cdots\eta_{d+1}!} t_1^{\eta_1} (1-t_1)^{\eta_2+\cdots+\eta_{d+1}} \cdots t_d^{\eta_d} (1-t_d)^{\eta_{d+1}} \\ &= B_{\eta_1}^{n}(t_1) B_{\eta_2}^{n-\eta_1}(t_2) \cdots B_{\eta_d}^{n-\eta_1-\cdots-\eta_{d-1}}(t_d), \end{aligned}$$

since $\sum_{k=1}^{d+1} \eta_k = n$.

Computation of Bernstein-Bézier Moments

Computing the moments by sum factorisation:

$$\rho_{\eta}^{n,d}(f) = \int_T f(x)\, B_{\eta}^{n,d}(x)\, \mathrm{d}x \approx d!\,|T| \sum_{i_d=1}^{q} w_{i_d}^{(0,0)}\, B_{\eta_d}^{n-\eta_1-\cdots-\eta_{d-1}}(\xi_{i_d}^{(0,0)}) \cdots$$
$$\sum_{i_2=1}^{q} w_{i_2}^{(d-2,0)}\, B_{\eta_2}^{n-\eta_1}(\xi_{i_2}^{(d-2,0)}) \sum_{i_1=1}^{q} w_{i_1}^{(d-1,0)}\, B_{\eta_1}^{n}(\xi_{i_1}^{(d-1,0)})\, f(x_{i_1,i_2,\ldots,i_d})$$

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- Precompute $B_j^m(\xi_i^{(k,0)})$ for all $m = 0, \ldots, n$, $j = 0, \ldots, m$, $i = 1, \ldots, q$, $k = 0, \ldots, d-1$, altogether $\mathcal{O}(dqn^2)$ values, and dq values of $w_i^{(k,0)}$, $i = 1, \ldots, q$, $k = 0, \ldots, d-1$.

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- k-th loop: sum over i_k with $\mathcal{O}(q)$ flops for each $(\eta_1,\ldots,\eta_k)$, $\eta_1+\cdots+\eta_k \le n$, and each $(i_{k+1}, i_2, \ldots, i_d)$, with $\mathcal{O}(qn^k q^{d-k})$ cost, for all $k = 1,\ldots,d$.

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- Hence, overall complexity $\sim n^{d+1}$ for any $d \ge 2$.

Computation of Bernstein-Bézier Moments

Fast evaluation of a polynomial $u(x) := \sum_{\eta \in \mathcal{I}_{n,d}} u_\eta B_\eta^{n,d}(x)$

at all Stroud points:

$$u(x_{i_1,i_2,\ldots,i_d}) = \sum_{\eta_1=0}^{n} B_{\eta_1}^{n}(\xi_{i_1}^{(d-1,0)}) \sum_{\eta_2=1}^{n-\eta_1} B_{\eta_2}^{n-\eta_1}(\xi_{i_2}^{(d-2,0)}) \cdots \sum_{\eta_d=1}^{n-\eta_1-\cdots-\eta_{d-1}} B_{\eta_d}^{n-\eta_1-\cdots-\eta_{d-1}}(\xi_{i_d}^{(0,0)}) u_\eta$$

Computation of Bernstein-Bézier Moments

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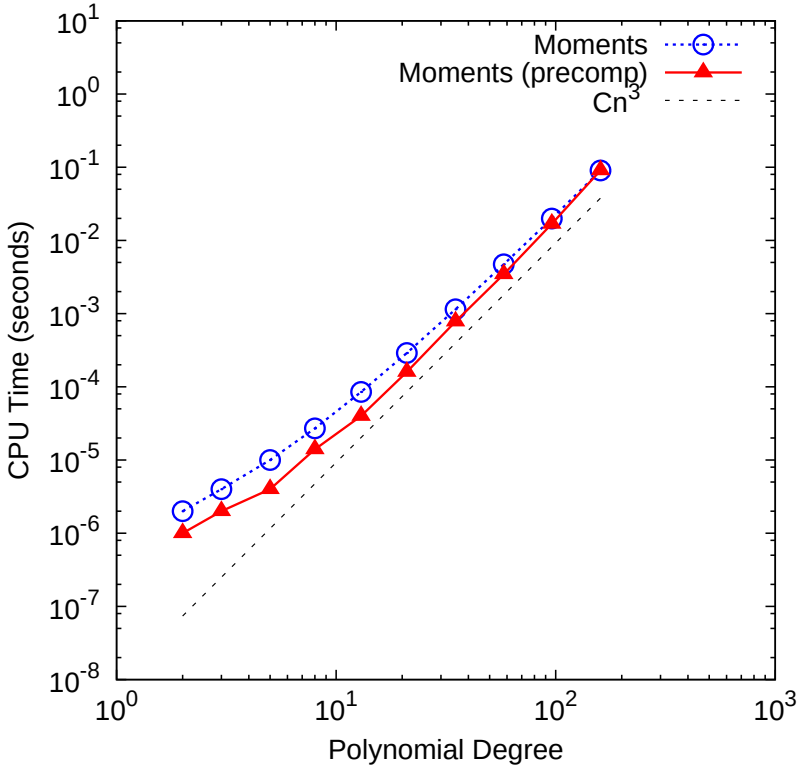
- Sum factorization: start from the i_d-loop.
- k-th loop: sum over η_k with $\mathcal{O}(1)$ flops for each $(\eta_1, \ldots, \eta_k)$, $\eta_1 + \cdots + \eta_k \leq n$, and each $(i_k, i_2, \ldots, i_d)$, with $\mathcal{O}(n^k q^{d-k+1})$ cost, for $k = d, \ldots, 1$.
- Overall complexity $\sim n^{d+1}$ (the same order as a single evaluation by de Casteljau algorithm).

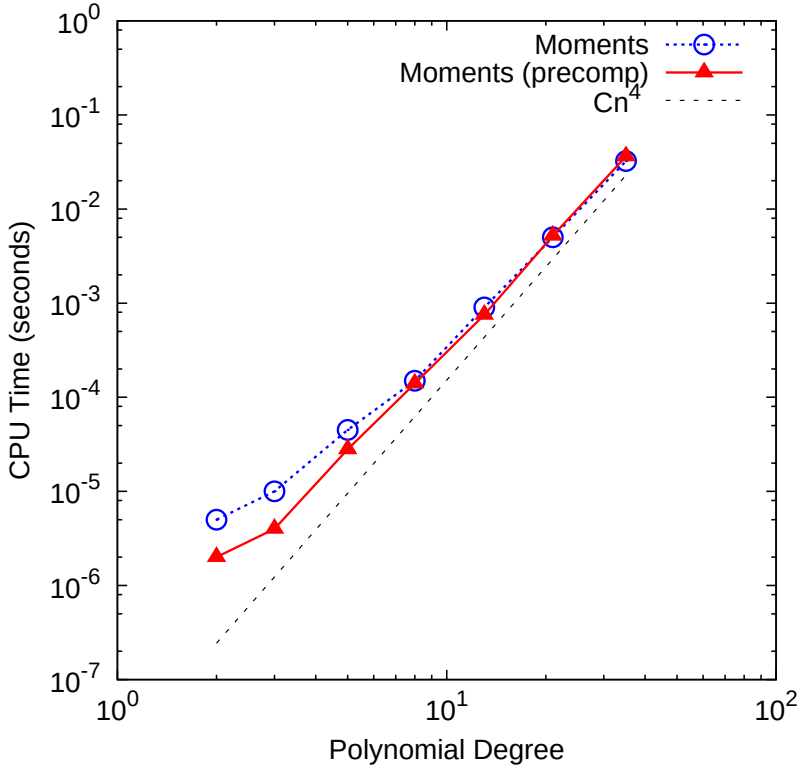
Computation of Bernstein-Bézier Moments

CPU time: computation of BB moments with $\mathcal{O}(n^{d+1})$ cost

2D



3D



C++ code available (GPL): **`www.bbfem.de`**

Outline

1. Bernstein Polynomials
2. FEM Assembly
3. Computation of Bernstein-Bézier Moments
4. Assembly of Element Matrices
5. Further Applications of BB Moments

Assembly of Element Matrices

Dirichlet problem for linear second order elliptic equation:

$$\begin{cases} -\operatorname{div}(A\nabla u) + \mathbf{b}\cdot\nabla u + cu &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{cases}$$

Weak formulation of the homogenuous problem ($g = 0$):
Find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$

$$\int_\Omega \Big[\nabla u \cdot A\nabla v + (\mathbf{b}\cdot\nabla u)v + cuv\Big]\mathrm{d}x = \int_\Omega fv\mathrm{d}x.$$

In Galerkin FEM with Bernstein basis shape functions on simplices, we need to compute for all $\xi, \zeta \in \mathcal{I}_{n,d}$

$$\int_T \Big[\nabla B_\xi^{n,d}\cdot A\nabla B_\zeta^{n,d} + (\mathbf{b}\cdot\nabla B_\xi^{n,d})B_\zeta^{n,d} + cB_\xi^{n,d}B_\zeta^{n,d}\Big]\mathrm{d}x, \quad \int_T f\,B_\zeta^{n,d}\mathrm{d}x.$$

Assembly of Element Matrices

Using Bernstein-Bézier moments

$$\rho_\xi^{n,d}(c) = \int_T c(x)\, B_\xi^{n,d}(x)\, \mathrm{d}x, \quad \xi \in \mathcal{I}_{n,d},$$

we have:

- Load vector $\int_T f(x) B_\xi^{n,d}(x) \mathrm{d}x = \rho_\xi^{n,d}(f), \quad \xi \in \mathcal{I}_{n,d},$

  This is just a moment vector, cost: $\mathcal{O}(n^{d+1})$ flops.

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- Mass matrix (thanks to the product formula)

$$\int_T c(x) B_\xi^{n,d}(x) B_\zeta^{n,d}(x) \mathrm{d}x = \frac{\binom{n}{\xi}\binom{n}{\zeta}}{\binom{2n}{\xi+\zeta}} \rho_{\xi+\zeta}^{2n,d}(c), \quad \xi, \zeta \in \mathcal{I}_{n,d}$$

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- Stiffness matrix (thanks to the gradient formula)

$$\int_T \nabla B_\xi^{n,d}(x) A(x) \nabla B_\zeta^{n,d}(x) \mathrm{d}x = n^2 \sum_{\alpha,\beta \in \mathcal{I}_{1,d}} \frac{\binom{n-1}{\xi-\alpha}\binom{n-1}{\zeta-\beta}}{\binom{2n-2}{\xi-\alpha+\zeta-\beta}} \rho_{\xi-\alpha+\zeta-\beta}^{2n-2,d}(\nabla b_\alpha^T A \nabla b_\beta)$$

Assembly of Element Matrices

Computation of the system matrices with $\mathcal{O}(1)$ cost per entry:

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- Number of entries is $|\mathcal{I}_{n,d}|^2 \sim n^{2d}$
- Multinomial coefficients may be computed recursively by a few divisions and multiplications per entry contributing $\mathcal{O}(n^{2d})$ flops.

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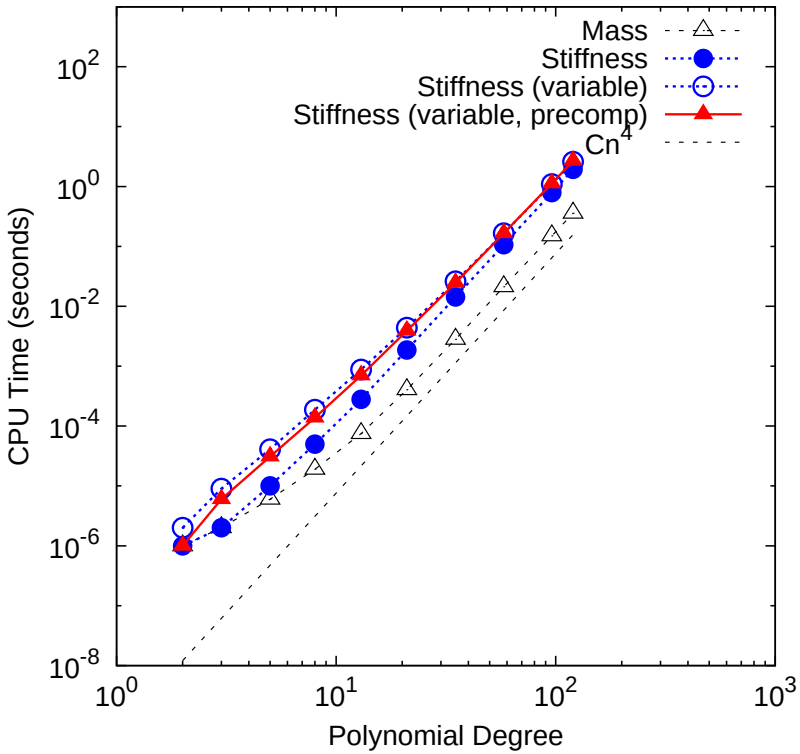
- As we know they can be computed with $\mathcal{O}(n^{d+1}) \ll n^{2d}$ flops! Hence overall cost is optimal: $\mathcal{O}(n^{2d})$

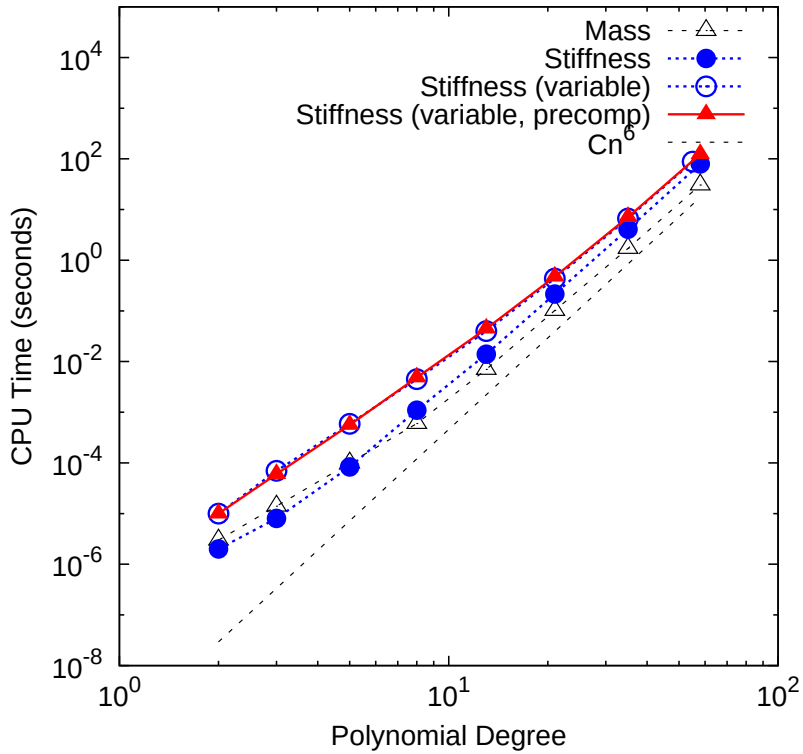
Assembly of Element Matrices

CPU time: assembly of mass and stiffness matrices

2D



3D



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History: Optimal assembly of element system matrices

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- [Ainsworth, Andriamaro & D., 2011]: Optimal complexity $\mathcal{O}(n^{2d})$ with Bernstein-Bézier shape functions.

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Matrix-vector products for iterative methods

- For example, let $M = [M_{\xi\zeta}]_{\xi,\zeta\in\mathcal{I}_{n,d}}$ be the element mass matrix,

$$M_{\xi\zeta} = \int_T c(x) B_\xi^{n,d}(x) B_\zeta^{n,d}(x) \mathrm{d}x.$$

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- [Kirby 2011, Kirby & Thinh 2012]: A different $\mathcal{O}(n^{d+1})$ algorithm for $M \cdot [u_\zeta]_{\zeta\in\mathcal{I}_{n,d}}$ implicitly relying on the same properties of Bernstein polynomials

Nonlinear problems

- Load vectors and linearized system matrices are computed as usual, with the exception that the coefficients depend on the current iterate u of the approximate solution.
- Reduces to the computation of the BB moments of the type

$$\rho_{\xi}^{n,d}(u, f) = \int_T f(x; u(x); \nabla u(x))\, B_{\xi}^{n,d}\, \mathrm{d}x, \quad \xi \in \mathcal{I}_{n,d}$$

- With u expressed in BB form, the cost of its evaluation and evaluation of its gradient at Stroud points is $\mathcal{O}(n^{d+1})$. Then f is evaluated with $\mathcal{O}(n^d)$ cost, and the usual algorithm for the moments requires again $\mathcal{O}(n^{d+1})$ flops, and we arrive at the total $\mathcal{O}(n^{d+1})$ cost for the moment vector.

Further Applications of BB Moments

Isogeometric elements

- Quadrilateral and hexahedral elements, isoparametric and other curved elements. Involve a nonlinear mapping from parameter to physical domain $\Phi : T \to \widetilde{T}$ with Jacobi matrix J_Φ.
- Stiffness matrix can be computed with the help of a BB moment vector at optimal $\mathcal{O}(n^{2d})$ cost:

$$\int_T \nabla B_\xi^{n,d}(x) J_\Phi^{-1}(x) \cdot A \nabla B_\zeta^{n,d}(x) J_\Phi^{-1}(x) |\det J_\Phi(x)| \, \mathrm{d}x$$

- Sum factorization is also useful for the assembly of high order B-spline system matrices (isogeometric analysis). Almost optimal algorithm: $\mathcal{O}(n)$ flops per non-zero entry[2]

[2] F. Calabrò, G. Sangalli and M. Tani, Fast formation of isogeometric Galerkin matrices by weighted quadrature, Comput. Methods Appl. Mech. Engrg., 316 (2017), 606–622.

Vector-valued finite elements

- Needed e.g. to solve Maxwell's equations (time harmonic):

$$\begin{cases} \operatorname{curl} \mathbf{E} + i\omega\mu\mathbf{H} &= \mathbf{0}, \\ \operatorname{curl} \mathbf{H} - i\omega\epsilon\mathbf{E} &= \mathbf{J}. \end{cases}$$

  $\mathbf{E}, \mathbf{H} \in H(\operatorname{curl}) := \{\mathbf{f} \in [L^2(\Omega)]^3 : \operatorname{curl}(\mathbf{f}) \in [L^2(\Omega)]^3\}$

- Nédélec's space for a triangulation $\triangle$ in 2D

$$\mathcal{N}_n(\triangle) = \Big\{\mathbf{s} \in [S_{n+1}(\triangle)]^2 : s \text{ is tangent continuous,}$$
$$\mathbf{s}|_T \in \mathbb{N}_n := [\mathbb{P}_n]^2 + \mathbf{x}^\perp \mathbb{H}_n \text{ for all } T \in \triangle\Big\}$$

  $\mathbf{x}^\perp = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\perp := \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$, $\mathbb{P}_n$: polynomials of total degree n,
  $\mathbb{H}_n$: homogeneous polynomials of degree n,
  $S_{n+1}(\triangle)$: (discont.) piecewise polynomials of degree $n+1$

- Shape functions are basis functions for $\mathbb{N}_n$ on triangles $T \in \triangle$. Combined together into basis functions for $\mathcal{N}_n(\triangle)$

Bernstein-Bézier shape functions for $H(\mathrm{curl})$ in 2D

- Shape functions for $\mathbb{N}_n$ are given by[3]

$$E_n^{(i)} = \{\nabla B_{\boldsymbol{\alpha}}^{n+1} : \boldsymbol{\alpha} \in \mathring{\gamma}_i^{n+1}\} \cup \{\boldsymbol{\omega}_i\}, \quad i = 1,2,3,$$
$$I_n^{\nabla} = \{\nabla B_{\boldsymbol{\alpha}}^{n+1} : \boldsymbol{\alpha} \in \mathring{\mathcal{I}}_{n+1,2}\},$$
$$I_n^{\sigma} = \left\{(n+1) B_{\boldsymbol{\alpha}}^{n} \sum_{i=1}^{3} \alpha_i \boldsymbol{\omega}_i : \boldsymbol{\alpha} \in \mathcal{I}_{n,2} \setminus \{\tilde{\boldsymbol{\alpha}}\}\right\},$$

  $\boldsymbol{\omega}_i := b_{i+1}\nabla b_{i-1} - b_{i-1}\nabla b_{i+1}$ are Whitney functions, $\mathring{\gamma}_i^{n+1}$, $\mathring{\gamma}_i^{n+1}$, $\mathring{\mathcal{I}}_{n+1,2}$, $\tilde{\boldsymbol{\alpha}}$: certain sets of indices

- High order gradient shape functions are separated
- Different shape functions based on barycentric coordinates have been suggested by [Arnold, Falk & Winther, 2009].

[3] M. Ainsworth and G. Andriamaro and O. Davydov, A Bernstein-Bézier basis for arbitrary order Raviart-Thomas finite elements, Constr. Approx., 41 (2015), 1-22.

Numerical example: Standing wave solutions of Maxwell's equations in 2D

- With $\Omega = [0,1]^2$, we want to solve the eigenvalue problem ($\lambda > 0$):

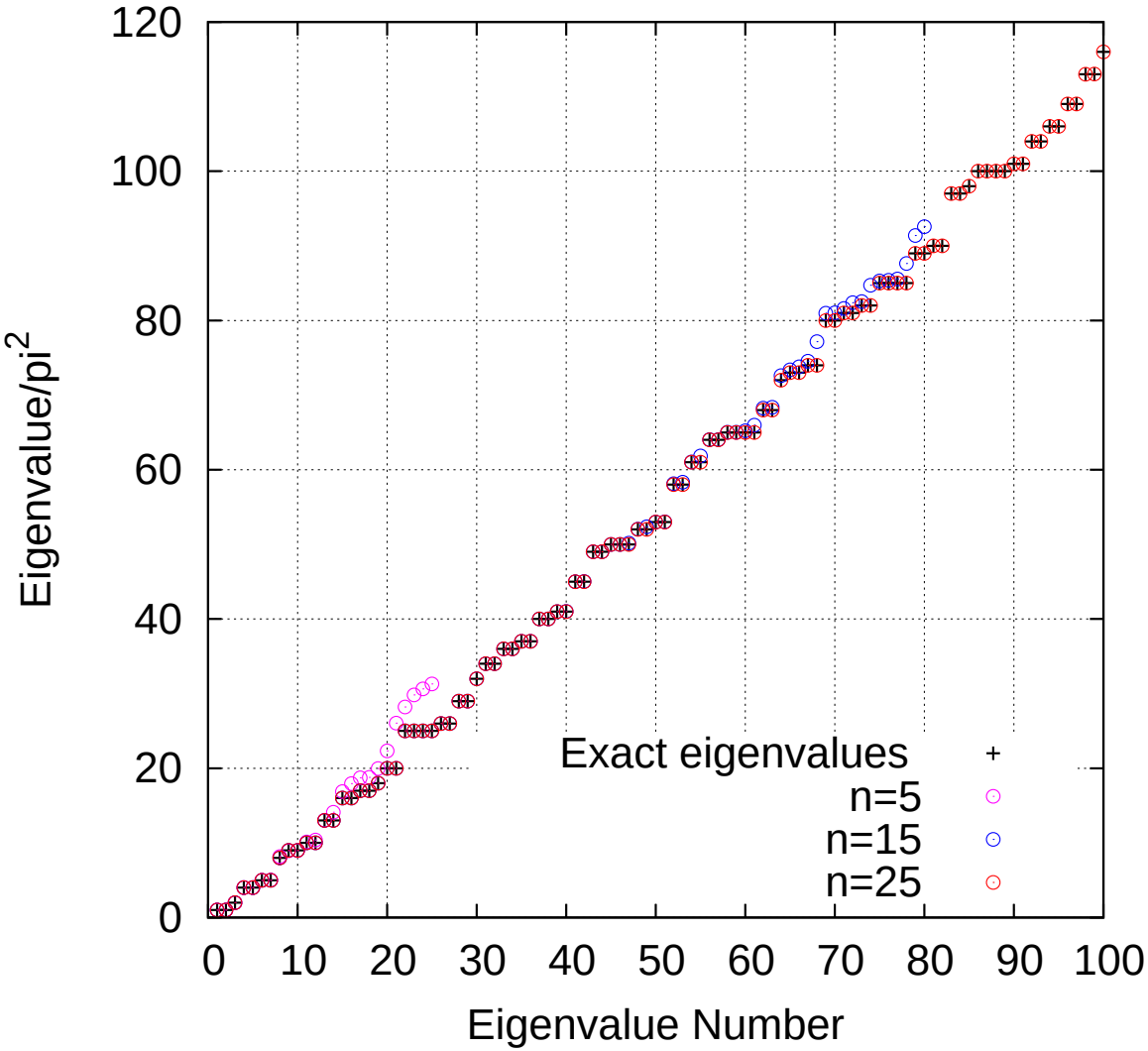
$$\begin{aligned} -\operatorname{curl}(\operatorname{curl}\mathbf{u}) &= \lambda^2 \mathbf{u} \quad \text{in } \Omega, \\ \mathbf{t}\cdot\mathbf{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

- The eigenpairs are given by $(\lambda_{k,\ell}^2, u_{k,\ell})$, $k, \ell = 0, 1, \ldots$, with

$$\lambda_{k,\ell}^2 := \pi^2(k^2+\ell^2), \quad \mathbf{u}_{k,\ell}(x,y) := \begin{pmatrix} k\sin(k\pi x)\cos(\ell\pi x) \\ \ell\sin(\ell\pi y)\cos(k\pi x) \end{pmatrix}^{\perp}.$$

- Divide the square into two triangles by a diagonal, and use BB shape functions of degree n up to 25.

Further Applications of BB Moments



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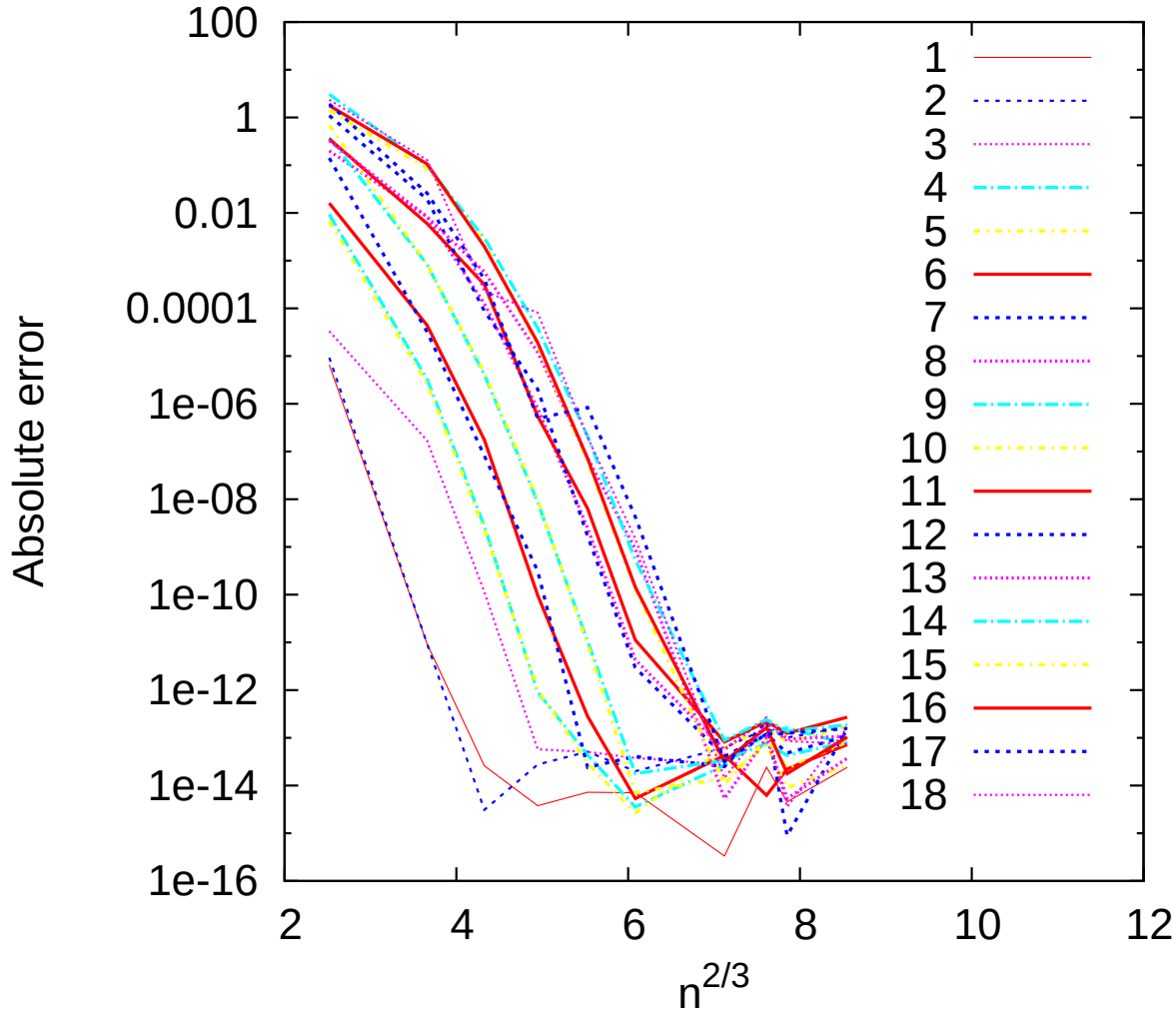

Convergence with rate $q^{n^{2/3}}$ for some $0 < q < 1$, in agreement with theoretical error bounds [Ainsworth & Coyle, 2003].

Further Applications of BB Moments

Condition numbers of simple eigenvalues

$\lambda_{1,1}$	$\lambda_{2,2}$	$\lambda_{3,3}$	$\lambda_{4,4}$	$\lambda_{6,6}$	$\lambda_{7,7}$
8.97e+00	3.69e+00	3.38e+01	8.00e+02	1.16e+06	5.25e+07

- Shown: the largest condition numbers of the simple eigenvalues that occured in our experiments with $n \leq 25$.
- The condition number ν of a simple eigenvalue λ of the generalized eigenvalue problem $\det(A - \lambda B) = 0$ is defined as

$$\nu = \frac{\|x\|_2 \|y\|_2}{\sqrt{|\alpha|^2 + |\beta|^2}}, \qquad \text{with} \quad \alpha = y^H A x, \quad \beta = y^H B x,$$

  where x and y are the right and left eigenvectors of λ [Stewart, Matrix Algorithms Vol.II, 2001].
- The numbers in the table suggest reasonable conditioning even for the 85th eigenvalue $\lambda_{7,7}$.

Further Applications of BB Moments

Some further papers on BBFEM

- **Mixed tetrahedral-hexahedral-pyramidal partitions:** M. Ainsworth, O. Davydov and L. L. Schumaker, Bernstein-Bézier finite elements on tetrahedral-hexahedral-pyramidal partitions, Comput. Methods Appl. Mech. Eng., 304 (2016), 140-170.
- **Inversion of BB mass matrix:** R. Kirby, Fast inversion of the simplicial Bernstein mass matrix. Numer. Math., 135 (2017), 73–95.
- **Comparison of GPU implementation of BB and Lagrange basis functions:** Jesse Chan and T. Warburton, GPU-accelerated Bernstein-Bezier discontinuous Galerkin methods for wave problems, SIAM J. Sci. Comp., 39 (2017), A628-A654.

Library BBFEM

C++ code for the BB-moments, system matrices and all experiments is available under GPL from

www.bbfem.de