Spline Fitting with Additional Normal Data

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CIME Summer School
July 3–7, 2017

JOHANNES KEPLER UNIVERSITY LINZ

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Agenda

1. The Fitting Problem
2. Some Theory and Examples
3. Next Steps

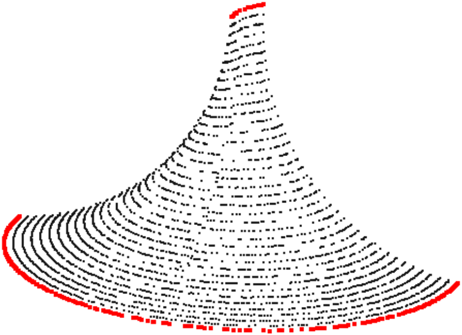
Motivation

Fitting:

- Approximate given function data (continuous or discrete) by splines
- Special treatment of interfaces or boundary edges possible

Industrial Application:

clamped fillet, generate smooth transitions across the top and bottom boundary



Least Squares: Introduction

Given (discrete case):

- Data samples $p_j \in \mathbb{R}^3$, $j = 1, \ldots, M$
- Parameter values $t_j = (u_j, v_j) \in \mathbb{R}^2, j = 1, \ldots, M$
- Spline basis $B_i : [0,1]^2 \mapsto \mathbb{R}, i = 1, \ldots, n$

Standard procedure: Find control points
$s = (d_1^1, d_1^2, d_1^3, \ldots, d_n^1, d_n^2, d_n^3)$, such that

$$F(s) = \sum_{j=1}^{M} \| \underbrace{\sum_{i=1}^{n} d_i B_i(t_j)}_{\text{spline surface } x_s(t_j)} - p_j \|_2^2$$

is minimal.
Next step: Generalize the objective function.

Smooth Transitions Across Boundaries

How to achieve smooth transitions across patch interfaces?

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$$\downarrow$$

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- n_k: given *unit* normal vector
- $n_{x_s}(\hat{t}_k)$: *unit* normal vector of the solution at a given parameter value $\hat{t}_k$
- γ: weight

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Find minimizer of $F(s)$ with Gauss-Newton.

Theory: Questions

- Existence of a solution - dependence on mesh parameter h
- Speed of convergence

Stability of a B-spline Basis

Theorem

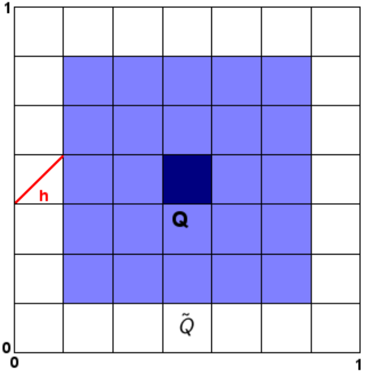
There exists a constant K such that all linear spline combinations $x_s(u) = \sum_{i=1}^{n} d_i B_i(u)$ with control point vector s fulfill

$$\frac{1}{K}\|s\|_\infty \le \|x_s\|_\infty \le \|s\|_\infty,$$

where $\|x\|_\infty = \sup_{(u,v)} \|x(u,v)\|_2$.

de Boor: Spline approximation by quasi interpolants.
Lyche, Peña: Optimally stable multivariate bases

Some Notation



Q: micro element
$\tilde{Q}$: support extension for $p = 2$
h: mesh size

$S_p(\Xi)$: spline space defined by degree p and knot vector Ξ
$\Pi_{p,\Xi}$: spline projector to $S_p(\Xi)$

Sobolev spaces:

- $H^r(Q)$: r-th derivative square integrable
- $\mathcal{H}^r_h(\tilde{Q})$: bent Sobolev space
  - r-th derivatives on single element Q square integrable
  - across element interfaces smooth like splines

Approximation Power of Splines

Theorem

Let the mesh $\mathcal{Q}$ induced by Ξ be locally quasi uniform with mesh parameter h. There is a constant $C > 0$ such that for all $s \in \mathbb{N}$, $s \leq p + 1$ and for all $f \in \mathcal{H}^s_h(\tilde{Q})$

$$|f - \Pi_{p,\Xi}(f)|_{H^r(Q)} \leq C \cdot h^{s-r} |f|_{\mathcal{H}^s(\tilde{Q})}$$

for $0 \leq r \leq s$.

Bazilevs, da Veiga, Cottrell, Hughes, Sangalli: Isogeometric Analysis: Approximation, stability and error estimates for h-refined meshes..

Application to our Problem

Continuous version of the problem:

$$F_h(s) = \|x_{s,h} - f\|_{L^2}^2 + \gamma_0 h^2 \|Nx_{s,h} - Nf\|_{L^2}^2 \to \min,$$

γ_0 constant

- $\varepsilon_h = \|x_{s,h} - f\|_{L^2}$ point error
- $\eta_h = \|Nx_{s,h} - Nf\|_{L^2}$ normal error

Theorem (Existence of a solution and convergence rate)

- *For every h this problem has a solution.*
- *The sequence of solutions realizes the optimal approximation order, i.e. there exists constants C_1, C_2 such that*
  - $\varepsilon_h \leq C_1 h^{p+1}$
  - $\eta_h \leq C_2 h^p$

Proof (Part 1, Sketch).

Compactness Argument: By stability, it suffices to consider $\min_s F_h(s)$ on a box with side length $K(C + \|f\|_2)$, i.e. on a compact domain.

Proof (Part 2, Sketch).

We know:

$$\|\Pi_{p,\Xi} f - f\|_{L^2} \leq c_1 h^{p+1} \text{ for some constant } c_1 \tag{1}$$

To show:

$$\|N\Pi_{p,\Xi} f - Nf\|_{L^2} \leq c_2 h^{p} \text{ for some constant } c_2 \tag{2}$$

(1) and (2) imply

$$\begin{aligned} F_h(s) &= \|x_{s,h} - f\|_{L^2}^2 + \gamma_0 h^2 \|N x_{s,h} - Nf\|_{L^2}^2 \\ &\leq c_1^2 h^{2p+2} + \gamma_0 h^2 c_2 h^{2p} \\ &= (c_1 + \gamma_0 c_2) h^{2p+2} \end{aligned}$$

Proof (Part 3, Sketch).

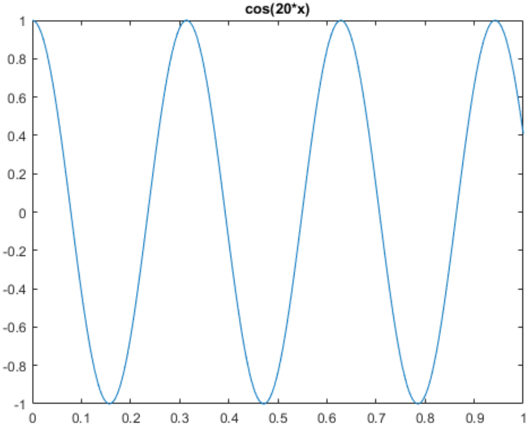
$$\begin{aligned}\varepsilon_h^2 &\leq F_h(s) \\ &\leq (c_1^2 + \gamma_0 c_2^2) h^{2p+2} \\ \Rightarrow \varepsilon_h &\leq \sqrt{c_1^2 + \gamma_0 c_2^2}\, h^{p+1}\end{aligned}$$

$$\begin{aligned}\gamma_0 h^2 \eta_h^2 &\leq F_h(s) \\ &\leq (c_1^2 + \gamma_0 c_2^2) h^{2p+2} \\ \Rightarrow \eta_h &\leq \sqrt{\frac{c_1^2 + \gamma_0 c_2^2}{\gamma_0}}\, h^p\end{aligned}$$

$\square$

Convergence Rates: Results

Sample data: $\cos(20x)$ on $[0, 1]$, 10 000 samples



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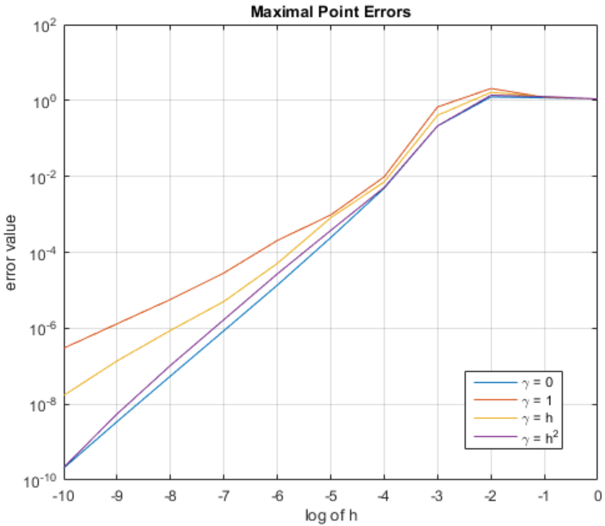

B-splines of degree 3, $\gamma_0 = 1$

Measured error: $\max_j \|x_{s,h}(t_j) - p_j\|_2$

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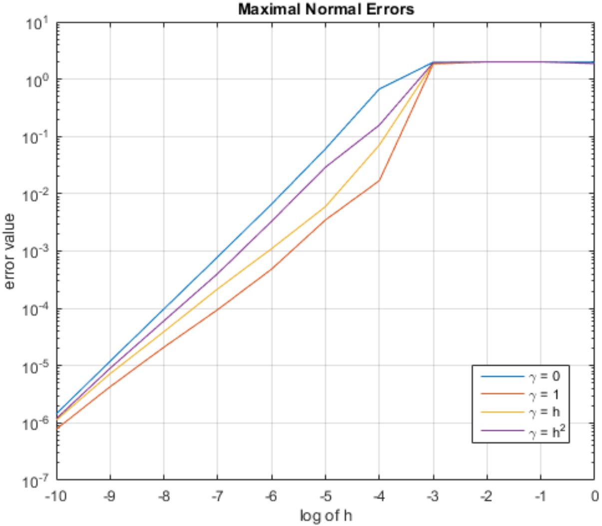

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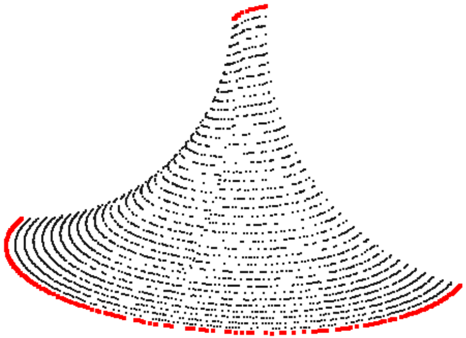
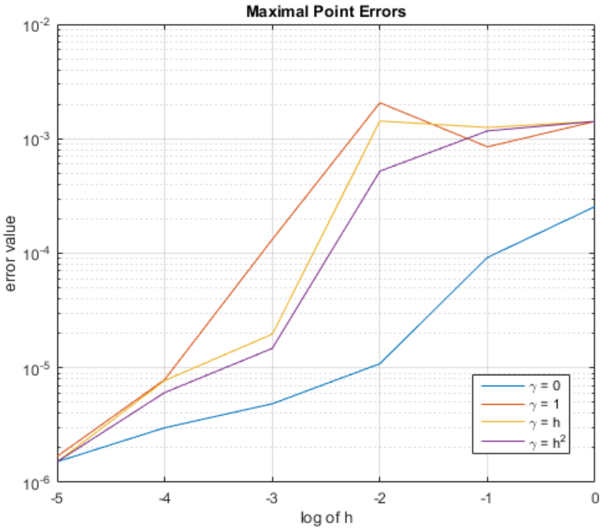
Recall: Data Set



Normal samples only along the red boundaries.

Convergence Rates: Industrial Application



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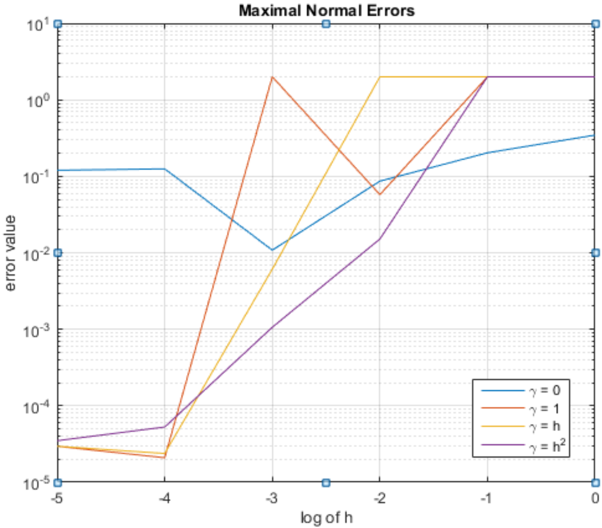

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Next Steps: Control of weight γ_0

Prescribed:

- error thresholds $\bar{\varepsilon}$ and $\bar{\eta}$ for the point and normal errors ε_h and η_h
- (induced) ratio $\varrho = \frac{\bar{\varepsilon}}{\bar{\eta}}$ between $\bar{\varepsilon}$ and $\bar{\eta}$

Idea:

- Use ϱ to reach $\bar{\varepsilon}$ and $\bar{\eta}$ quickly, i.e. without using unnecessarily many dofs.
- Control γ_0 to reach and/or keep ϱ in each refinement step.
    - For each h, test if $\varepsilon_h \leq \bar{\varepsilon}$ and $\eta_h \leq \bar{\eta}$.
    - If this is not the case, choose γ_0 s.t. $\varrho_h = \frac{\varepsilon_h}{\eta_h} \approx \varrho$

How to find a suitable value γ_0:
Find a function δ which describes the dependence $\frac{\varepsilon_h}{\eta_h} = \delta(\gamma_0)$ for each h.

Extension: Replace $\|\cdot\|_2^2$ by a norm-like function

$$F(s) = \sum_{j=1}^{M} \|x_s(t_j) - p_j\|_2^2$$

$$\downarrow$$

$$F(s) = \sum_{j=1}^{M} N(\|x_s(t_j) - p_j\|_2),$$

$N : \mathbb{R}^+ \to \mathbb{R}^+, N \in \mathcal{C}^2$, is a norm-like function

Advantages:

- different norms in the objective function treat outliers differently
- non-differentiable norms can be approximated
- least squares problem contained as a special case

Done: Implementation - Missing: Theory

Thank you for your attention!