Spectral analysis for pure and stabilized 2D curl-curl operator with applications to the related iterative solutions

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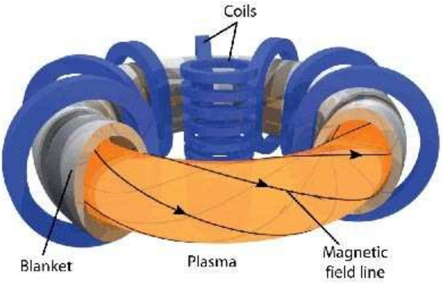
Problem setting

Lightning is an example of plasma present at Earth's surface.

- **Plasma**: a gaseous mixture of negatively charged electrons and highly charged positive ions, being created by heating a gas or by subjecting gas to a strong electromagnetic field.
- **Fusion energy**: a major area of plasma physics research which attempts to harness fusion reactions as a source of large scale sustainable energy.

Problem setting



A device that uses magnetic fields to confine plasma in the shape of a torus.

- **Tokamak**: the most well-developed and well-funded approach to fusion energy. This method races hot plasma around in a magnetically confined, donut-shaped ring, with an internal current.
- **ITER**: (International Thermonuclear Experimental Reactor) when completed, ITER will be the world's largest tokamak able to produce more energy than is required to initiate and sustain a fusion reaction.

Problem setting

MHD: (MagnetoHydroDynamics) a combination of the Navier-Stokes and Maxwell's equations which model the Plasma.

Involved operators in a 2D vector setting:

- curl-curl: $(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v})$, with $\boldsymbol{u}, \boldsymbol{v} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ in

$$H(\mathrm{curl}, \Omega) := \{\boldsymbol{u} \in (L^2(\Omega))^2 \text{ s.t. } \nabla \times \boldsymbol{u} \in L^2(\Omega)\};$$

- div-div: $(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})$ with $\boldsymbol{u}, \boldsymbol{v} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ in

$$H(\mathrm{div}, \Omega) := \{\boldsymbol{u} \in (L^2(\Omega))^2 \text{ s.t. } \nabla \cdot \boldsymbol{u} \in L^2(\Omega)\};$$

- zero order: $(\boldsymbol{u}, \boldsymbol{v})$ with $\boldsymbol{u}, \boldsymbol{v} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ either in $H(\mathrm{curl}, \Omega)$ or $H(\mathrm{div}, \Omega)$;
- combinations of the previous ones.

2D stabilized curl-curl

- **Our focus**: compatible B-Splines discretization based on the following discrete De Rham sequence[1]

$$\mathbb{R} \longrightarrow \mathcal{S}^{p,p} \xrightarrow{\text{grad}} \begin{pmatrix} \mathcal{S}^{p-1,p} \\ \mathcal{S}^{p,p-1} \end{pmatrix} \xrightarrow{\text{curl}} \mathcal{S}^{p-1,p-1} \longrightarrow 0$$

of this variational problem:

Find $\boldsymbol{u} \in H(\text{curl}, [0,1]^2)$ such that

$$(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + \mu\,(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in H(\text{curl}, [0,1]^2),$$

where $\mu \geqslant 0$.

- $S^{p_1,p_2} :=$ span $\left\{ N_{i_1}^{p_1}(t_1) N_{i_2}^{p_2}(t_2) \right\}_{i_1,i_2}$ 2D tensor-product B-spline space;
- $\nabla \times \boldsymbol{u} = \partial_x u^2 - \partial_y u^1$ for any $\boldsymbol{u} = [u^1(x,y), u^2(x,y)]^T$.

[1] Buffa, Sangalli, Vázquez, *Comput. Methods Appl. Mech. Engrg.*, 2010

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Compatible B-spline discretization

- **Coefficient matrix** $\mathcal{A}^{\mu}_{\boldsymbol{n}}$: is a 2×2 block matrix (rectangular blocks may occur)

$$\mathcal{A}^{\mu}_{\boldsymbol{n}} = \begin{bmatrix} M^{p-1}_{n_1} \otimes S^{p}_{n_2} & -A^{p}_{n_1} \otimes A^{p}_{n_2} \\ -(A^{p}_{n_1} \otimes A^{p}_{n_2})^T & S^{p}_{n_1} \otimes M^{p-1}_{n_2} \end{bmatrix} + \mu \begin{bmatrix} M^{p-1}_{n_1} \otimes M^{p}_{n_2} & \mathbf{0} \\ \mathbf{0}^T & M^{p}_{n_1} \otimes M^{p-1}_{n_2} \end{bmatrix},$$

with

- $\boldsymbol{n} = (n_1, n_2)$, where n_1, n_2 are the mesh-sizes in x, y-direction, respectively;
- $S^p_n = \int_0^1 (N^p_i(t))'(N^p_j(t))'\, dt$ stiffness matrix;
- $A^p_n = \int_0^1 N^{p-1}_i(t)(N^p_j(t))'\, dt$ 'advection' matrix;
- $M^p_n = \int_0^1 N^p_i(t) N^p_j(t)\, dt$ mass matrix.

$$\Downarrow$$

Symbol of the matrix-sequence $\{\mathcal{A}^{\mu}_{\boldsymbol{n}}\}_{\boldsymbol{n}}$

Spectral tools: symbol

- **A rather informal definition of symbol:**
  - $\{A_n\}_n$ = matrix-sequence, $\dim(A_n) = d_n \to \infty$
  - $f : D \subset \mathbb{R}^d \to \mathbb{C}$ measurable, $\quad 0 <$ measure$(D) < \infty$

$\{A_n\}_n$ has a **spectral distribution** described by f means that for n large enough

> the eigenvalues of A_n are approximately a uniform sampling of f over D.

f = **spectral symbol** of $\{A_n\}_n$. Notation: $\boxed{\{A_n\}_n \sim_\lambda (f, D)}$

- **Remark:** this definition can also be given is the singular values sense (replacing $f \to |f|$). Notation: $\{A_n\}_n \sim_\sigma (f, D)$.

Spectral tools: symbol

- **A rather informal definition of symbol:**
  - $\{A_n\}_n$ = matrix-sequence, $\dim(A_n) = d_n \to \infty$
  - $f : D \subset \mathbb{R}^d \to \mathbb{C}^{s\times s}$ measurable, $\quad 0 <$ measure$(D) < \infty$

  $\{A_n\}_n$ has a **spectral distribution** described by f means that for n large enough

  > the eigenvalues of A_n are approximately a uniform sampling of $\lambda_j(f)$ over D.

  f = **spectral symbol** of $\{A_n\}_n$. Notation: $\boxed{\{A_n\}_n \sim_\lambda (f, D)}$

- **Remark:**
  - $d_{\boldsymbol{n}}/s$ eigenvalues can be approximated by a sampling of $\lambda_1(f)$ on a uniform equispaced grid of the domain G

    $\vdots$

  - $d_{\boldsymbol{n}}/s$ eigenvalues can be approximated by a sampling of $\lambda_s(f)$ on a uniform equispaced grid of the domain G

Toeplitz sequences and GLT

- **Toeplitz sequences** $\{T_n(f)\}_n$ generated by $f \in L^1[-\pi,\pi]$ are such that

$$\{T_n(f)\}_n \sim_{\sigma,\lambda} (f, [-\pi,\pi]),$$

  under the hypothesis that f is real-valued.

- **GLT sequences** $\{A_n\}_n$ are equipped with a symbol in the singular value sense

$$\{A_n\}_n \sim_\sigma \chi, \qquad \chi : [0,1] \times [-\pi,\pi] \to \mathbb{C}. \quad \text{(1D case)}$$

  (GLT $=$ algebra containing Toeplitz, low-rank+small-norm and special diagonal matrices)

Symbol for the curl-curl problem, $\mu = 0$

- **Structure of $\mathcal{A}_{\boldsymbol{n}}^0$:** submatrix of a 2-level block Toeplitz+low-rank → **GLT**
- **Symbol:** assume $(n_1, n_2) = (\nu_1, \nu_2)n$ with $\nu_1, \nu_2 \in \mathbb{Q}$, $n \in \mathbb{N}$, then

$$\{\mathcal{A}_{\boldsymbol{n}}^0\}_{\boldsymbol{n}} \sim_\lambda (f^0, [-\pi, \pi]^2),$$

where $f^0 : [-\pi, \pi]^2 \to \mathbb{C}^{2\times 2}$ is the following dyad

$$f^0(\theta_1, \theta_2) = \frac{1}{\nu_1\nu_2} m_{p-1}(\theta_1) m_{p-1}(\theta_2) \underbrace{\begin{pmatrix} \nu_2(\mathrm{e}^{-\mathbf{i}\theta_2} - 1) \\ -\nu_1(\mathrm{e}^{\mathbf{i}\theta_1} - 1) \end{pmatrix}}_{v(\theta_1,\theta_2)} \underbrace{\begin{pmatrix} \nu_2(\mathrm{e}^{\mathbf{i}\theta_2} - 1) & -\nu_1(\mathrm{e}^{-\mathbf{i}\theta_1} - 1) \end{pmatrix}}_{v^H(\theta_1,\theta_2)}$$

with

- $m_p(\theta) = \phi_{2p+1}(p+1) + 2\sum_{k=1}^{p} \phi_{2p+1}(p+1-k)\cos(k\theta)$, symbol of the mass matrix-sequence $\{nM_n^p\}_n$;
- ϕ_q = cardinal B-spline of degree q on the nodes $0, 1, \ldots, q+1$.

Eigenvalue functions for curl-curl problem, $\mu = 0$

Since f^0 is a dyad

- $\lambda_1(f^0) \equiv 0$;
- $$\begin{aligned} \lambda_2(f^0) &= \frac{1}{\nu_1\nu_2} m_{p-1}(\theta_1) m_{p-1}(\theta_2) \left[v^H(\theta_1,\theta_2) v(\theta_1,\theta_2) \right] \\ &= \frac{1}{\nu_1\nu_2} m_{p-1}(\theta_1) m_{p-1}(\theta_2) \left[\nu_2^2(2 - 2\cos(\theta_2)) + \nu_1^2(2 - 2\cos(\theta_1)) \right]. \end{aligned}$$

A nice connection between continuous problem and spectral information:

- **Continuum**: the curl-curl operator has infinite dimensional kernel and on the complement behaves as a second order operator.

$$\Downarrow$$

- **Spectral counterpart**: $\lambda_1(f^0) \equiv 0$, while $\lambda_2(f^0)$ is the symbol of the 2D Laplacian operator[1].

[1] Donatelli, Garoni, Manni, Serra-Capizzano, Speleers, *Comput. Methods Appl. Mech. Engrg.*, 2015

Eigenvalue functions for curl-curl problem, $\mu > 0$

Eigenvalue functions of f^{μ}:

$$\lambda_1\left(f^{\mu}\right) \approx m_{p-1}(\theta_1)m_{p-1}(\theta_2)\frac{\mu}{n^2}$$

$$\lambda_2\left(f^{\mu}\right) \approx m_{p-1}(\theta_1)m_{p-1}(\theta_2)\left[\frac{1}{\nu_1\nu_2}(\nu_2^2(2-2\cos(\theta_2))+\nu_1^2(2-2\cos(\theta_1)))+\frac{\mu}{n^2}\right]$$

$$m_k = \min_{[-\pi,\pi]^2}\lambda_k(f^{\mu}), \qquad M_k = \max_{[-\pi,\pi]^2}\lambda_k(f^{\mu}),$$

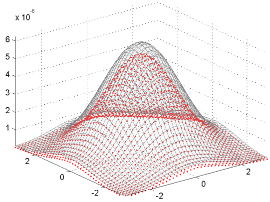
we expect the eigenvalues of $\mathcal{A}_{\boldsymbol{n}}^{\mu}$ to identify 2 blocks and to verify

$$\#\left\{i\,:\,\lambda_i(\mathcal{A}_{\boldsymbol{n}}^{\mu})\in[m_1,M_1]\right\} = \frac{\dim(\mathcal{A}_{\boldsymbol{n}}^{\mu})}{2}+o(\dim(\mathcal{A}_{\boldsymbol{n}}^{\mu})),$$

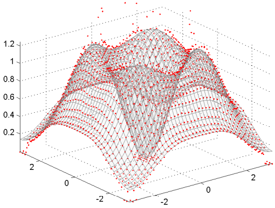
$$\#\left\{i\,:\,\lambda_i(\mathcal{A}_{\boldsymbol{n}}^{\mu})\in[m_2,M_2]\right\} = \frac{\dim(\mathcal{A}_{\boldsymbol{n}}^{\mu})}{2}+o(\dim(\mathcal{A}_{\boldsymbol{n}}^{\mu})).$$

Eigenvalue functions for curl-curl problem, $\mu > 0$

- An equispaced sampling of the eigenvalues functions in $[-\pi, \pi]^2$ gives an approximation of the eigenvalues of $\mathcal{A}_{\boldsymbol{n}}^{\mu}$.



$\lambda_1(f^{\mu})$



$\lambda_2(f^{\mu})$

Comparison between the eigenvalues of $\mathcal{A}_{\boldsymbol{n}}^{\mu}$ (red dots) and $\lambda_k(f^{\mu})$, $k = 1, 2$ (grey surface), when $n = 40$, $p = 3$, $\mu = 10^{-2}$ (matrix-size 3612).

- $\#\{i : \lambda_i(\mathcal{A}_{\boldsymbol{n}}^{\mu}) \in [m_1, M_1]\} = 1848, \qquad \#\{i : \lambda_i(\mathcal{A}_{\boldsymbol{n}}^{\mu}) \in [m_2, M_2]\} = 1659.$

Eigenvalue functions for curl-curl problem, $\mu > 0$

n	eigs in $[m_1, M_1]$	$\dim(\mathcal{A}_{\boldsymbol{n}}^{\mu})/2$	Out.	Out./$\dim(\mathcal{A}_{\boldsymbol{n}}^{\mu})$
10	168	156	12	0.0385
20	528	506	22	0.0217
30	1088	1056	32	0.0152
40	1848	1806	42	0.0116
50	2808	2756	52	0.0094
60	3968	3906	62	0.0079

n	eigs in $[m_2, M_2]$	$\dim(\mathcal{A}_{\boldsymbol{n}}^{\mu})/2$	Out.	Out./$\dim(\mathcal{A}_{\boldsymbol{n}}^{\mu})$
10	117	156	39	0.1250
20	431	506	75	0.0741
30	945	1056	111	0.0526
40	1659	1806	147	0.0407
50	2572	2756	184	0.0334
60	3687	3906	219	0.0280

Comparison of the effective number of eigenvalues of $\mathcal{A}_{\boldsymbol{n}}^{\mu}$ contained in the interval $[m_1, M_2]$ (up) $[m_2, M_2]$ (down) with the expected number $\dim(\mathcal{A}_{\boldsymbol{n}}^{\mu})/2$.

Sources of ill-conditioning for the curl-curl problem

A study of the eigenvalue functions tell us that there are three sources of ill-conditioning for the curl-curl problem:

- $\mu = 0$**:**

  (1) $\lambda_1(f^\mu) \equiv 0$, that is the number of eigenvalues in a neighborhood of zero is given by $\frac{\dim(\mathcal{A}_{\boldsymbol{n}})}{2} + o(\dim(\mathcal{A}_{\boldsymbol{n}}))$.

  (2) $\lambda_2(f^\mu)$ has an analytic zero in $(\theta_1, \theta_2) = (0, 0)$ of order 2 $\Rightarrow$ ill-conditioning in the low frequencies.

  (3) $\lambda_2(f^\mu)$ possesses infinitely many numerical exponential zeros at the π-edges when p becomes large $\Rightarrow$ ill-conditioning in the high frequencies.

- $\mu > 0$**, but 'small':** similar scenario.

Another stabilization term: the curl-div problem

- B-spline discretization of the following variational problem

Find $\boldsymbol{u} \in H(\mathrm{curl}, [0,1]^2) \cap H(\mathrm{div}, [0,1]^2)$ such that

$$\alpha(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + \beta(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}), \quad \forall \boldsymbol{v} \in H(\mathrm{curl}, [0,1]^2) \cap H(\mathrm{div}, [0,1]^2),$$

with $0 < \alpha, \beta \leqslant 1$ and $H(\mathrm{curl}, [0,1]^2) \cap H(\mathrm{div}, [0,1]^2) = H^1([0,1]^2)$.

- **Coefficient matrix:**

$$\mathcal{A}_{\boldsymbol{n}}^{\alpha,\beta} = \alpha \begin{bmatrix} M_{n_1}^p \otimes S_{n_2}^p & -A_{n_1}^p \otimes A_{n_2}^p \\ -(A_{n_1}^p \otimes A_{n_2}^p)^T & S_{n_1}^p \otimes M_{n_2}^p \end{bmatrix} + \beta \begin{bmatrix} S_{n_1}^p \otimes M_{n_2}^p & (A_{n_1}^p \otimes A_{n_2}^p)^T \\ A_{n_1}^p \otimes A_{n_2}^p & M_{n_1}^p \otimes S_{n_2}^p \end{bmatrix}$$

with $A_n^p = \int_0^1 N_i^p(t)(N_j^p(t))' \, dt$ advection matrix.

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with $A_n^p = \int_0^1 N_i^p(t)(N_j^p(t))'\,dt$ advection matrix.

Symbol for the curl-div problem

- **Structure of $\mathcal{A}_{\boldsymbol{n}}^{\alpha,\beta}$:** a permutation of $\mathcal{A}_{\boldsymbol{n}}^{\alpha,\beta}$ is a 2-level block Toeplitz+low-rank $\rightarrow$ **GLT**
- **Symbol:** assume $(n_1, n_2) = (\nu_1, \nu_2)n$ with $\nu_1, \nu_2 \in \mathbb{Q}$, $n \in \mathbb{N}$, then

$$\left\{\mathcal{A}_{\boldsymbol{n}}^{\alpha,\beta}\right\}_{\boldsymbol{n}} \sim_\lambda (f^{\alpha,\beta}, [-\pi,\pi]^2),$$

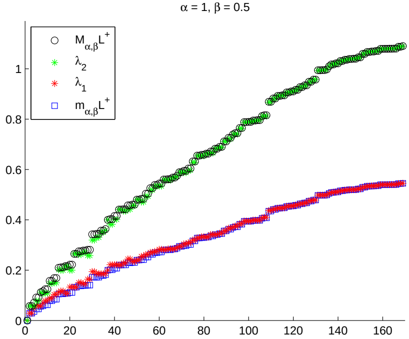
where $f^{\alpha,\beta} : [-\pi,\pi]^2 \rightarrow \mathbb{C}^{2\times 2}$ with

$$f_{11}^{\alpha,\beta}(\theta_1,\theta_2) = \alpha\frac{\nu_2}{\nu_1}m_p(\theta_1)m_{p-1}(\theta_2)(2-2\cos(\theta_2)) + \beta\frac{\nu_1}{\nu_2}m_{p-1}(\theta_1)m_p(\theta_2)(2-2\cos(\theta_1)),$$
$$f_{12}^{\alpha,\beta}(\theta_1,\theta_2) = (\alpha-\beta)a_p(\theta_1)a_p(\theta_2),$$
$$f_{21}^{\alpha,\beta}(\theta_1,\theta_2) = f_{12}^{\alpha,\beta}(\theta_1,\theta_2),$$
$$f_{22}^{\alpha,\beta}(\theta_1,\theta_2) = \alpha\frac{\nu_1}{\nu_2}m_{p-1}(\theta_1)m_p(\theta_2)(2-2\cos(\theta_1)) + \beta\frac{\nu_2}{\nu_1}m_p(\theta_1)m_{p-1}(\theta_2)(2-2\cos(\theta_2)).$$

with $a_p(\theta) = -2\sum_{k=1}^{p} \phi'_{2p+1}(p+1-k)\sin(k\theta)$ symbol of the advection matrix-sequence $\{-\mathbf{i}A_n^p\}_n$.

Bounds for the eigenvalue functions

Here, $n = 10$, $p = 3$.



$$0 \leqslant \underbrace{\min(\alpha,\beta)}_{m_{\alpha,\beta}} L(\theta_1,\theta_2) \leqslant \lambda_1(f^{\alpha,\beta}) \leqslant \lambda_2(f^{\alpha,\beta}) \leqslant \underbrace{\max(\alpha,\beta)}_{M_{\alpha,\beta}} L(\theta_1,\theta_2).$$

$L(\theta_1, \theta_2)$ is the symbol of the Laplacian operator.

A subcase: $\alpha = \beta = 1$ (vector Laplacian)

Case $\alpha = \beta = 1$: vector Laplacian matrix-sequence

$$\{\mathcal{L}_{\boldsymbol{n}}\}_{\boldsymbol{n}} = \left\{ \begin{pmatrix} (\mathcal{A}_{\boldsymbol{n}}^{1,1})_{(1,1)} & 0 \\ 0 & (\mathcal{A}_{\boldsymbol{n}}^{1,1})_{(2,2)} \end{pmatrix} \right\}_{\boldsymbol{n}} \sim_\lambda \left(\begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, [-\pi, \pi]^2 \right)$$

(1) L is nonnegative and has a zero of order 2 in zero $\longrightarrow$ $\mathcal{L}_{\boldsymbol{n}}$ is ill-conditioned in the low frequencies. Classical problem solved by MGM preconditioning.

(2) L has infinitely many exponential numerical zeros at the π-edges when p becomes large $\longrightarrow$ $\mathcal{L}_{\boldsymbol{n}}$ is ill-conditioned in the high frequencies. Non-canonical problem solvable by GLT theory.

[1] Donatelli, Garoni, Manni, Serra-Capizzano, Speleers, *Comput. Methods Appl. Mech. Engrg.*, 2015

A subcase: $\alpha = \beta = 1$ (vector Laplacian)

- **MGM-GLT**[1]: A suitable smoother for MGM is suggested by the symbol of $\mathcal{L}_{\boldsymbol{n}}$:

PCG or PGMRES with preconditioner

$$\begin{pmatrix} T(m_{p-1}(\theta_1)) \otimes T(m_{p-1}(\theta_2)) & 0 \\ 0 & T(m_{p-1}(\theta_1)) \otimes T(m_{p-1}(\theta_2)) \end{pmatrix}$$

- **Remark:** such a post-smoothing is used only at the finest level (few iterations), while at the other levels we use standard Gauss-Seidel pre- and post-smoothing.
- **Remark:** such a preconditioner is a tensor product of banded matrices then only a linear computational cost is required.

[1] Donatelli, Garoni, Manni, Serra-Capizzano, Speleers, *Comput. Methods Appl. Mech. Engrg.*, 2015

Numerical proposal

Thanks to the following relation,

$$0 \leqslant \underbrace{\min(\alpha,\beta)}_{m_{\alpha,\beta}} L(\theta_1,\theta_2) \leqslant \lambda_1(f^{\alpha,\beta}) \leqslant \lambda_2(f^{\alpha,\beta}) \leqslant \underbrace{\max(\alpha,\beta)}_{M_{\alpha,\beta}} L(\theta_1,\theta_2).$$

$L(\theta_1,\theta_2)$ is the symbol of the Laplacian operator.

we can apply the **MGM-GLT** method to the coefficient matrix $\mathcal{A}_{\boldsymbol{n}}^{\alpha,\beta}$ for general α and β.

- We expect robustness with respect to p;
- We expect optimality with respect to the matrix-size;
- A first attempt to guarantee robustness with respect to α,β is to use the MGM-GLT as preconditioner for the CG.

Numerical results

Test example: tol$= 10^{-7}$. Here n is the mesh-size.

$p=1$				$p=2$				$p=3$			
n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG
16	3	5	28	15	2	4	19	14	2	4	31
32	3	5	55	31	2	4	34	30	2	5	39
64	3	5	107	63	2	4	66	62	2	5	68
$p=4$				$p=5$				$p=6$			
n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG
13	2	4	62	12	2	4	103	11	3	5	160
29	2	4	70	28	2	4	142	27	3	5	281
61	3	5	88	60	3	5	155	59	3	5	325

$\alpha = 1,\ \beta = 1$

"P_{MGM}" is the preconditioner given by one iteration of the multigrid "MGM-GLT" applied to the coefficient matrix $\mathcal{A}_{\boldsymbol{n}}^{\alpha,\beta}$.

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$p=1$				$p=2$				$p=3$			
n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG
16	6	18	76	15	5	12	49	14	4	10	52
32	6	17	149	31	5	11	94	30	5	10	94
64	6	16	280	63	5	11	180	62	5	10	187
$p=4$				$p=5$				$p=6$			
n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG
13	4	10	105	12	5	11	206	11	6	13	330
29	4	10	119	28	5	11	231	27	6	13	475
61	5	10	192	60	5	11	265	59	5	13	522

$\alpha = 1,\ \beta = 10^{-1}$

"P_{MGM}" is the preconditioner given by one iteration of the multigrid "MGM-GLT" applied to the coefficient matrix $\mathcal{A}_{\boldsymbol{n}}^{\alpha,\beta}$.

Numerical results

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$p=1$				$p=2$				$p=3$			
n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG
16	16	105	166	15	15	94	128	14	13	77	124
32	17	115	352	31	15	83	237	30	14	73	234
64	17	118	698	63	14	72	449	62	14	68	460
$p=4$				$p=5$				$p=6$			
n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG	n	P_{MGM}	MGM	CG
13	13	67	149	12	14	70	273	11	16	78	No conv.
29	13	65	240	28	13	60	330	27	14	66	602
61	13	65	476	60	13	62	510	59	13	60	667

$\alpha = 1,\ \beta = 10^{-2}$

"P_{MGM}" is the preconditioner given by one iteration of the multigrid "MGM-GLT" applied to the coefficient matrix $\mathcal{A}_{\boldsymbol{n}}^{\alpha,\beta}$.

Conclusions

Summary:

- We use the GLT theory to spectrally analyse matrices coming from a IgA discretization of the curl-curl and curl-div problems.
- We exploit the obtained spectral information to suggest a suitable solver for the corresponding linear systems.

Ongoing works and future tasks:

- Multidimensional spectral analysis and related iterative strategies.
- Deal with the parameters μ, α, β: depending on their value the coefficient matrix can be highly ill-conditioned, so a regularization strategy should be applied.
- Extend this approach to the case with mapping (general geometries).

THANK YOU FOR YOUR ATTENTION!