An optimal-order collocation method in Isogeometric analysis

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4 July 2017

joint work with G. Sangalli and L. Tamellini

CIME-EMS Summer School on Splines and PDEs
Recent Advances from Approximation Theory to Structured Numerical Linear Algebra

Outline

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Poisson problem

Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $f : \Omega \to \mathbb{R}$ is a sufficiently regular function.
We suppose that the problem has an unique solution.

The idea of collocation method

- Find a suitable finite dimensional space for the solution:

$$\mathcal{V}_n = \mathrm{span}\{u_1, ..., u_n\};$$

- carefully choose n points in Ω (collocation points):

$$\Theta = \{\tau_1, ..., \tau_n\};$$

- collocate the equations of the system at these points:

find $\tilde{u} \in \mathcal{V}_n$ such that

$$-\Delta \tilde{u}(\tau_i) = f(\tau_i) \quad \forall \tau_i \in \Theta$$

$$\tilde{u}(x) = \sum_{i=1}^{n} c_i u_i(x) \Rightarrow \mathbf{K}c = F$$

$$\mathbf{K}_{i,j} = -\Delta u_j(\tau_i) \qquad F_i = f(\tau_i).$$

High regular functions needed: at least C^2 at each collocation points.

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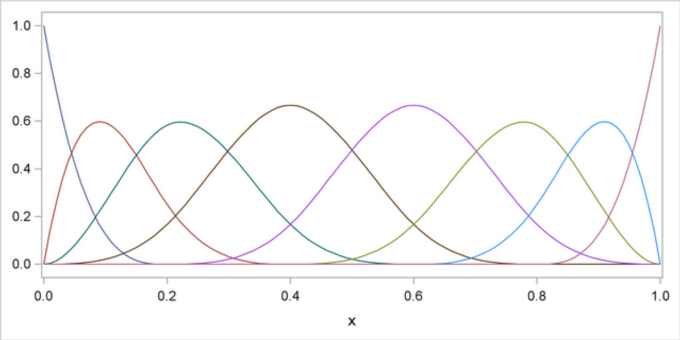
B-Splines

- *Uniform* knot vector $\Rightarrow$ no repetitions on the internal knots
  $\Rightarrow$ maximal regularity B-splines C^{p-1};
- *open* knot vector.



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An insight on C-CSP Collocation

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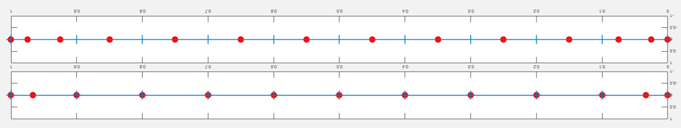
An insight on C-CSP Collocation

- IGA standard Galerkin: *slow* ; *optimal* ;
- IGA standard collocation: *fast* ; *suboptimal* ;

  Greville points: $\tau_i = \frac{\xi_{i+1}+\ldots+\xi_{i+p}}{p}$ $i = 1, \ldots, n$



	Galerkin	Greville	
		odd p	even p
L^2	$p+1$	$p-1$	p
H^1	p	$p-1$	p
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Why are we interested in efficient IGA collocation?

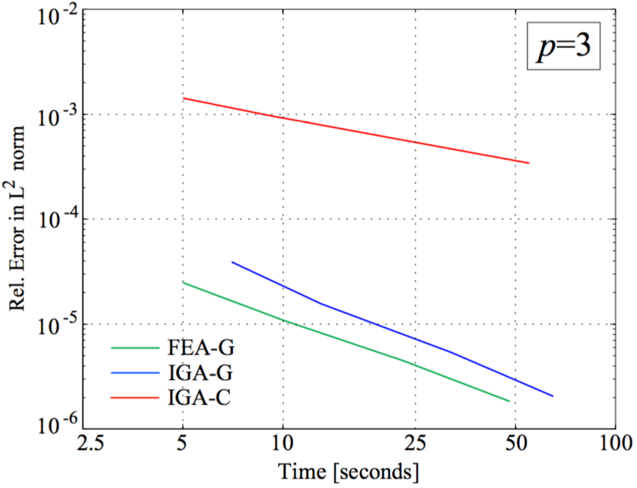

Figure courtesy of [Schillinger, Evans, Reali, Scott and Hughes, 2013]

C-CSP collocation

New schemes based on superconvergent points: [Gomez and De Lorenzis, 2016] ; in particular Collocation at Clustered Superconvergent Points (C-CSP) [Montardini, 2016] [Montardini, Sangalli and Tamellini, 2017] .

$$-u''(x) = f(x)$$
$$u(0) = u(1) = 0$$

Galerkin residual: $f - u_h'' = u'' - u_h''$.

$u_h \rightarrow$ Galerkin solution;

$u \rightarrow$ exact solution;

$h \rightarrow$ mesh-size.

Galerkin superconvergent points:

$$\Big(\sum_{\psi_{h,i} \in \Psi_h} (u'' - u_h'')^2(\psi_{h,i})\Big)^{\frac{1}{2}} \leq Ch^p.$$

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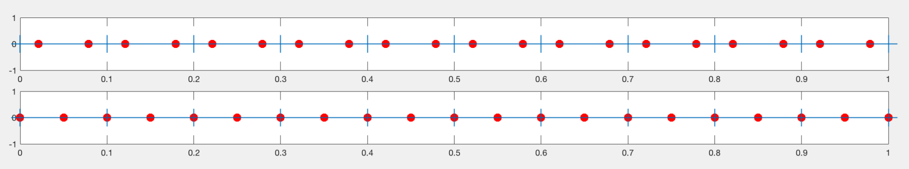
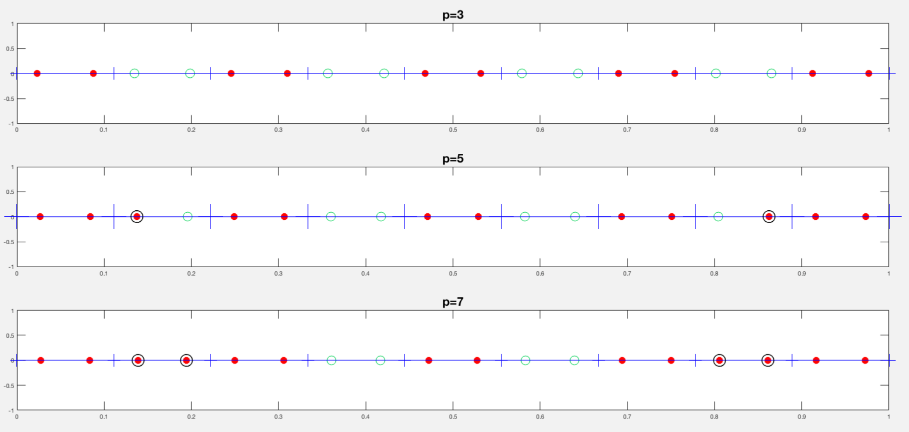
Estimated Superconvergent Points

Degree	Superconvergent Points in [-1,1]
p=3	$\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
p=4	-1,0,1
p=5	$\pm\frac{\sqrt{225-30\sqrt{30}}}{15}$
p=6	-1,0,1
p=7	$\pm$0.504918567512

[Gomez and De Lorenzis, 2016]

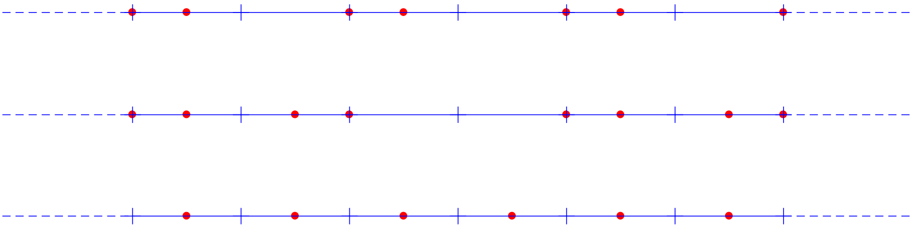


Odd p: C-CSP points



In 2D: tensor product of univariate ones.

Even p: C-CSP attempts



SUBOPTIMALITY in L^2 norm

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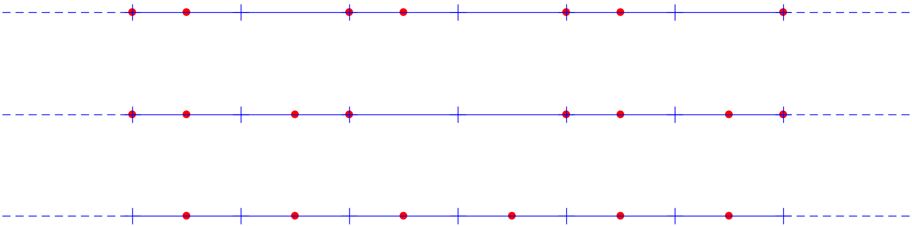

SUBOPTIMALITY in L^2 norm

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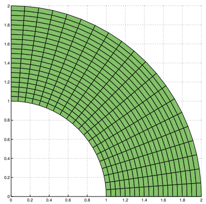
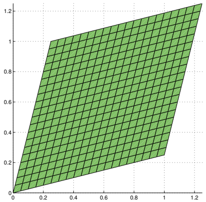
2D domains

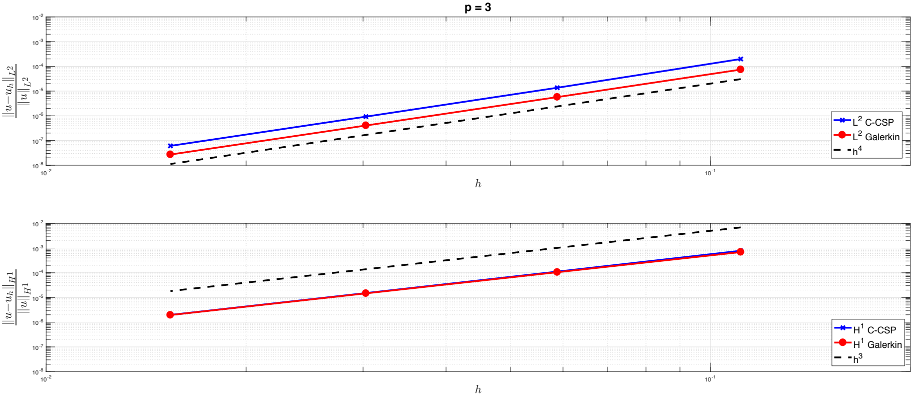


$$u(x,y) = -(x^2+y^2-1)(x^2+y^2-4)xy^2$$

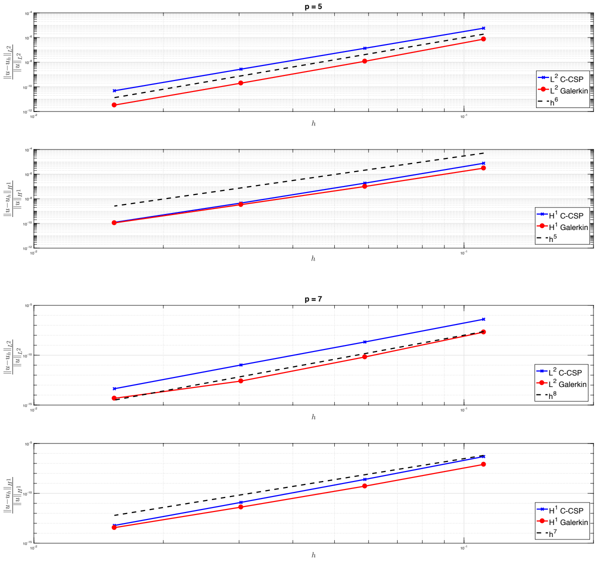


$$u(x,y) = \sin\left(-\frac{4}{15}\pi(y-4x)\right) \sin\left(-\frac{16}{15}\pi\left(\frac{x}{4}-y\right)\right)(x^3+y^3)$$

Annulus domain: p=3



Annulus domain: p=5 and p=7



Orders of convergence

	Galerkin	Greville		C-CSP	
		odd p	even p	odd p	even p
L^2	$p+1$	$p-1$	p	$p+1$	p
H^1	p	$p-1$	p	p	p
H^2	$p-1$	$p-1$	$p-1$	$p-1$	$p-1$

Conclusions

Advantages

- fast;
- optimal for odd p;
- optimal for odd p and for both periodic and Neumann problems;

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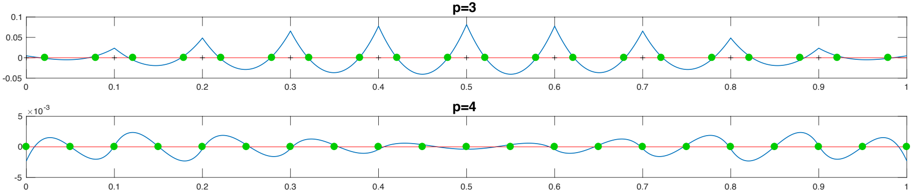

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Advantages

- fast;
- optimal for odd p;
- optimal for odd p and for both periodic and Neumann problems;

Problems

- suboptimal for even p:
- lack of mathematical explanation.

References

Schillinger, Evans, Reali, Scott and Hughes (2013). Isogeometric collocation: Cost comparison with Galerkin methods and extension to adaptive hierarchical NURBS discretizations. *Comput. Methods Appl. Mech. Engrg. 267*, 170–232.

Gomez and De Lorenzis (2016). The variational collocation method. *Comput. Methods Appl. Mech. Engrg. 309*, 152–181.

Montardini (2016). A new isogeometric collocation method based on Galerkin superconvergent points. *Master's Thesis, Università di Pavia.*.

Montardini, Sangalli and Tamellini (2017). Optimal-order isogeometric collocation at Galerkin superconvergent points. *Comput. Methods Appl. Mech. Engrg. 316*, 741–757.

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Thank you!