B-Splines and Spline Approximation

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Abstract After presenting a detailed summary of the main analytic properties of B-splines, we discuss in details the approximation power of the spline space they span. More precisely, we analyze the distance of any sufficiently smooth function from the considered spline space. Using the properties of the B-spline basis, we explicitly construct a (local) quasi-interpolant based on integral averages which achieves the optimal accuracy for approximating the function and its derivatives, and we determine the corresponding error bounds.

1 B-splines and piecewise polynomials

This section introduces one of the most powerful tools in CAGD: B-splines. We present the definition and main properties of the B-spline basis as well as the properties of the space they span.

1.1 B-splines

We start by defining B-spline functions (in short B-splines) [1] and derive some of their most fundamental properties. B-splines are piecewise polynomials with a certain global smoothness. The positions where the pieces meet are known as knots.

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[1] The original meaning of the word "spline" is a flexible ruler used to draw curves, mainly in the aircraft and shipbuilding industries. The "B" in B-splines stands for basis or basic.

1.1.1 Definition and basic properties

In order to define B-splines we need the concept of knot sequences.

Definition 1. *A* **knot sequence** $\boldsymbol{\xi}$ *is a nondecreasing sequence of real numbers,*

$$\boldsymbol{\xi} := \{\xi_i\}_{i=1}^m = \{\xi_1 \le \xi_2 \le \cdots \le \xi_m\}, \quad m \in \mathbb{N}.$$

The elements ξ_i are called **knots***.*

Provided that $m \ge p+2$ we can define B-splines of degree p over the knot-sequence $\boldsymbol{\xi}$.

Definition 2. *Suppose for a nonnegative integer p and some integer j that $\xi_j \le \xi_{j+1} \le \cdots \le \xi_{j+p+1}$ are $p+2$ real numbers taken from a knot sequence $\boldsymbol{\xi}$. The j-th* **B-spline** $B_{j,p,\boldsymbol{\xi}} : \mathbb{R} \to \mathbb{R}$ *of degree p is identically zero if $\xi_{j+p+1} = \xi_j$ and otherwise defined recursively by*[2]

$$B_{j,p,\boldsymbol{\xi}}(x) := \frac{x-\xi_j}{\xi_{j+p}-\xi_j} B_{j,p-1,\boldsymbol{\xi}}(x) + \frac{\xi_{j+p+1}-x}{\xi_{j+p+1}-\xi_{j+1}} B_{j+1,p-1,\boldsymbol{\xi}}(x), \tag{1}$$

starting with

$$B_{i,0,\boldsymbol{\xi}}(x) := \begin{cases} 1, & \text{if } x \in [\xi_i, \xi_{i+1}), \\ 0, & \text{otherwise}. \end{cases}$$

Here we used the convention that fractions with zero denominator have value zero.

We start with some preliminary remarks.

- For degree 0 the B-spline $B_{0,p,\boldsymbol{\xi}}$ is simply the characteristic function of the half open interval $[\xi_j, \xi_{j+1})$. This implies that a B-spline is continuous except possibly at a knot ξ. We have $B_{j,p,\boldsymbol{\xi}}(\xi) = B_{j,p,\boldsymbol{\xi}}(\xi_+)$, where

$$x_+ := \lim_{\substack{t \to x \\ t > x}} t, \quad x_- := \lim_{\substack{t \to x \\ t < x}} t, \quad x \in \mathbb{R}.$$

  Thus a B-spline is **right continuous**, i.e., the value at a point x is obtained by taking limits from the right.

- We also use the notation

$$B[\xi_j, \ldots, \xi_{j+p+1}] := B_{j,p,\boldsymbol{\xi}},$$

  showing explicitly on which knots the B-spline depends.

[2] The recurrence relation is due to Cox, de Boor and Mansfield [6, 2]. However, it appears already in works by Popoviciu and Chakalov in the 1930's; see [5] for an account of the early history of splines. For the modern theory of splines we refer to the seminal papers by Schoenberg [16, 17, 18] and Curry/Schoenberg [7, 8]. In their works, B-splines were studied using divided differences.

- We say that a knot has **multiplicity** μ if it occurs exactly μ times in the knot sequence. A knot is called **simple**, **double**, **triple**, $\dots$ if its multiplicity is equal to $1, 2, 3, \dots$, and a **multiple knot** in general.

Example 3. A B-spline of degree 1 is also called a **linear B-spline** or a **hat function**. The recurrence relation (1) takes the form

$$B_{j,1,\boldsymbol{\xi}}(x) = \frac{x-\xi_j}{\xi_{j+1}-\xi_j} B_{j,0,\boldsymbol{\xi}}(x) + \frac{\xi_{j+2}-x}{\xi_{j+2}-\xi_{j+1}} B_{j+1,0,\boldsymbol{\xi}}(x),$$

resulting in

$$B_{j,1,\boldsymbol{\xi}}(x) = \begin{cases} \dfrac{x-\xi_j}{\xi_{j+1}-\xi_j}, & \text{if } x \in [\xi_j, \xi_{j+1}), \\ \dfrac{\xi_{j+2}-x}{\xi_{j+2}-\xi_{j+1}}, & \text{if } x \in [\xi_{j+1}, \xi_{j+2}), \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

The linear B-spline is discontinuous at a double knot and continuous at a simple knot.

Example 4. A B-spline of degree 2 is also called a **quadratic B-spline**. Using the recurrence relation (1), the three pieces of the quadratic B-spline $B_{j,2,\boldsymbol{\xi}}$ are given by

$$B_{j,2,\boldsymbol{\xi}}(x) = \begin{cases} \dfrac{(x-\xi_j)^2}{(\xi_{j+2}-\xi_j)(\xi_{j+1}-\xi_j)}, & \text{if } x \in [\xi_j, \xi_{j+1}), \\ \dfrac{(x-\xi_j)(\xi_{j+2}-x)}{(\xi_{j+2}-\xi_j)(\xi_{j+2}-\xi_{j+1})} + \dfrac{(x-\xi_{j+1})(\xi_{j+3}-x)}{(\xi_{j+2}-\xi_{j+1})(\xi_{j+3}-\xi_{j+1})}, & \text{if } x \in [\xi_{j+1}, \xi_{j+2}), \\ \dfrac{(\xi_{j+3}-x)^2}{(\xi_{j+3}-\xi_{j+1})(\xi_{j+3}-\xi_{j+2})}, & \text{if } x \in [\xi_{j+2}, \xi_{j+3}), \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

The general explicit expression for a B-spline quickly becomes complicated. Applying the recurrence relation repeatedly we find

$$B_{j,p,\boldsymbol{\xi}}(x) = \sum_{i=j}^{j+p} B_{j,p,\boldsymbol{\xi}}^{\{i\}}(x) B_{i,0,\boldsymbol{\xi}}(x), \quad p \geq 0, \tag{4}$$

where each $B_{j,p,\boldsymbol{\xi}}^{\{i\}}$ is a polynomial of degree p which is zero if $\xi_i = \xi_{i+1}$. In particular, for the nontrivial cases we have

$$B_{j,0,\boldsymbol{\xi}}^{\{j\}}(x) = 1, \quad B_{j,1,\boldsymbol{\xi}}^{\{j\}}(x) = \frac{x-\xi_j}{\xi_{j+1}-\xi_j}, \quad B_{j,1,\boldsymbol{\xi}}^{\{j+1\}}(x) = \frac{\xi_{j+2}-x}{\xi_{j+2}-\xi_{j+1}}.$$

For $p \geq 2$, in the nontrivial cases, it follows from Definition 2 that the first and last polynomial pieces in (4) are given by

$$
\begin{aligned}
B^{\{j\}}_{j,p,\boldsymbol{\xi}}(x) &= (x-\xi_j)^p \Big/ \prod_{i=1}^{p}(\xi_{j+i}-\xi_j),\\
B^{\{j+p\}}_{j,p,\boldsymbol{\xi}}(x) &= (\xi_{j+p+1}-x)^p \Big/ \prod_{i=1}^{p}(\xi_{j+p+1}-\xi_{j+i}).
\end{aligned} \tag{5}
$$

These expressions are valid for multiple knots. Indeed, if $\xi_{k+1} = \xi_k$ for some k then $B_{k,0,\boldsymbol{\xi}} = 0$ and the corresponding polynomial piece is not used.

Using induction on the recurrence relation (1), we deduce immediately the following basic properties of a B-spline.

- **Local support.** A B-spline is locally supported on the interval given by the extreme knots used in its definition, i.e.,

$$B_{j,p,\boldsymbol{\xi}}(x) = 0, \quad x \notin [\xi_j, \xi_{j+p+1}). \tag{6}$$

- **Piecewise structure.** A B-spline has a piecewise polynomial structure, i.e.,

$$B^{\{m\}}_{j,p,\boldsymbol{\xi}} \in \mathbb{P}_p, \quad m = j, \dots, j+p. \tag{7}$$

- **Nonnegativity.** A B-spline is nonnegative everywhere, and positive inside its support, i.e.,

$$B_{j,p,\boldsymbol{\xi}}(x) \geq 0, \quad x \in \mathbb{R}, \quad \text{and} \quad B_{j,p,\boldsymbol{\xi}}(x) > 0, \quad x \in (\xi_j, \xi_{j+p+1}). \tag{8}$$

- **Translation and scaling invariance.** A B-spline is invariant under a translation and/or scaling transformation of its knot sequence, i.e.,

$$B_{j,p,\alpha\boldsymbol{\xi}+\beta}(\alpha x+\beta) = B_{j,p,\boldsymbol{\xi}}(x), \quad \alpha,\beta \in \mathbb{R}, \quad \alpha \neq 0, \tag{9}$$

  where $\alpha\boldsymbol{\xi}+\beta := (\alpha\xi_j+\beta, \dots, \alpha\xi_{j+p+1}+\beta)$.

  Further properties will be considered in the next sections.

1.1.2 Dual polynomials

To each B-spline $B_{j,p,\boldsymbol{\xi}}$ of degree p, there corresponds a polynomial $\psi_{j,p,\boldsymbol{\xi}}$ of degree p with roots at the interior knots of the B-spline. We define $\psi_{j,0,\boldsymbol{\xi}} := 1$ and

$$\psi_{j,p,\boldsymbol{\xi}}(y) := (y-\xi_{j+1})\cdots(y-\xi_{j+p}), \quad y \in \mathbb{R}, \quad p \in \mathbb{N}. \tag{10}$$

This polynomial is called **dual polynomial**. Many of the B-spline properties can be proved in an elegant way by exploiting a recurrence relation for these dual polynomials.

Theorem 5. *For $p \in \mathbb{N}$, $x,y \in \mathbb{R}$ and $\xi_{j+p} > \xi_j$, we have the* **dual recurrence relation**

$$(y-x)\psi_{j,p-1,\boldsymbol{\xi}}(y) = \frac{x-\xi_j}{\xi_{j+p}-\xi_j}\psi_{j,p,\boldsymbol{\xi}}(y) + \frac{\xi_{j+p}-x}{\xi_{j+p}-\xi_j}\psi_{j-1,p,\boldsymbol{\xi}}(y), \tag{11}$$

and the **dual difference formula**

$$\psi_{j,p-1,\boldsymbol{\xi}}(y) = \frac{\psi_{j-1,p,\boldsymbol{\xi}}(y)}{\xi_{j+p}-\xi_j} - \frac{\psi_{j,p,\boldsymbol{\xi}}(y)}{\xi_{j+p}-\xi_j}. \tag{12}$$

Proof. For fixed $y \in \mathbb{R}$ let us define the function $\ell_y : \mathbb{R} \to \mathbb{R}$ given by $\ell_y(x) = y - x$. By linear interpolation, we have

$$\ell_y(x) = \frac{x-\xi_j}{\xi_{j+p}-\xi_j}\ell_y(\xi_{j+p}) + \frac{\xi_{j+p}-x}{\xi_{j+p}-\xi_j}\ell_y(\xi_j).$$

Multiplying both sides by $\psi_{j,p-1,\boldsymbol{\xi}}(y)$ we obtain (11). Moreover, (12) follows by differentiating with respect to x in (11). □

Proposition 6. *The r-th derivative of the dual polynomial $\psi_{j,p,\boldsymbol{\xi}}$ for $0 \le r \le p$ can be bounded as follows:*

$$|D^r\psi_{j,p,\boldsymbol{\xi}}(y)| \le \frac{p!}{(p-r)!}(\xi_{j+p+1}-\xi_j)^{p-r}, \quad \xi_j \le y \le \xi_{j+p+1}. \tag{13}$$

Moreover,

$$|D^r\psi_{j,p,\boldsymbol{\xi}}(y)| \le \frac{p!}{(p-r)!}(\xi_{j+p}-\xi_{j+1})^{p-r}, \quad \xi_{j+1} \le y \le \xi_{j+p}. \tag{14}$$

Here we define $0^0 := 1$ if $r = p$ and $\xi_{j+p} = \xi_{j+1}$.

Proof. Clearly (13) holds for all $p \in \mathbb{N}_0$ if $r = 0$. Using induction on r, p and the product rule for differentiation, we get

$$\begin{aligned}
|D^r\psi_{j,p,\boldsymbol{\xi}}(y)| &= |D^r(\psi_{j,p-1,\boldsymbol{\xi}}(y)(y-\xi_{j+p}))| \\
&= |(D^r\psi_{j,p-1,\boldsymbol{\xi}}(y))(y-\xi_{j+p}) + rD^{r-1}\psi_{j,p-1,\boldsymbol{\xi}}(y)| \\
&\le \left(\frac{(p-1)!}{(p-1-r)!} + r\frac{(p-1)!}{(p-r)!}\right)(\xi_{j+p+1}-\xi_j)^{p-r},
\end{aligned}$$

and (13) follows. The proof of (14) is similar. □

1.1.3 Local Marsden identity and linear independence

In this and the following sections (unless specified otherwise) we will extend the knots $\xi_j \le \cdots \le \xi_{j+p+1}$ of $B_{j,p,\boldsymbol{\xi}}$ by defining p extra knots at each end, and we will assume

$$\boldsymbol{\xi} := \{\xi_{j-p} \le \cdots \le \xi_{j-1} < \xi_j \le \cdots \le \xi_{j+p+1} < \xi_{j+p+2} \le \cdots \le \xi_{j+2p+1}\}. \tag{15}$$

These extra knots can be defined in any way we like. One possibility is

$$\xi_{j-p} = \cdots = \xi_{j-1} := \xi_j - 1, \quad \xi_{j+p+1} + 1 =: \xi_{j+p+2} = \cdots = \xi_{j+2p+1}. \tag{16}$$

On such a knot sequence $2p+1$ B-splines $B_{i,p,\boldsymbol{\xi}} = B[\xi_i, \ldots, \xi_{i+p+1}]$, $i = j-p, \ldots, j+p$ are well defined.

The following identity was first proved by Marsden [13] and simplifies many dealings with B-splines.

Theorem 7 (Local Marsden identity). *For $j \le m \le j+p$ and $\xi_m < \xi_{m+1}$, we have*

$$(y-x)^p = \sum_{i=m-p}^{m} \psi_{i,p,\boldsymbol{\xi}}(y) B_{i,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_m, \xi_{m+1}), \quad y \in \mathbb{R}. \tag{17}$$

If $B_{i,p,\boldsymbol{\xi}}^{\{m\}}$ is the polynomial which is equal to $B_{i,p,\boldsymbol{\xi}}(x)$ for $x \in [\xi_m, \xi_{m+1})$ then

$$(y-x)^p = \sum_{i=m-p}^{m} \psi_{i,p,\boldsymbol{\xi}}(y) B_{i,p,\boldsymbol{\xi}}^{\{m\}}(x), \quad x, y \in \mathbb{R}. \tag{18}$$

Proof. Suppose $x \in [\xi_m, \xi_{m+1})$. The equality (17) can be proved by induction. It is clearly true for $p=0$. Let us now assume it holds for degree $p-1$. Then, by means of the dual recurrence (11) and the B-spline recurrence relation we obtain

$$\begin{aligned}
(y-x)^p &= (y-x)(y-x)^{p-1} = (y-x) \sum_{i=m-p+1}^{m} \psi_{i,p-1,\boldsymbol{\xi}}(y) B_{i,p-1,\boldsymbol{\xi}}(x) \\
&= \sum_{i=m-p+1}^{m} \left(\frac{x-\xi_i}{\xi_{i+p}-\xi_i} \psi_{i,p,\boldsymbol{\xi}}(y) + \frac{\xi_{i+p}-x}{\xi_{i+p}-\xi_i} \psi_{i-1,p,\boldsymbol{\xi}} \right) B_{i,p-1,\boldsymbol{\xi}}(x) \\
&= \sum_{i=m-p}^{m} \left(\frac{x-\xi_i}{\xi_{i+p}-\xi_i} B_{i,p-1,\boldsymbol{\xi}}(x) + \frac{\xi_{i+p+1}-x}{\xi_{i+p+1}-\xi_{i+1}} B_{i+1,p-1,\boldsymbol{\xi}}(x) \right) \psi_{i,p,\boldsymbol{\xi}}(y) \\
&= \sum_{i=m-p}^{m} \psi_{i,p,\boldsymbol{\xi}}(y) B_{i,p,\boldsymbol{\xi}}(x).
\end{aligned}$$

Here we used that $\frac{x-\xi_i}{\xi_{i+p}-\xi_i} B_{i,p-1,\boldsymbol{\xi}}(x) = 0$ for $i = m-p, m+1$. □

The local Marsden identity immediately leads to the following properties, where we suppose $\xi_m < \xi_{m+1}$ for some $j \le m \le j+p$.

- **Local representation of monomials.** We have for $p \ge k$,

$$x^k = \sum_{i=m-p}^{m} \left((-1)^k \frac{k!}{p!} D^{p-k} \psi_{j,p,\boldsymbol{\xi}}(0) \right) B_{i,p,\boldsymbol{\xi}}(x) \quad x \in [\xi_m, \xi_{m+1}). \tag{19}$$

Proof. Differentiating $p-k$ times with respect to y in (18) results in

$$\frac{(y-x)^k}{k!} = \sum_{i=m-p}^{m} \left(\frac{1}{p!} D^{p-k}\psi_{i,p,\boldsymbol{\xi}}(y)\right) B_{i,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_m, \xi_{m+1}), \quad y \in \mathbb{R}, \tag{20}$$

for $k = 0, 1, \ldots, p$. Setting $y = 0$ in (20) results in (19). □

- **Local partition of unity.** Taking $k = 0$ in (19) gives

$$\sum_{i=m-p}^{m} B_{i,p,\boldsymbol{\xi}}(x) = 1, \quad x \in [\xi_m, \xi_{m+1}). \tag{21}$$

- **Local linear independence.** The two sets $\{B_{i,p,\boldsymbol{\xi}}\}_{i=m-p}^{m}$ and $\{\psi_{i,p,\boldsymbol{\xi}}\}_{i=m-p}^{m}$ form both a basis for the polynomial space $\mathbb{P}_p$ on $[\xi_m, \xi_{m+1})$.

  Proof. From (20) we see that on $[\xi_m, \xi_{m+1})$ every polynomial of degree at most p can be written as a linear combination of the $p+1$ polynomials $B_{i,p,\boldsymbol{\xi}}(x)$, $i = m-p, \ldots, m$. Since the dimension of the space $\mathbb{P}_p$ is $p+1$, these polynomials must be linearly independent and a basis. The result for $\{\psi_{i,p,\boldsymbol{\xi}}\}_{i=m-p}^{m}$ follows by symmetry. □

1.1.4 Smoothness, differentiation and integration

The derivative of a B-spline can be expressed by means of a simple difference formula.

Theorem 8 (Differentiation formula). *We have*

$$D_+ B_{j,p,\boldsymbol{\xi}}(x) = p\left(\frac{B_{j,p-1,\boldsymbol{\xi}}(x)}{\xi_{j+p} - \xi_j} - \frac{B_{j+1,p-1,\boldsymbol{\xi}}(x)}{\xi_{j+p+1} - \xi_{j+1}}\right), \quad p \geq 1, \tag{22}$$

where fractions with zero denominator have value zero.

Proof. If $\xi_{j+p+1} = \xi_j$ then both sides of (22) are zero, so we can assume $\xi_{j+p+1} > \xi_j$. We continue to use the extra knots (15). If $x < \xi_j$ or $x \geq \xi_{j+p+1}$ then both sides of (22) are zero. Otherwise $x \in [\xi_m, \xi_{m+1})$ for some m with $j \leq m \leq j+p$ and it is enough to prove (22) for such an interval. Differentiating both sides of (17) with respect to x gives

$$-p(y-x)^{p-1} = \sum_{i=m-p}^{m} DB_{i,p,\boldsymbol{\xi}}(x)\psi_{i,p}(y), \quad x \in [\xi_m, \xi_{m+1}). \tag{23}$$

On the other hand, using the local Marsden identity (17) for degree $p-1$ and the difference formula for dual polynomials (12) results in

$$
\begin{aligned}
-p(y-x)^{p-1} &= -p \sum_{i=m-p+1}^{m} \psi_{i,p-1}(y) B_{i,p-1,\boldsymbol{\xi}}(x) \\
&= p \sum_{i=m-p+1}^{m} \left(\frac{\psi_{i,p}(y)}{\xi_{i+p}-\xi_i} - \frac{\psi_{i-1,p}(y)}{\xi_{i+p}-\xi_i} \right) B_{i,p-1,\boldsymbol{\xi}}(x) \\
&= \sum_{i=m-p}^{m} p \left(\frac{B_{i,p-1,\boldsymbol{\xi}}(x)}{\xi_{i+p}-\xi_i} - \frac{B_{i+1,p-1,\boldsymbol{\xi}}(x)}{\xi_{i+p+1}-\xi_{i+1}} \right) \psi_{i,p}(y).
\end{aligned}
$$

By comparing this with (23) and using the linear independence of the dual polynomials, it follows that (22) holds for $i = m-p, \dots, m$. Since $m-p \le j \le m$, (22) holds for $i = j$. $\square$

Example 9. The differentiation formula (22) for $p = 2$ together with the expression (2) immediately gives the piecewise form of the derivative of the quadratic B-spline $B_{j,2,\boldsymbol{\xi}}$:

$$
D_+ B_{j,2,\boldsymbol{\xi}}(x) = \begin{cases}
\dfrac{2(x-\xi_j)}{(\xi_{j+2}-\xi_j)(\xi_{j+1}-\xi_j)}, & \text{if } x \in [\xi_j, \xi_{j+1}), \\[2ex]
\dfrac{2(\xi_{j+2}-x)}{(\xi_{j+2}-\xi_j)(\xi_{j+2}-\xi_{j+1})} - \dfrac{2(x-\xi_{j+1})}{(\xi_{j+3}-\xi_{j+1})(\xi_{j+2}-\xi_{j+1})}, & \text{if } x \in [\xi_{j+1}, \xi_{j+2}), \\[2ex]
-\dfrac{2(\xi_{j+3}-x)}{(\xi_{j+3}-\xi_{j+1})(\xi_{j+3}-\xi_{j+2})}, & \text{if } x \in [\xi_{j+2}, \xi_{j+3}), \\[2ex]
0, & \text{otherwise.}
\end{cases}
$$

This is in agreement with taking the derivative of the piecewise expression (3) of $B_{j,2,\boldsymbol{\xi}}$ given in Example 4.

Proposition 10. *The r-th derivative of the B-spline $B_{j,p,\boldsymbol{\xi}}$ for $0 \le r \le p$ can be bounded as follows. For any $x \in [\xi_m, \xi_{m+1})$ with $j \le m \le j+p$ we have*

$$
|D^r B_{j,p,\boldsymbol{\xi}}(x)| \le 2^r \frac{p!}{(p-r)!} \prod_{k=p-r+1}^{p} \frac{1}{\Delta_{m,k}}, \tag{24}
$$

where

$$
\Delta_{m,k} := \min_{m-k+1 \le i \le m} h_{i,k}, \quad h_{i,k} := \xi_{i+k} - \xi_i, \quad k = 1, \dots, p. \tag{25}
$$

Proof. This holds for $r = 0$ because of the nonnegativity of $B_{j,p,\boldsymbol{\xi}}$ and the partition of unity property (21). By the differentiation formula (22) and the local support property (6) we have

$$
\begin{aligned}
&D^r B_{j,p,\boldsymbol{\xi}}(x) \\
&= p \begin{cases}
-D^{r-1} B_{j+1,p-1,\boldsymbol{\xi}}(x)/h_{j+1,p}, & \text{if } m = j+p, \\
D^{r-1} B_{j,p-1,\boldsymbol{\xi}}(x)/h_{j,p} - D^{r-1} B_{j+1,p-1,\boldsymbol{\xi}}(x)/h_{j+1,p}, & \text{if } j < m < j+p, \\
D^{r-1} B_{j,p-1,\boldsymbol{\xi}}(x)/h_{j,p}, & \text{if } m = j.
\end{cases}
\end{aligned}
$$

It follows that

$$|D^r B_{j,p,\boldsymbol{\xi}}(x)| \le 2p \max_{m-p+1\le i\le m} |D^{r-1}B_{i,p-1,\boldsymbol{\xi}}(x)|/\Delta_{m,p},$$

and by induction on r we obtain (24). $\square$

Note that the upper bound in (24) is well defined since $\Delta_{m,k} \ge \xi_{m+1} - \xi_m > 0$.

Theorem 11 (Smoothness property). *If ξ is a knot of $B_{j,p,\boldsymbol{\xi}}$ of multiplicity $\mu \le p+1$ then*

$$B_{j,p,\boldsymbol{\xi}} \in C^{p-\mu}(\xi),$$

i.e., its derivatives of order $0,1,\dots,p-\mu$ are continuous at ξ.

Proof. Suppose ξ is a knot of $B_{j,p,\boldsymbol{\xi}}$ of multiplicity μ. We first consider the smoothness property when $\mu = p+1$. For $x \in [\xi_j, \xi_{j+p+1})$ it follows immediately from the first and last piece in (4) and (5) that

$$B_{j,p,\boldsymbol{\xi}}(x) = (x-\xi_j)^p/(\xi_{j+p+1}-\xi_j)^p, \quad \xi_j < \xi_{j+1} = \cdots = \xi_{j+p+1}, \tag{26}$$

$$B_{j,p,\boldsymbol{\xi}}(x) = (\xi_{j+p+1}-x)^p/(\xi_{j+p+1}-\xi_j)^p, \quad \xi_j = \cdots = \xi_{j+p} < \xi_{j+p+1}. \tag{27}$$

These two B-splines are discontinuous with a jump of absolute size one at the multiple knot showing the smoothness property for $\mu = p+1$.

Let us now consider the case where $B_{j,p,\boldsymbol{\xi}}$ has an interior knot of multiplicity equal to $\mu = p$, i.e., $\xi_j < \xi_{j+1} = \cdots = \xi_{j+p} < \xi_{j+p+1}$. For $x \in [\xi_j, \xi_{j+p+1})$ it follows from the first and last pieces in (4) and (5) that

$$B_{j,p,\boldsymbol{\xi}}(x) = \frac{(x-\xi_j)^p}{(\xi_{j+p}-\xi_j)^p} B_{j,0,\boldsymbol{\xi}}(x) + \frac{(\xi_{j+p+1}-x)^p}{(\xi_{j+p+1}-\xi_{j+1})^p} B_{j+p,0,\boldsymbol{\xi}}(x). \tag{28}$$

The two nontrivial pieces have both value one at the center knot $\xi_{j+1} = \xi_{j+p}$, and $B_{j,p,\boldsymbol{\xi}}$ is continuous on $\mathbb{R}$. Moreover, the first derivative has a nonzero jump at the center knot.

For the remaining cases we use induction on p to show that $B_{j,p,\boldsymbol{\xi}} \in C^{p-\mu}(\xi)$. The case $p=1$ follows from Example 3. Suppose for some $p \ge 2$ that $B_{j,p-1,\boldsymbol{\xi}} \in C^{p-1-\mu}(\xi)$ at a knot ξ of multiplicity μ. For the multiplicity p case $\xi = \xi_j = \cdots = \xi_{j+p-1} < \xi_{j+p} \le \xi_{j+p+1}$ we use the recurrence relation

$$B_{j,p,\boldsymbol{\xi}}(x) = \frac{x-\xi_j}{\xi_{j+p}-\xi_j} B_{j,p-1,\boldsymbol{\xi}}(x) + \frac{\xi_{j+p+1}-x}{\xi_{j+p+1}-\xi_{j+1}} B_{j+1,p-1,\boldsymbol{\xi}}(x).$$

The first term vanishes at $x = \xi = \xi_j$. Since $B_{j+1,p-1,\boldsymbol{\xi}}$ has a knot of multiplicity $p-1$ at ξ, it follows from the induction hypothesis that it is continuous there. We conclude that $B_{j,p,\boldsymbol{\xi}}$ is continuous at ξ. The case where the right end knot of $B_{j,p,\boldsymbol{\xi}}$ has multiplicity p is handled similarly. Finally, if $\mu \le p-1$ then both terms in the differentiation formula (22) has a knot of multiplicity at most μ at ξ and by the induction hypothesis we obtain $D_+B_{j,p,\boldsymbol{\xi}} \in C^{p-1-\mu}(\xi)$. Moreover, by the recurrence relation and the induction hypothesis it follows that $B_{j,p,\boldsymbol{\xi}}$ is continuous at ξ, and so we also conclude that $B_{j,p,\boldsymbol{\xi}} \in C^{p-\mu}(\xi)$ if $\mu \le p-1$. $\square$

The B-spline $B_{j,p,\boldsymbol{\xi}}$ is supported on the interval $[\xi_j, \xi_{j+p+1}]$. Hence, Theorem 11 implies that $B_{j,p,\boldsymbol{\xi}}$ is continuous on $\mathbb{R}$ whenever $\xi_{j+p} > \xi_j$ and $\xi_{j+p+1} > \xi_{j+1}$. Similarly, $B_{j,p,\boldsymbol{\xi}}$ is C^r-continuous on $\mathbb{R}$ whenever $\xi_{j+p-r+i} > \xi_{j+i}$ for each $i = 0, \dots, r+1$ and $-1 \le r < p$.

Theorem 12 (Integration formula). *We have*

$$\gamma_{j,p,\boldsymbol{\xi}} := \int_{\xi_j}^{\xi_{j+p+1}} B_{j,p,\boldsymbol{\xi}}(x)\,\mathrm{d}x = \frac{\xi_{j+p+1} - \xi_j}{p+1}. \tag{29}$$

Proof. This time we define $p+1$ extra knots at each end, and we assume

$$\boldsymbol{\xi} := \{\xi_{j-p-1} = \cdots = \xi_{j-1} < \xi_j \le \cdots \le \xi_{j+p+1} < \xi_{j+p+2} = \cdots = \xi_{j+2p+2}\}.$$

On this knot sequence we consider $p+1$ B-splines $B_{i,p+1,\boldsymbol{\xi}}$, $i = j-p-1, \dots, j-1$ of degree $p+1$. From Theorem 11 we know that these B-splines are continuous on $\mathbb{R}$. Therefore, we get for $i = j-p-1, \dots, j-1$,

$$0 = B_{i,p+1,\boldsymbol{\xi}}(\xi_{i+p+2}) - B_{i,p+1,\boldsymbol{\xi}}(\xi_i) = \int_{\xi_i}^{\xi_{i+p+2}} D_+ B_{i,p+1,\boldsymbol{\xi}}(x)\,\mathrm{d}x = E_i - E_{i+1},$$

where by the local support and the differentiation formula (22),

$$E_i := \frac{p+1}{\xi_{i+p+1} - \xi_i} \int_{\xi_i}^{\xi_{i+p+1}} B_{i,p,\boldsymbol{\xi}}(x)\,\mathrm{d}x, \quad i = j-p-1, \dots, j.$$

This means that $E_j = E_{j-1} = \cdots = E_{j-p-1}$. Moreover, since $\xi_{j-p-1} = \cdots = \xi_{j-1}$, we obtain from (27) that

$$E_{j-p-1} = \frac{p+1}{\xi_j - \xi_{j-p-1}} \int_{\xi_{j-p-1}}^{\xi_j} \frac{(\xi_j - x)^p}{(\xi_j - \xi_{j-p-1})^p}\,\mathrm{d}x = 1,$$

and the integration formula (29) follows. □

1.2 *Linear combinations of B-splines*

We now analyse linear combinations of a given set of consecutive B-splines and their properties.

1.2.1 The spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$ and some properties

Suppose for integers $n > p \ge 0$ that a knot sequence

$$\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1} = \{\xi_1 \le \xi_2 \le \cdots \le \xi_{n+p+1}\}, \quad n \in \mathbb{N}, \quad p \in \mathbb{N}_0,$$

is given. This knot sequence allows us to define a set of n B-splines of degree p, namely

$$\{B_{1,p,\boldsymbol{\xi}},\ldots,B_{n,p,\boldsymbol{\xi}}\}. \tag{30}$$

We consider the space

$$\mathbb{S}_{p,\boldsymbol{\xi}} := \left\{ s : [\xi_{p+1},\xi_{n+1}] \to \mathbb{R} : s = \sum_{j=1}^{n} c_j B_{j,p,\boldsymbol{\xi}},\ c_j \in \mathbb{R} \right\}. \tag{31}$$

This is the space of **splines** spanned by the B-splines in (30) over the interval $[\xi_{p+1},\xi_{n+1}]$, which is called the **basic interval**.

We now introduce some terminology to identify certain properties of knot sequences which are crucial in the study of the space (31).

- A knot sequence $\boldsymbol{\xi}$ is called $(p+1)$**-regular** if $\xi_j < \xi_{j+p+1}$ for $j = 1,\ldots,n$. By the local support (6) such a knot sequence ensures that all the B-splines in (30) are not identically zero.
- A knot sequence $\boldsymbol{\xi}$ is called $(p+1)$**-basic** if it is $(p+1)$-regular with $\xi_{p+1} < \xi_{p+2}$ and $\xi_n < \xi_{n+1}$. As we will show later, the B-splines in (30) defined on a $(p+1)$-basic knot sequence are linearly independent on the basic interval $[\xi_{p+1},\xi_{n+1}]$.
- A knot sequence $\boldsymbol{\xi}$ is called $(p+1)$**-open** on an interval $[a,b]$ if it is $(p+1)$-regular and it has end knots of multiplicity $p+1$, i.e.,

$$a := \xi_1 = \cdots = \xi_{p+1} < \xi_{p+2} \le \cdots \le \xi_n < \xi_{n+1} = \cdots = \xi_{n+p+1} =: b. \tag{32}$$

  This sequence is often used in practice. In particular, it turns out to be natural to construct open curves, clamped at two given points.

  Some further preliminary remarks are in order here.

- We consider B-splines on a closed basic interval $[\xi_{p+1},\xi_{n+1}]$. In order to avoid the asymmetry at the right endpoint we define the B-splines to be **left continuous** at the right endpoint, i.e., its value at ξ_{n+1} is obtained by taking limits from the left:

$$B_{j,p,\boldsymbol{\xi}}(\xi_{n+1}) := \lim_{\substack{x\to\xi_{n+1}\\ x<\xi_{n+1}}} B_{j,p,\boldsymbol{\xi}}(x), \quad j = 1,\ldots,n. \tag{33}$$

  Note that for a $(p+1)$-open knot sequence the end condition (33) means that $B_{n,p,\boldsymbol{\xi}}(\xi_{n+p+1}) = 1$ and (6) has to be modified for this B-spline.
- We define a **multiplicity function** $\mu_{\boldsymbol{\xi}} : \mathbb{R} \to \mathbb{N}_0$ given by $\mu_{\boldsymbol{\xi}}(\xi_i) = \mu_i$ if $\xi_i \in \boldsymbol{\xi}$ occurs exactly $\mu_i \ge 1$ times in $\boldsymbol{\xi}$, and $\mu_{\boldsymbol{\xi}}(x) = 0$ if $x \notin \boldsymbol{\xi}$. If $\boldsymbol{\xi}$ and $\check{\boldsymbol{\xi}}$ are two knot sequences we say that $\boldsymbol{\xi} \subseteq \check{\boldsymbol{\xi}}$ if $\mu_{\xi}(x) \le \mu_{\check{\boldsymbol{\xi}}}(x)$ for all $x \in \mathbb{R}$.
- Without loss of generality we can always assume that the end knots have multiplicity $p+1$. If this is not the case, then we can add extra knots at the ends and assume the extra B-splines to have coefficients zero. This observation simplifies many proofs.

From the properties of B-splines, we immediately conclude the following properties of the spline representation in (31).

- **Smoothness.** If ξ is a knot of multiplicity μ then $s \in C^r(\xi)$ for any $s \in \mathbb{S}_{p,\boldsymbol{\xi}}$, where $r + \mu = p$. This follows from the smoothness property of the B-splines (Theorem 11). The relation between smoothness, multiplicity and degree is as follows:
$$\text{“smoothness} + \text{multiplicity} = \text{degree”}. \tag{34}$$
- **Local support.** The local support (6) of the B-splines implies
$$\sum_{j=1}^{n} c_j B_{j,p,\boldsymbol{\xi}}(x) = \sum_{j=m-p}^{m} c_j B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_m, \xi_{m+1}), \quad p+1 \le m \le n, \tag{35}$$
and
$$\sum_{j=1}^{n} c_j B_{j,p,\boldsymbol{\xi}}(\xi_m) = \sum_{j=m-p}^{m-1} c_j B_{j,p,\boldsymbol{\xi}}(\xi_m), \quad p+1 \le m \le n+1. \tag{36}$$
- **Minimal support.** From the smoothness properties it can be proved that if the support of $s \in \mathbb{S}_{p,\boldsymbol{\xi}}$ is a proper subset of $[\xi_j, \xi_{j+p+1}]$ for some j then $s = 0$. Therefore, the B-splines have minimal support.
- **Coefficient recurrence.** For $x \in [\xi_{p+1}, \xi_{n+1}]$, by the recurrence relation (1) we have
$$\sum_{j=1}^{n} c_j B_{j,p,\boldsymbol{\xi}}(x) = \sum_{j=2}^{n} c_j^{[1]}(x) B_{j,p-1,\boldsymbol{\xi}}(x), \tag{37}$$
where
$$c_j^{[1]}(x) := \frac{x - \xi_j}{\xi_{j+p} - \xi_j} c_j + \frac{\xi_{j+p} - x}{\xi_{j+p} - \xi_j} c_{j-1}, \tag{38}$$
and $c_j^{[1]}(x) B_{j,p-1,\boldsymbol{\xi}}(x) = 0$ if $\xi_{j+p} = \xi_j$.
- **Differentiation formula.** By (22) we have
$$D_+\left(\sum_{j=1}^{n} c_j B_{j,p,\boldsymbol{\xi}}\right) = \sum_{j=2}^{n} c_j^{(1)} B_{j,p-1,\boldsymbol{\xi}}, \quad p \ge 1, \tag{39}$$
where
$$c_j^{(1)} := p\left(\frac{c_j - c_{j-1}}{\xi_{j+p} - \xi_j}\right), \tag{40}$$
and as usual fractions with zero denominator have value zero.
- **Linear independence.** If $\boldsymbol{\xi}$ is $(p+1)$-basic, then the B-splines in (30) are linearly independent on the basic interval. Thus, the spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$ is a vector space of dimension n.

  Proof. We must show that if $s(x) = \sum_{j=1}^{n} c_j B_{j,p,\boldsymbol{\xi}}(x) = 0$ for $x \in [\xi_{p+1}, \xi_{n+1}]$ then $c_j = 0$ for all j. Let us fix $1 \le j \le n$. Since $\boldsymbol{\xi}$ is $(p+1)$-regular, there is an integer m_j with $j \le m_j \le j+p$ such that $\xi_{m_j} < \xi_{m_j+1}$. Moreover, the assumptions $\xi_{p+1} < \xi_{p+2}$ and $\xi_n < \xi_{n+1}$ guarantee that $[\xi_{m_j}, \xi_{m_j+1})$ can be chosen in the basic interval. From the local support property (35) we know

$$s(x) = \sum_{i=m_j-p}^{m_j} c_i B_{i,p,\boldsymbol{\xi}}(x) = 0, \quad x \in [\xi_{m_j}, \xi_{m_j+1}).$$

The local linear independence property (see Section 1.1.3) implies $c_{m_j-p} = \cdots = c_{m_j} = 0$, and in particular $c_j = 0$. $\square$

1.2.2 The piecewise polynomial space $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$

In this section we focus on the spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$. We prove that such a space is nothing else than a space of piecewise polynomials of degree p defined by a given sequence of break points and by some prescribed smoothness. The set of knots $\boldsymbol{\xi}$ must be suitably selected according to the break points and the smoothness conditions. Therefore, the B-splines are a basis of such a space of piecewise polynomials.

Let Δ be a sequence of distinct real numbers,

$$\Delta := \{\eta_0 < \eta_1 < \cdots < \eta_{\ell+1}\}.$$

The elements in Δ are called **break points**. Moreover, let $\boldsymbol{r} := (r_1, \ldots, r_\ell)$ be a vector of integers such that $-1 \le r_i \le p$ for $i = 1, \ldots, \ell$. The space $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$ of piecewise polynomials of degree p with smoothness $\boldsymbol{r}$ over the partition Δ is defined by

$$\begin{aligned}\mathbb{S}_p^{\boldsymbol{r}}(\Delta) := \big\{ s : [\eta_0, \eta_{\ell+1}] \to \mathbb{R} : s \in \mathbb{P}_p([\eta_i, \eta_{i+1})),\ i = 0, \ldots, \ell, \\ s \in C^{r_i}(\eta_i),\ i = 1, \ldots, \ell \big\}.\end{aligned} \tag{41}$$

This space is denoted by $\mathbb{S}_p^r(\Delta)$ when $r = r_1 = \cdots = r_\ell$.

Suppose that $s^{\{i\}} \in \mathbb{P}_p$ is the polynomial equal to the restriction of a given spline $s \in \mathbb{S}_p^{\boldsymbol{r}}(\Delta)$ to the interval $[\eta_i, \eta_{i+1})$, $i = 0, \ldots, \ell$. Since $s \in C^{r_i}(\eta_i)$, we have

$$s^{\{i\}}(x) - s^{\{i-1\}}(x) = \sum_{j=r_i+1}^{p} c_{i,j}(x - \eta_i)^j,$$

for some coefficients $c_{i,j}$. It follows that $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$ is spanned by the set of functions

$$\big\{1, x, \ldots, x^p, (x-\eta_1)_+^{r_1+1}, \ldots, (x-\eta_1)_+^p, \ldots, (x-\eta_\ell)_+^{r_\ell+1}, \ldots, (x-\eta_\ell)_+^p\big\}, \tag{42}$$

where the **truncated power** function $(\cdot)_+^p$ is defined by

$$(x)_+^p := \begin{cases} x^p, & x > 0, \\ 0, & x < 0, \end{cases} \tag{43}$$

and the value at zero is defined by taking the right limit.

It is easy to see that the functions in (42) are linearly independent. Indeed, let

$$s(x) := \sum_{j=0}^{p} c_{0,j} x^j + \sum_{i=1}^{\ell} \sum_{j=r_i+1}^{p} c_{i,j} (x-\eta_i)_+^j = 0, \quad x \in [\eta_0, \eta_{\ell+1}].$$

On $[\eta_0, \eta_1]$ we have $s(x) = \sum_{j=0}^{p} c_{0,j} x^j$ and it follows that $c_{0,0} = \cdots = c_{0,p} = 0$. Suppose for some $k \leq \ell$ that $c_{i,j} = 0$ for $i < k$. Then, on $[\eta_k, \eta_{k+1}]$ we have $s(x) = \sum_{j=r_k+1}^{p} c_{k,j}(x-\eta_k)^j = 0$ showing that all $c_{k,j} = 0$.

This implies that the set of functions in (42) forms a basis for $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$, the so-called **truncated power basis**. As a consequence,

$$\dim(\mathbb{S}_p^{\boldsymbol{r}}(\Delta)) = p + 1 + \sum_{i=1}^{\ell} (p - r_i).$$

The next theorem shows that the set of B-splines in (30) defined over a specific knot sequence $\boldsymbol{\xi}$ forms an alternative basis for $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$. This was first proved by Curry and Schoenberg in [8].

Theorem 13 (Curry–Schoenberg). *The piecewise polynomial space* $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$ *is characterized in terms of B-splines by*

$$\mathbb{S}_p^{\boldsymbol{r}}(\Delta) = \mathbb{S}_{p,\boldsymbol{\xi}},$$

where the knot sequence $\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1}$ *with* $n := \dim(\mathbb{S}_p^{\boldsymbol{r}}(\Delta))$ *is constructed such that*

$$\xi_1 \leq \cdots \leq \xi_{p+1} := \eta_0, \quad \eta_{\ell+1} =: \xi_{n+1} \leq \cdots \leq \xi_{n+p+1},$$

and

$$\xi_{p+2}, \ldots, \xi_n := \overbrace{\eta_1, \ldots, \eta_1}^{p-r_1}, \ldots, \overbrace{\eta_\ell, \ldots, \eta_\ell}^{p-r_\ell}.$$

Proof. From the piecewise polynomial and smoothness properties of B-splines it follows that the space $\mathbb{S}_{p,\boldsymbol{\xi}}$ of B-splines restricted to the basic interval $[\xi_{p+1}, \xi_{n+1}]$ is a subspace of $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$. Moreover, $\dim(\mathbb{S}_{p,\boldsymbol{\xi}}) = n$ since $\xi_{j+p+1} > \xi_j$ for $j = 1, \ldots, n$ and $\xi_{p+2} > \xi_{p+1}$, $\xi_{n+1} > \xi_n$. Since $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$ is spanned by n functions we obtain $\mathbb{S}_p^{\boldsymbol{r}}(\Delta) = \mathbb{S}_{p,\boldsymbol{\xi}}$. □

Note that the knot sequence in the above theorem is $(p+1)$-basic.

Example 14. Consider $\Delta := \{\eta_0 < \eta_1 < \eta_2 < \eta_3\}$ and the space $\mathbb{S}_3^{\boldsymbol{r}}(\Delta)$ with $\boldsymbol{r} = (r_1, r_2) = (2, 1)$. Then it follows from Theorem 13 that $\mathbb{S}_3^{\boldsymbol{r}}(\Delta) = \mathbb{S}_{3,\boldsymbol{\xi}}$, where

$$\boldsymbol{\xi} = \{\xi_i\}_{i=1}^{7+3+1} = \{\eta_0 = \eta_0 = \eta_0 = \eta_0 < \eta_1 < \eta_2 = \eta_2 < \eta_3 = \eta_3 = \eta_3 = \eta_3\}.$$

This knot sequence is 4-open.

We now give a characterization for the space spanned by the r-th derivatives of B-splines for $0 \leq r \leq p$, i.e.,

$$D_+^r \mathbb{S}_{p,\boldsymbol{\xi}} := \left\{ s : [\xi_{p+1}, \xi_{n+1}] \to \mathbb{R} : s = D_+^r \left(\sum_{j=1}^n c_j B_{j,p,\boldsymbol{\xi}} \right),\ c_j \in \mathbb{R} \right\}.$$

Theorem 15. *Given a knot sequence* $\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1}$*, we have for* $0 \le r \le p$*,*

$$D_+^r \mathbb{S}_{p,\boldsymbol{\xi}} = \mathbb{S}_{p-r,\boldsymbol{\xi}_r},$$

where $\boldsymbol{\xi}_r := \{\xi_i\}_{i=r+1}^{n+p+1-r}$.

Proof. The result is obvious for $r = 0$. Let us now consider the case $r = 1$, for which we note that

$$\{B_{1,p-1,\boldsymbol{\xi}_1}, \dots, B_{n-1,p-1,\boldsymbol{\xi}_1}\} = \{B_{2,p-1,\boldsymbol{\xi}}, \dots, B_{n,p-1,\boldsymbol{\xi}}\}.$$

By the differentiation formula (39) it is clear that

$$D_+ \left(\sum_{j=1}^n c_j B_{j,p,\boldsymbol{\xi}} \right) = p \sum_{j=2}^n \left(\frac{c_j - c_{j-1}}{\xi_{j+p} - \xi_j} \right) B_{j,p-1,\boldsymbol{\xi}} \in \mathbb{S}_{p-1,\boldsymbol{\xi}_1}.$$

On the other hand, suppose $s \in \mathbb{S}_{p-1,\boldsymbol{\xi}_1}$, represented as $s = \sum_{j=2}^n d_j B_{j,p-1,\boldsymbol{\xi}}$. Then, by using again the differentiation formula, we can write $s = D_+\big(\sum_{j=1}^n c_j B_{j,p,\boldsymbol{\xi}}\big)$, where c_1 can be any real number and

$$c_j = c_{j-1} + \frac{\xi_{j+p} - \xi_j}{p} d_j, \quad j = 2, \dots, n.$$

For $r > 1$ we use the relation $D_+^r = D_+ D_+^{r-1}$. □

By combining Theorem 13 and Theorem 15 it follows that for $0 \le r \le p$,

$$\mathbb{S}_{p-r}^{\boldsymbol{r}-r}(\Delta) = D_+^r \mathbb{S}_{p,\boldsymbol{\xi}},$$

where $\boldsymbol{r} - r := \big(\max(r_1 - r, -1), \dots, \max(r_\ell - r, -1)\big)$ and the knot sequence $\boldsymbol{\xi}$ is constructed as in Theorem 13.

1.2.3 B-spline representation of polynomials

Polynomials can be represented in terms of B-splines of at least the same degree. We now derive an explicit expression for their B-spline coefficients by using the dual polynomials and the (local) Marsden identity.

Theorem 16 (Marsden identity). *We have*

$$(y - x)^p = \sum_{j=1}^n \psi_{j,p,\boldsymbol{\xi}}(y) B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_{p+1}, \xi_{n+1}], \quad y \in \mathbb{R}, \tag{44}$$

where $\psi_{j,p,\boldsymbol{\xi}}(y) := (y-\xi_{j+1})\cdots(y-\xi_{j+p})$ *is the polynomial of degree* p *that is dual to* $B_{j,p,\boldsymbol{\xi}}$.

Proof. This follows immediately from the local version (17). Indeed, if $x \in [\xi_{p+1}, \xi_{n+1})$ then $x \in [\xi_m, \xi_{m+1})$ for some $p+1 \leq m \leq n$, and by the local support property (35) we get

$$(y-x)^p = \sum_{j=m-p}^{m} \psi_{j,p,\boldsymbol{\xi}}(y) B_{j,p,\boldsymbol{\xi}}(x) = \sum_{j=1}^{n} \psi_{j,p,\boldsymbol{\xi}}(y) B_{j,p,\boldsymbol{\xi}}(x).$$

Taking into account the left continuity of B-splines at the endpoint ξ_{n+1}, see (33), we arrive at the Marsden identity (44). □

Differentiating $p-k$ times with respect to y in (44) results in the following formula.

Corollary 17. *For* $k = 0, 1, \ldots, p$ *we have*

$$\frac{(y-x)^k}{k!} = \sum_{j=1}^{n} \left(\frac{1}{p!} D^{p-k} \psi_{j,p,\boldsymbol{\xi}}(y) \right) B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_{p+1}, \xi_{n+1}], \quad y \in \mathbb{R}. \tag{45}$$

Corollary 17 immediately leads to the following properties.

- **Representation of monomials.** For $k = 0, 1, \ldots, p$ we have

$$x^k = \sum_{j=1}^{n} \xi_{j,p,\boldsymbol{\xi}}^{*,k} B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_{p+1}, \xi_{n+1}], \tag{46}$$

  where

$$\xi_{j,p,\boldsymbol{\xi}}^{*,k} := (-1)^k \frac{k!}{p!} D^{p-k} \psi_{j,p,\boldsymbol{\xi}}(0). \tag{47}$$

  This follows from (45) with $y = 0$.

- **Partition of unity.** Taking $k = 0$ in (46) gives

$$\sum_{j=1}^{n} B_{j,p,\boldsymbol{\xi}}(x) = 1, \quad x \in [\xi_{p+1}, \xi_{n+1}]. \tag{48}$$

  Since the B-splines are nonnegative it follows that they form a **nonnegative partition of unity** on $[\xi_{p+1}, \xi_{n+1}]$.

- **Greville points.** Taking $k = 1$ in (46) gives for $p \geq 1$,

$$x = \sum_{j=1}^{n} \xi_{j,p,\boldsymbol{\xi}}^{*} B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_{p+1}, \xi_{n+1}], \tag{49}$$

  where

$$\xi_{j,p,\boldsymbol{\xi}}^{*} := \xi_{j,p,\boldsymbol{\xi}}^{*,1} = \frac{\xi_{j+1} + \cdots + \xi_{j+p}}{p}. \tag{50}$$

The number $\xi^*_{j,p,\boldsymbol{\xi}}$ is called a **Greville point** [3]. It is also known as a **knot average** or a **node**.

Example 18. For $p = 3$ the equation (46) gives

$$
\begin{aligned}
1 &= \sum_{j=1}^{n} B_{j,3,\boldsymbol{\xi}}, \\
x &= \sum_{j=1}^{n} \frac{\xi_{j+1}+\xi_{j+2}+\xi_{j+3}}{3} B_{j,3,\boldsymbol{\xi}}, \\
x^2 &= \sum_{j=1}^{n} \frac{\xi_{j+1}\xi_{j+2}+\xi_{j+1}\xi_{j+3}+\xi_{j+2}\xi_{j+3}}{3} B_{j,3,\boldsymbol{\xi}}, \\
x^3 &= \sum_{j=1}^{n} \xi_{j+1}\xi_{j+2}\xi_{j+3} B_{j,3,\boldsymbol{\xi}}.
\end{aligned}
$$

We finally present an expression for the B-spline coefficients of a general polynomial.

Proposition 19 (Representation of polynomials). *Any polynomial g of degree p can be represented as*

$$g(x) = \sum_{j=1}^{n} \Lambda_{j,p,\boldsymbol{\xi}}(g) B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_{p+1}, \xi_{n+1}], \tag{51}$$

where

$$\Lambda_{j,p,\boldsymbol{\xi}}(g) := \frac{1}{p!} \sum_{r=0}^{p} (-1)^{p-r} D^r \psi_{j,p,\boldsymbol{\xi}}(\tau_j) D^{p-r} g(\tau_j), \quad \tau_j \in \mathbb{R}. \tag{52}$$

Proof. The polynomial g can be represented in Taylor form (79) as

$$g(x) = \sum_{r=0}^{p} \frac{(x-\tau_j)^{p-r}}{(p-r)!} D^{p-r} g(\tau_j), \quad \tau_j \in \mathbb{R}.$$

The result follows when we apply (45) with $k = p - r$. □

Note that, if τ_j is a root of ψ_j of multiplicity μ_j then $D^r \psi_i(\tau_j) = 0$, $r = 0, 1, \ldots, \mu_j - 1$ and (52) becomes

$$\Lambda_{j,p,\boldsymbol{\xi}}(g) = \frac{1}{p!} \sum_{r=\mu_j}^{p} (-1)^{p-r} D^r \psi_{j,p,\boldsymbol{\xi}}(\tau_j) D^{p-r} g(\tau_j), \quad \tau_j \in \mathbb{R}. \tag{53}$$

Example 20. The polynomial $g(x) = ax^2 + bx + c$ can be represented in terms of quadratic B-splines:

$$ax^2 + bx + c = \sum_{j=1}^{n} c_j B_{j,2,\boldsymbol{\xi}}.$$

[3] An explicit expression of (50) was given by Greville in [10]. According to Schoenberg [18], Greville reviewed the paper [18] introducing some elegant simplifications.

From (51)–(52) with $\psi_{j,2,\boldsymbol{\xi}}(y) := (y-\xi_{j+1})(y-\xi_{j+2})$, we obtain that

$$
\begin{aligned}
c_j = \Lambda_{j,2,\boldsymbol{\xi}}(g) &= \frac{1}{2}\left[(\tau_j-\xi_{j+1})(\tau_j-\xi_{j+2})2a-(2\tau_j-\xi_{j+1}-\xi_{j+2})(2a\tau_j+b)+2(a\tau_j^2+b\tau_j+c)\right] \\
&= a\,\xi_{j+1}\xi_{j+2}+b\,\frac{\xi_{j+1}+\xi_{j+2}}{2}+c.
\end{aligned}
$$

1.2.4 B-spline representation of splines

In the previous section we have derived an explicit expression for the B-spline coefficients of polynomials; see (51). The next theorem extends this result by providing an explicit expression for the B-spline coefficients of any spline in $\mathbb{S}_{p,\boldsymbol{\xi}}$.

Theorem 21 (Representation of B-spline coefficients). *Any element s in the space $\mathbb{S}_{p,\boldsymbol{\xi}}$ can be represented as*[4]

$$
s(x) = \sum_{j=1}^{n} \Lambda_{j,p,\boldsymbol{\xi}}(s) B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_{p+1}, \xi_{n+1}], \tag{54}
$$

where

$$
\Lambda_{j,p,\boldsymbol{\xi}}(s) := \frac{1}{p!}
\begin{cases}
\sum_{r=\mu_j}^{p}(-1)^{p-r}D^r\psi_{j,p,\boldsymbol{\xi}}(\tau_j)D_+^{p-r}s(\tau_j), & \textit{if } \tau_j=\xi_j, \\
\sum_{r=\mu_j}^{p}(-1)^{p-r}D^r\psi_{j,p,\boldsymbol{\xi}}(\tau_j)D^{p-r}s(\tau_j), & \textit{if } \xi_j<\tau_j<\xi_{j+p+1}, \\
\sum_{r=\mu_j}^{p}(-1)^{p-r}D^r\psi_{j,p,\boldsymbol{\xi}}(\tau_j)D_-^{p-r}s(\tau_j), & \textit{if } \tau_j=\xi_{j+p+1},
\end{cases}
\tag{55}
$$

and where $\mu_j \ge 0$ is the number of times τ_j appears in $\xi_{j+1},\dots,\xi_{j+p}$.

Proof. Suppose $\xi_j \le \tau_j < \xi_{j+p+1}$ and let $I_j := [\xi_{m_j}, \xi_{m_j+1})$ be the interval containing τ_j. The restriction of s to I_j is a polynomial and so by Proposition 19 we find

$$
s(x) = \sum_{i=m_j-p}^{m_j}\left(\frac{1}{p!}\sum_{r=0}^{p}(-1)^{p-r}D^r\psi_{i,p,\boldsymbol{\xi}}(\tau_j)D_+^{p-r}s(\tau_j)\right)B_{i,p,\boldsymbol{\xi}}(x), \quad x \in I_j. \tag{56}
$$

Note that since $\xi_j \le \tau_j < \xi_{j+p+1}$ we have $j \le m_j \le j+p$ which implies $m_j - p \le j \le m_j$. By taking $i=j$ in (56) and using the local linear independence of the B-splines, we obtain

$$
\Lambda_{j,p,\boldsymbol{\xi}}(s) := \frac{1}{p!}\sum_{r=0}^{p}(-1)^{p-r}D^r\psi_{j,p,\boldsymbol{\xi}}(\tau_j)D_+^{p-r}s(\tau_j).
$$

Since $D^r\psi_{j,p,\boldsymbol{\xi}}(\tau_j)=0$ for $r<\mu_j$ we obtain the top term in (55). In the middle term we can replace $D_+^{p-r}s(\tau_j)$ by $D^{p-r}s(\tau_j)$ since $s \in C^{p-\mu_j}(\tau_j)$. The proof of the last term is similar using D_- instead of D_+. □

[4] The number $\Lambda_{j,p,\boldsymbol{\xi}}(s)$ is known as the **de Boor–Fix functional** [4] applied to s.

Note that the operator $\Lambda_{j,p,\boldsymbol{\xi}}$ in (53) is identical to $\Lambda_{j,p,\boldsymbol{\xi}}$ in (55). However, in the spline case we need the restriction $\tau_j \in [\xi_j, \xi_{j+p+1}]$.

Because the set of B-splines $\{B_{j,p,\boldsymbol{\xi}}\}_{j=1}^n$ is a basis for the space $\mathbb{S}_{p,\boldsymbol{\xi}}$, the coefficients $\Lambda_{j,p,\boldsymbol{\xi}}(s)$ are uniquely determined for any $s \in \mathbb{S}_{p,\boldsymbol{\xi}}$. Thus, the right-hand side in (55) does not depend on the choice of τ_j. This is an astonishing property considering the complexity of the expression.

For example, one could take the Greville point $\xi^*_{j,p,\boldsymbol{\xi}}$ defined in (50) as a valid choice for the point τ_j. It is easy to verify that $\xi^*_{j,p,\boldsymbol{\xi}} \in [\xi_j, \xi_{j+p+1}]$, and moreover, $\xi^*_{j,p,\boldsymbol{\xi}} \in (\xi_j, \xi_{j+p+1})$ if $B_{j,p,\boldsymbol{\xi}}$ is a continuous function.

Example 22. We consider the quadratic spline

$$s(x) = \sum_{j=1}^n c_j B_{j,2,\boldsymbol{\xi}}(x),$$

and we illustrate that some derivative terms in the expression (55) can be canceled by specific choices of τ_j.

- If τ_j is the Greville point $\xi^*_{j,2,\boldsymbol{\xi}} := (\xi_{j+1} + \xi_{j+2})/2$, then there is no first derivative term. Indeed, we have

$$c_j = \Lambda_{j,2,\boldsymbol{\xi}}(s) = s(\xi^*_{j,2,\boldsymbol{\xi}}) - \frac{(\xi_{j+2} - \xi_{j+1})^2}{8} D^2 s(\xi^*_{j,2,\boldsymbol{\xi}}).$$

  Moreover, since $s \in \mathbb{P}_2$ on $[\xi_{j+1}, \xi_{j+2}]$, we can replace $D^2 s(\xi^*_{j,2,\boldsymbol{\xi}})$ by a difference quotient

$$D^2 s(\xi^*_{j,2,\boldsymbol{\xi}}) = \big(s(\xi_{j+2}) - 2s(\xi^*_{j,2,\boldsymbol{\xi}}) + s(\xi_{j+1})\big) \Big/ \left(\frac{\xi_{j+2} - \xi_{j+1}}{2}\right)^2,$$

  to obtain

$$c_j = -\frac{1}{2}s(\xi_{j+1}) + 2s(\xi^*_{j,2,\boldsymbol{\xi}}) - \frac{1}{2}s(\xi_{j+2}). \tag{57}$$

- If τ_j is equal to ξ_{j+1} or ξ_{j+2}, then there is no second derivative term. Indeed, we have

$$c_j = \Lambda_{j,2,\boldsymbol{\xi}}(s) = s(\tau_j) + \frac{\xi^*_{j,2,\boldsymbol{\xi}} - \tau_j}{2} Ds(\tau_j).$$

  A similar property holds for any p: if τ_j is chosen as one of the interior knots $\xi_{j+1}, \dots, \xi_{j+p}$, then there is no p-th derivative term in the expression of $\Lambda_{j,p,\boldsymbol{\xi}}(s)$.

1.3 Cardinal B-splines

A particularly interesting case of B-spline functions is obtained when the knot sequence is uniformly spaced. Without loss of generality we can assume that the knot sequence is given by the set of integers $\mathbb{Z}$. It is natural to index the knots as $\xi_j = j$, $j \in \mathbb{Z}$. Due to the translation invariance property (9) we have

$$B_{j,p,\mathbb{Z}}(x) = B_{0,p,\mathbb{Z}}(x - j), \quad j \in \mathbb{Z}. \tag{58}$$

Therefore, all the B-splines on the knot sequence $\mathbb{Z}$ are integer translates of a single function. This motivates the following definition.

Definition 23. *The function $M_p := B[0,1,\ldots,p+1]$ is the* **cardinal B-spline** *of degree p.*

Cardinal B-splines possess several nice properties.

- **Recurrence relation.** From Definition 2 we obtain

$$M_0(x) = \begin{cases} 1, & \text{if } x \in [0,1), \\ 0, & \text{otherwise}, \end{cases} \tag{59}$$

$$M_p(x) = \frac{x}{p} M_{p-1}(x) + \frac{p+1-x}{p} M_{p-1}(x-1), \quad p \geq 1. \tag{60}$$

- **Differentiation and integration.** The formulas (22) and (29) simplify in the case of cardinal B-splines to

$$DM_p(x) = M_{p-1}(x) - M_{p-1}(x-1), \tag{61}$$

and

$$\int_{\mathbb{R}} M_p(x)\,\mathrm{d}x = 1. \tag{62}$$

- **Convolution.** The convolution of two functions f and g is given by

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y.$$

We have

$$M_p(x) = (M_{p-1} * M_0)(x) = \int_0^1 M_{p-1}(x-y)\,\mathrm{d}y, \tag{63}$$

and

$$M_p(x) = \big(\overbrace{M_0 * \cdots * M_0}^{p+1}\big)(x). \tag{64}$$

Proof. From (61) we deduce

$$\begin{aligned} M_p(x) &= \int_0^x (M_{p-1}(y) - M_{p-1}(y-1))\,\mathrm{d}y = \int_0^x M_{p-1}(y)\,\mathrm{d}y - \int_{-1}^{x-1} M_{p-1}(y)\,\mathrm{d}y \\ &= \int_{x-1}^x M_{p-1}(y)\,\mathrm{d}y = \int_0^1 M_{p-1}(x-y)\,\mathrm{d}y. \end{aligned}$$

Applying recursively (63) immediately gives (64). □

- **Symmetry.** The cardinal B-spline M_p is symmetric with respect to the midpoint of the support, namely $(p+1)/2$. More generally,

$$D^r M_p\left(\frac{p+1}{2} + x\right) = (-1)^r D^r M_p\left(\frac{p+1}{2} - x\right), \quad r = 0,\ldots,p-1, \tag{65}$$

and

$$D_-^p M_p\left(\frac{p+1}{2}+x\right) = (-1)^p D_+^p M_p\left(\frac{p+1}{2}-x\right). \tag{66}$$

Proof. From repeated differentiations, it is sufficient to prove $M_p(p+1-x) = M_p(x)$. We proceed by induction. It is easy to check that it is true for $p=0$. Assuming the symmetry property holds for degree $p-1$ and using the convolution property, we get

$$\begin{aligned} M_p(p+1-x) &= \int_0^1 M_{p-1}(p+1-x-t)\,\mathrm{d}t = \int_0^1 M_{p-1}(x-1+t)\,\mathrm{d}t \\ &= -\int_0^1 M_{p-1}(x-t)\,\mathrm{d}t = \int_0^1 M_{p-1}(x-t)\,\mathrm{d}t = M_p(x). \end{aligned}$$

□

- **Fourier transform.** The Fourier transform of $f \in L_2(\mathbb{R})$ is given by

$$\widehat{f}(\theta) := \int_{\mathbb{R}} f(x)\,\mathrm{e}^{-\mathrm{i}\,\theta x}\,\mathrm{d}x,$$

where $\mathrm{i} := \sqrt{-1}$ denotes the imaginary unit. From (59), a direct computation immediately gives

$$\widehat{M_0}(\theta) = \frac{1-\mathrm{e}^{-\mathrm{i}\,\theta}}{\mathrm{i}\,\theta}.$$

An interesting property of the Fourier transform of a convolution is

$$(\widehat{f*g})(\theta) = \widehat{f}(\theta)\widehat{g}(\theta), \quad \forall f,g \in L_2(\mathbb{R});$$

see, e.g., [14]. So, the convolution property (63) gives

$$\widehat{M_p}(\theta) = \big(\widehat{M_0}(\theta)\big)^{p+1} = \left(\frac{1-\mathrm{e}^{-\mathrm{i}\,\theta}}{\mathrm{i}\,\theta}\right)^{p+1}.$$

From (58) it follows that the set of integer translates

$$\big\{M_p(x-i),\ i \in \mathbb{Z}\big\} \tag{67}$$

consists of locally linearly independent functions. They span the space of piecewise polynomials of degree p and smoothness $p-1$ with integer break points, see (41), and they have the following properties.

- **Partition of unity.** From (48) and (58) we get

$$\sum_{i\in\mathbb{Z}} M_p(x-i) = 1, \quad x \in \mathbb{R}.$$

Due to the local support of cardinal B-splines, the above series reduces to a finite sum for any x. More precisely, referring to (21), we have

$$\sum_{i=m-p}^{m} M_p(x-i) = 1, \quad x \in [m, m+1).$$

- **Greville points.** From (50)–(49) and (58) we have

$$x = \sum_{i \in \mathbb{Z}} \zeta_{i,p}^* M_p(x-i), \quad x \in \mathbb{R},$$

with

$$\zeta_{i,p}^* := \frac{(1+i)+\cdots+(p+i)}{p} = \frac{p+1}{2} + i. \tag{68}$$

We now provide an expression for inner products of the cardinal B-spline and its translates.

Theorem 24 (Inner product). *Given $p_1, p_2 \geq 0$, we have*

$$\int_{\mathbb{R}} M_{p_1}(y) M_{p_2}(y+x)\,\mathrm{d}y = M_{p_1+p_2+1}(p_1+1+x) = M_{p_1+p_2+1}(p_2+1-x).$$

Proof. From the symmetry property (65) with $r=0$ and the convolution relation (63) of cardinal B-splines, we get

$$\begin{aligned}
\int_{\mathbb{R}} M_{p_1}(y) M_{p_2}(y+x)\,\mathrm{d}y &= \int_{\mathbb{R}} M_{p_1}(y) M_{p_2}(p_2+1-y-x)\,\mathrm{d}y \\
&= \big(M_{p_1} * M_{p_2}\big)(p_2+1-x) \\
&= \big(\overbrace{M_0 * \cdots * M_0}^{p_1+1} * \overbrace{M_0 * \cdots * M_0}^{p_2+1}\big)(p_2+1-x) \\
&= M_{p_1+p_2+1}(p_2+1-x).
\end{aligned}$$

Finally, again by symmetry of cardinal B-splines, we have

$$M_{p_1+p_2+1}(p_1+1+x) = M_{p_1+p_2+1}(p_2+1-x),$$

which completes the proof. □

A generalization towards inner products of derivatives can be found in [9].

Theorem 25 (Inner product of derivatives). *Given $p_1 \geq r_1 \geq 0$ and $p_2 \geq r_2 \geq 0$, we have*

$$\begin{aligned}
\int_{\mathbb{R}} D^{r_1} M_{p_1}(y)\, D^{r_2} M_{p_2}(y+x)\,\mathrm{d}y &= (-1)^{r_1} D^{r_1+r_2} M_{p_1+p_2+1}(p_1+1+x) \\
&= (-1)^{r_2} D^{r_1+r_2} M_{p_1+p_2+1}(p_2+1-x).
\end{aligned}$$

Due to the relevance of the set (67), the results in Theorem 24 and Theorem 25 are of particular interest when we consider integer shifts, i.e., $x \in \mathbb{Z}$. In this case, the above inner products reduce to evaluations of cardinal B-splines and their derivatives at either integer or half-integer points. Moreover, there is a relation with the Greville points (68). Indeed, if $p_1 = p_2 = p$ and $x = i$ in Theorem 24, then

$$\int_{\mathbb{R}} M_p(x)M_p(x+i)\,\mathrm{d}x = M_{2p+1}(p+1+i) = M_{2p+1}(\zeta^*_{i,2p+1}).$$

A similar relation holds for the inner products of derivatives in Theorem 25. Thanks to the recurrence relation for derivatives (61), the inner products of derivatives of cardinal B-splines and its integer translates reduce to evaluations of cardinal B-splines at either integer or half-integer points.

1.4 Condition number

A basis $\{B_j\}$ of a normed space is said to be **stable** with respect to a vector norm if there are positive constants K_L and K_U such that

$$K_L^{-1}\|\boldsymbol{c}\| \leq \Big\|\sum_j c_j B_j\Big\| \leq K_U\|\boldsymbol{c}\|, \tag{69}$$

for all coefficient vectors $\boldsymbol{c} := (c_j)$. For simplicity we use the same symbol $\|\cdot\|$ for the norm in the space and the vector norm. The number

$$\kappa := \inf\{K_L K_U :\ K_L \text{ and } K_U \text{ satisfy (69)}\} \tag{70}$$

is called the **condition number** of the basis $\{B_j\}$ with respect to $\|\cdot\|$.

Such condition numbers give an upper bound for how much an error in coefficients can be magnified in function values. Indeed, if $f = \sum_j c_j B_j \neq 0$ and $g = \sum_j d_j B_j$ then it follows immediately from (69) that

$$\frac{\|f-g\|}{\|f\|} \leq \kappa \frac{\|\boldsymbol{c}-\boldsymbol{d}\|}{\|\boldsymbol{c}\|},$$

where $\boldsymbol{c} := (c_j)$ and $\boldsymbol{d} := (d_j)$. Many other applications are given in [3] and it is interesting to have estimates for the size of κ.

We consider the L_q-norm for functions and the q-norm for vectors with $1 \leq q \leq \infty$. We focus on a scaled version of the B-spline basis defined on $[\xi_1, \xi_{n+p+1})$,

$$\{N_{j,p,q,\boldsymbol{\xi}}\}_{j=1}^n := \{\gamma_{j,p,\boldsymbol{\xi}}^{-1/q} B_{j,p,\boldsymbol{\xi}}\}_{j=1}^n, \tag{71}$$

where $\gamma_{j,p,\boldsymbol{\xi}}$ is defined in (29). Note that the knot sequence $\boldsymbol{\xi}$ has to be $(p+1)$-basic in order to have linearly independent B-splines. This also ensures that $\gamma_{j,p,\boldsymbol{\xi}} > 0$. The q**-norm condition number** of the basis in (71) will be denoted by $\kappa_{p,q,\boldsymbol{\xi}}$, i.e.,

$$\kappa_{p,q,\boldsymbol{\xi}} := \sup_{\boldsymbol{c}\neq 0} \frac{\big\|\sum_{j=1}^n c_j N_{j,p,q,\boldsymbol{\xi}}\big\|_{L_q([\xi_1,\xi_{n+p+1}])}}{\|\boldsymbol{c}\|_q} \sup_{\boldsymbol{c}\neq 0} \frac{\|\boldsymbol{c}\|_q}{\big\|\sum_{j=1}^n c_j N_{j,p,q,\boldsymbol{\xi}}\big\|_{L_q([\xi_1,\xi_{n+p+1}])}}. \tag{72}$$

The next theorem shows that the scaled B-spline basis above is stable in any L_q-norm independently on the knot sequence $\boldsymbol{\xi}$. It also provides an upper bound for the q-norm condition number which does not depend on $\boldsymbol{\xi}$. To this end, we first state the **Hölder inequality for sums**:

$$\sum_{j=1}^{n} |x_j y_j| \leq \|\boldsymbol{x}\|_q \, \|\boldsymbol{y}\|_{q'}, \tag{73}$$

where q, q' are integers so that

$$\frac{1}{q} + \frac{1}{q'} = 1, \quad 1 \leq q \leq \infty. \tag{74}$$

In particular, $q' = \infty$ if $q = 1$ and $q' = 2$ if $q = 2$.

Theorem 26. *For any $p \geq 0$ there exists a positive constant K_p depending only on p, such that for any vector $\boldsymbol{c} := (c_1, \ldots, c_n)$ and for any $1 \leq q \leq \infty$ we have*

$$K_p^{-1} \|\boldsymbol{c}\|_q \leq \left\| \sum_{j=1}^{n} c_j N_{j,p,q,\boldsymbol{\xi}} \right\|_{L_q([\xi_1, \xi_{n+p+1}])} \leq \|\boldsymbol{c}\|_q. \tag{75}$$

Proof. We first prove the upper inequality. Using the nonnegative partition of unity property of B-splines the upper bound for $q = \infty$ is straightforward, so we consider $1 \leq q < \infty$ in the following. By applying the Hölder inequality (73) and again the nonnegative partition of unity property of B-splines, we obtain for $x \in [\xi_1, \xi_{n+p+1})$,

$$\begin{aligned}
\left| \sum_{j=1}^{n} c_j N_{j,p,q,\boldsymbol{\xi}}(x) \right| &\leq \sum_{j=1}^{n} \left| c_j \gamma_{j,p,\boldsymbol{\xi}}^{-1/q} B_{j,p,\boldsymbol{\xi}}(x)(x)^{1/q} \right| \left| B_{j,p,\boldsymbol{\xi}}(x) \right|^{1-1/q} \\
&\leq \left(\sum_{j=1}^{n} |c_j|^q \gamma_{j,p,\boldsymbol{\xi}}^{-1} B_{j,p,\boldsymbol{\xi}}(x) \right)^{1/q} \left(\sum_{j=1}^{n} B_{j,p,\boldsymbol{\xi}}(x) \right)^{1-1/q} \\
&\leq \left(\sum_{j=1}^{n} |c_j|^q \gamma_{j,p,\boldsymbol{\xi}}^{-1} B_{j,p,\boldsymbol{\xi}}(x) \right)^{1/q}.
\end{aligned}$$

Raising both sides of this inequality to the q-th power and integrating gives the inequality

$$\int_{\xi_1}^{\xi_{n+p+1}} \left| \sum_{j=1}^{n} c_j N_{j,p,q,\boldsymbol{\xi}}(x) \right|^q \mathrm{d}x \leq \sum_{j=1}^{n} |c_j|^q \gamma_{j,p,\boldsymbol{\xi}}^{-1} \int_{\xi_j}^{\xi_{j+p+1}} B_{j,p,\boldsymbol{\xi}}(x) \, \mathrm{d}x = \|\boldsymbol{c}\|_q^q.$$

Taking the q-th roots on both sides proves the upper inequality in (75).

We now focus on the lower inequaltity. We extend $\boldsymbol{\xi}$ to a $(p+1)$-open knot sequence $\hat{\boldsymbol{\xi}}$ by possibly increasing the multiplicity of ξ_1 and ξ_{n+p+1} to $p+1$. Clearly, the set of B-splines on $\boldsymbol{\xi}$ is a subset of the set of B-splines on $\hat{\boldsymbol{\xi}}$, and any linear combination of the B-splines on $\boldsymbol{\xi}$ is a linear combination of the B-splines on $\hat{\boldsymbol{\xi}}$ where the extra B-splines have coefficients zero. Therefore, without loss of generality, we can assume that the knot sequence is open with the basic interval $[\xi_1, \xi_{n+p+1}]$. The lower bound then follows from Lemma 45. $\square$

We define a condition number which is independent of the knot sequence as follows

$$\kappa_{p,q} := \sup_{\boldsymbol{\xi}} \kappa_{p,q,\boldsymbol{\xi}}. \tag{76}$$

Theorem 26 shows that

$$\kappa_{p,q} \leq K_p < \infty.$$

It is known that $\kappa_{p,q}$ grows like 2^p for all $1 \leq q \leq \infty$; see [12, 15] where it is proved that

$$\frac{1}{p+1} 2^{p-1/2} \leq \kappa_{p,q} \leq (p+1)2^{p+1}, \quad 1 \leq q \leq \infty. \tag{77}$$

2 Spline approximation

In this section we discuss how well a sufficiently smooth function can be approximated in the spline space spanned by a given set of B-splines. Exploiting the properties of the B-spline basis presented in the previous section, we explicitly construct a spline which achieves optimal approximation accuracy for the function and its derivatives, and we determine the corresponding error estimates. The construction method we are going to present is local and linear.

2.1 *Preliminaries*

The **Hölder inequality for integrals** is given by

$$\int_a^b |f(x)g(x)|\, \mathrm{d}x \leq \|f\|_{L_q(I)} \|g\|_{L_{q'}(I)}, \tag{78}$$

where q, q' are integers satisfying (74).

The **Taylor polynomial** of degree p at the point a to a function $f \in W_\infty^{p+1}([a,b])$ is defined by

$$\mathscr{T}_{a,p} f(x) := \sum_{j=0}^{p} \frac{(x-a)^j}{j!} D^j f(a), \tag{79}$$

and its approximation error can be expressed in integral form for $x \in [a,b]$ as

$$f(x) - \mathscr{T}_{a,p} f(x) = \frac{1}{p!} \int_a^b (x-y)_+^p D^{p+1} f(y)\, \mathrm{d}y. \tag{80}$$

Every polynomial $g \in \mathbb{P}_p$ can be written in **Taylor form** as $g = \mathscr{T}_{a,p}g$.

Theorem 27. *Let $f \in W_\infty^{p+1}([a,b])$ with $1 \le q \le \infty$, and let $\mathscr{T}_{a,p}f$ be the Taylor polynomial of degree p to f at the point a. Then, for any $x \in [a,b]$ and $0 \le r \le p$,*

$$|D^r(f - \mathscr{T}_{a,p}f)(x)| \le \frac{(b-a)^{p+1-r-1/q}}{(p-r)!} \|D^{p+1}f\|_{L_q([a,b])}, \tag{81}$$

and

$$\|D^r(f - \mathscr{T}_{a,p}f)\|_{L_q([a,b])} \le \frac{(b-a)^{p+1-r}}{(p-r)!} \|D^{p+1}f\|_{L_q([a,b])}. \tag{82}$$

Proof. By differentiating the integral form of the Taylor approximation error (80) and using the Hölder inequality (78), we obtain

$$\begin{aligned}
|D^r(f - \mathscr{T}_{a,p}f)(x)| &= \frac{1}{(p-r)!} \int_a^b (x-y)_+^{p-r} D^{p+1}f(y)\,\mathrm{d}y \\
&\le \frac{1}{(p-r)!} \left[\int_a^b (x-y)_+^{(p-r)q'}\,\mathrm{d}y\right]^{1/q'} \|D^{p+1}f\|_{L_q([a,b])} \\
&\le \frac{(b-a)^{p-r+1/q'}}{(p-r)!\,((p-r)q'+1)^{1/q'}} \|D^{p+1}f\|_{L_q([a,b])}.
\end{aligned}$$

Since $1/q + 1/q' = 1$ and $(p-r)q' \ge 0$, we obtain (81). Finally, taking the L_q-norm shows (82). □

For the sake of simplicity one can use the following weaker, but simpler upper bound

$$\|D^r(f - \mathscr{T}_{a,p}f)\|_{L_q([a,b])} \le (b-a)^{p+1-r} \|D^{p+1}f\|_{L_q([a,b])}. \tag{83}$$

2.2 *Spline quasi-interpolation*

In general, a spline approximating a function f can be written in terms of B-splines as

$$\mathscr{Q}f(x) := \sum_{j=1}^{n} \lambda_j f B_{j,p,\boldsymbol{\xi}}(x) \tag{84}$$

for suitable coefficients $\lambda_j f$. The spline in (84) will be referred to as a **quasi-interpolant** to f whenever it provides a “reasonable” approximation to f.

Both interpolation and least squares are examples of quasi-interpolation methods. They are global methods since we have to solve an n by n system of linear equations to find the coefficients $\lambda_j f$. It follows that the value of the spline (84) at a point depends on all the data. In this section we focus on **local linear methods**, i.e., methods where each λ_j is a linear functional only depending on the values of f “near” the support of $B_{j,p,\boldsymbol{\xi}}$. This implies that the value of the spline approximation $\mathscr{Q}f$ at a point depends only on the data in a local neighborhood of the point.

In order to deal with point evaluator functionals we assume in this section that $f \in C^{-1}([a,b])$, where $[a,b]$ is a bounded interval. We consider a spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$, where the knot sequence $\boldsymbol{\xi}$ is $(p+1)$-basic and the basic interval $[\xi_{p+1}, \xi_{n+1}]$ is equal to $[a,b]$.

2.2.1 Definition and basic approximation properties

In view of constructing a spline quasi-interpolant with optimal accuracy, we present some basic approximation properties of quasi-interpolants of the form (84). Since we are interested in local methods, we start with the following definition.

Definition 28. *We say that a linear functional* $\lambda : C^{-1}([a,b]) \to \mathbb{R}$ *is* **supported on a nonempty set** $\mathscr{S} \subset [a,b]$ *if* $\lambda f = 0$ *for any* $f \in C^{-1}([a,b])$ *which vanishes on* $\mathscr{S}$.

Note that the set $\mathscr{S}$ in this definition is not uniquely defined and is not necessary minimal.

To construct our quasi-interpolant we use linear functionals which are supported on intervals consisting of a few knot intervals, where few means independent on the dimension n of the spline space, but can depend on the degree p. This will ensure that $\mathscr{Q}f$ only depends locally on f. To ensure a good approximation power we require polynomial reproduction up to a given degree. Moreover, to bound the error a boundedness assumption on the linear functionals is needed. This leads to the following definitions.

Definition 29. *The quasi-interpolant* $\mathscr{Q}$ *given by* (84) *is called a* **local quasi-interpolant** *if*

(i) each λ_j *is supported on the interval* I_j*, where*

$$I_j := [\xi_{j-\nu_L}, \xi_{j+p+1+\nu_U}] \cap [a,b], \tag{85}$$

for some integers $\nu_L, \nu_U \geq -p$ *such that* I_j *has nonempty interior;*

(ii) the λ_j *are chosen so that* (84) *reproduces* $\mathbb{P}_l$*, i.e.,*

$$\mathscr{Q}g(x) = g(x) \text{ for all } x \in [a,b] \text{ and all } g \text{ in } \mathbb{P}_l, \tag{86}$$

for some l *with* $0 \leq l \leq p$.

Note that the number of knot intervals in I_j is bounded above by $\nu_L + \nu_U + p + 1$.

Definition 30. *A local quasi-interpolant* $\mathscr{Q}$ *is called* **bounded** *in an* L_q*-norm,* $1 \leq q \leq \infty$*, if there is a constant* $C_{\mathscr{Q}}$ *such that for each* λ_j *we have*

$$|\lambda_j f| \leq C_{\mathscr{Q}} h_{j,p,\boldsymbol{\xi}}^{-1/q} \|f\|_{L_q(I_j)} \text{ for all } f \in C^{-1}(I_j), \tag{87}$$

where

$$h_{j,p,\boldsymbol{\xi}} := \max_{\max(j,p+1) \leq k \leq \min(n,j+p)} \xi_{k+1} - \xi_k. \tag{88}$$

Note that $h_{j,p,\boldsymbol{\xi}}$ is the largest length of a knot interval in the intersection of the basic interval with the support of $B_{j,p,\boldsymbol{\xi}}$. The requirement (85) ensures that the spline in (84) provides a local approximation to f. The polynomial reproduction as stated in (86) coupled with the boundedness of the linear functionals are the main ingredients to prove the approximation power of any bounded local quasi-interpolant.

We now give both a local and a global version of the approximation power of bounded local quasi-interpolants. To turn a local bound into a global bound we first state the following lemma.

Lemma 31. *Suppose that $f \in L_q([\xi_{p+1}, \xi_{n+1}])$ for some q, $1 \le q < \infty$, and that $m_{i_1}, \dots, m_{i_2}$ are integers with $m_{i_1} < \cdots < m_{i_2}$, $\xi_{p+1} \le \xi_{m_{i_1}}$ and $\xi_{m_{i_2}+k} \le \xi_{n+1}$ for some positive integer k and integers $i_1 \le i_2$. Then*

$$\left(\sum_{j=i_1}^{i_2} \|f\|^q_{L_q([\xi_{m_j}, \xi_{m_j+k}])} \right)^{1/q} \le k^{1/q} \|f\|_{L_q([\xi_{p+1}, \xi_{n+1}])}. \tag{89}$$

Proof. With the stated assumptions each knot interval in $[\xi_{p+1}, \xi_{n+1}]$ is counted at most k times and moreover all the local intervals $[\xi_{m_j}, \xi_{m_j+k}]$ are contained in $[\xi_{p+1}, \xi_{n+1}]$. The definition of the L_q-norm gives immediately (89). □

Theorem 32. *Let $\mathscr{Q}$ be a bounded local quasi-interpolant in an L_q-norm, $1 \le q \le \infty$, as in Definitions 29 and 30. Let l, p be integers with $0 \le l \le p$. Suppose $\xi_m < \xi_{m+1}$ for some $p+1 \le m \le n$ and let $f \in W_q^{l+1}(J_m)$ with*

$$J_m := [\xi_{m-p-\nu_L}, \xi_{m+p+1+\nu_U}] \cap [a,b].$$

Then,

$$\|f - \mathscr{Q}f\|_{L_q([\xi_m, \xi_{m+1}])} \le \frac{(2p+\nu_L+\nu_U+1)^{l+1}}{l!} (1+C_{\mathscr{Q}}) h^{l+1}_{m,\boldsymbol{\xi}} \|D^{l+1} f\|_{L_q(J_m)}, \tag{90}$$

where $h_{m,\boldsymbol{\xi}}$ is the largest length of a knot interval in J_m. Moreover, if $f \in W_q^{l+1}([a,b])$ then

$$\|f - \mathscr{Q}f\|_{L_q([a,b])} \le \frac{(2p+\nu_L+\nu_U+1)^{l+1+1/q}}{l!} (1+C_{\mathscr{Q}}) h^{l+1}_{\boldsymbol{\xi}} \|D^{l+1} f\|_{L_q([a,b])}, \tag{91}$$

where

$$h_{\boldsymbol{\xi}} := \max_{p+1 \le j \le n} \xi_{j+1} - \xi_j.$$

Proof. Note that f is continuous since $l \ge 0$. Suppose $x \in [\xi_m, \xi_{m+1})$. By the local partition of unity (21) and by (87) we have

$$|\mathscr{Q}f(x)| \le \max_{m-p \le j \le m} |\lambda_j f| \le C_{\mathscr{Q}} \max_{m-p \le j \le m} h^{-1/q}_{j,p,\boldsymbol{\xi}} \|f\|_{L_q(I_j)}.$$

Since $\xi_{m+1} - \xi_m \le \min_{m-p \le j \le m} h_{j,p,\boldsymbol{\xi}}$ and $J_m = \cup_{m-p \le j \le m} I_j$ we find

$$\|\mathscr{Q}f\|_{L_q([\xi_m, \xi_{m+1}])} \le C_{\mathscr{Q}} \|f\|_{L_q(J_m)}. \tag{92}$$

¿From (86) we know that $\mathscr{Q}$ reproduces any polynomial $g \in \mathbb{P}_l$, and so the triangle inequality gives

$$\|f - \mathscr{Q}f\|_{L_q([\xi_m,\xi_{m+1}])} \leq \|f - g\|_{L_q([\xi_m,\xi_{m+1}])} + \|\mathscr{Q}(f-g)\|_{L_q([\xi_m,\xi_{m+1}])}.$$

Since $\nu_L, \nu_U \geq -p$ we have $[\xi_m, \xi_{m+1}] \subset J_m$. Therefore, by (92) for any $g \in \mathbb{P}_l$,

$$\|f - \mathscr{Q}f\|_{L_q([\xi_m,\xi_{m+1}])} \leq (1 + C_{\mathscr{Q}})\|f - g\|_{L_q(J_m)}. \tag{93}$$

Let us now choose $g := \mathscr{T}_{\xi_{m-p-\nu_L},l} f$, where $\mathscr{T}_{\xi_{m-p-\nu_L},l} f$ is the Taylor polynomial of degree l defined in (79) with $a = \xi_{m-p-\nu_L}$. Then, by (82) with $r = 0$ we have

$$\|f - g\|_{L_q(J_m)} \leq \frac{(2p + \nu_L + \nu_U + 1)^{l+1}}{l!} h_{m,\boldsymbol{\xi}}^{l+1} \|D^{l+1} f\|_{L_q(J_m)}. \tag{94}$$

Combining the inequalities (93) and (94) gives the local bound.

Since each J_m is contained in the basic interval $[a,b]$ the global bound follows immediately from the local one and Lemma 31. □

Example 33. Let $\boldsymbol{\xi}$ be a $(p+1)$-open knot sequence. The Schoenberg operator

$$\mathscr{V}_{p,\boldsymbol{\xi}} f(x) := \sum_{j=1}^{n} f(\xi^*_{j,p,\boldsymbol{\xi}}) B_{j,p,\boldsymbol{\xi}}(x), \tag{95}$$

where $\xi^*_{j,p,\boldsymbol{\xi}}$ is the j-th Greville point of degree p, see (50), is a bounded local quasi-interpolant in the L_∞-norm with $l = 1$ and $C_{\mathscr{Q}} = 1$. Since $\xi^*_{j,p,\boldsymbol{\xi}}$ belongs to a knot interval $[\xi_{m_j}, \xi_{m_j+1})$ of $[\xi_{j+1}, \xi_{j+p}]$ we can choose $\nu_L = \nu_U = 0$ in (85). Therefore, Theorem 32 implies for any $f \in W^2_\infty([a,b])$,

$$\|f - \mathscr{V}_{p,\boldsymbol{\xi}} f\|_{L_\infty([a,b])} \leq 2(2p+1)^2 h_{\boldsymbol{\xi}}^2 \|D^2 f\|_{L_\infty([a,b])}. \tag{96}$$

The next proposition can be used to find the degree l of polynomials reproduced by a linear quasi-interpolant. We will formulate another condition in Proposition 38.

Proposition 34. *Let*

$$\{\varphi_{j,0}, \ldots, \varphi_{j,l}\}, \quad j = 1, \ldots, n, \quad 0 \leq l \leq p \tag{97}$$

be n sets of basis functions for $\mathbb{P}_l$ and let

$$\varphi_{j,r} = \sum_{m=1}^{n} c_{j,r,m} B_{m,p,\boldsymbol{\xi}} \tag{98}$$

be their B-spline representations. The linear quasi-interpolant (84) *reproduces $\mathbb{P}_l$ provided the corresponding linear functionals satisfy*

$$\lambda_j(\varphi_{j,r}) = c_{j,r,j}, \quad j = 1, \ldots, n, \quad r = 0, \ldots, l. \tag{99}$$

Proof. Any $g \in \mathbb{P}_l$ can be written both in terms of the B-splines and the φ's, say

$$g=\sum_{m=1}^{n} b_m B_{m,p,\boldsymbol{\xi}}=\sum_{r=0}^{l} b_{j,r}\varphi_{j,r}, \quad j=1,\ldots,n. \tag{100}$$

By (98) and (100) for $j=1,\ldots,n$,

$$g=\sum_{r=0}^{l} b_{j,r}\left(\sum_{m=1}^{n} c_{j,r,m}B_{m,p,\boldsymbol{\xi}}\right)=\sum_{m=1}^{n}\left(\sum_{r=0}^{l} b_{j,r}c_{j,r,m}\right)B_{m,p,\boldsymbol{\xi}}=\sum_{m=1}^{n} b_m B_{m,p,\boldsymbol{\xi}}.$$

By linear independence of the B-splines and choosing $i=m$ we obtain

$$b_m=\sum_{r=0}^{l} b_{m,r}c_{m,r,m}. \tag{101}$$

Similarly, for $\mathscr{Q}g$ using (100) with $j=m$,

$$\mathscr{Q}g:=\sum_{m=1}^{n}\lambda_m(g)B_{m,p,\boldsymbol{\xi}}=\sum_{m=1}^{n}\lambda_m\left(\sum_{r=0}^{l} b_{m,r}\varphi_{m,r}\right)B_{m,p,\boldsymbol{\xi}}.$$

¿From the linearity of λ_m and (99), (101) and finally (100) again we obtain

$$\mathscr{Q}g=\sum_{m=1}^{n}\sum_{r=0}^{l} b_{m,r}\lambda_m(\varphi_{m,r})B_{m,p,\boldsymbol{\xi}}=\sum_{m=1}^{n}\sum_{r=0}^{l} b_{m,r}c_{m,r,m}B_{m,p,\boldsymbol{\xi}}=\sum_{m=1}^{n} b_m B_{m,p,\boldsymbol{\xi}}=g.$$

□

The next proposition gives a sufficient condition for a quasi-interpolant to reproduce the whole spline space. We will formulate another sufficient condition in Proposition 40.

Proposition 35. *The linear quasi-interpolant* (84) *reproduces the whole spline space, i.e.,*

$$\mathscr{Q}s(x)=s(x), \quad s\in\mathbb{S}_{p,\boldsymbol{\xi}}, \quad x\in[\xi_{p+1},\xi_{n+1}), \tag{102}$$

if $\mathscr{Q}$ reproduces $\mathbb{P}_p$ and each linear functional λ_j is supported on one knot interval[5]

$$[\xi_{m_j}^{+},\xi_{m_j+1}^{-}]\subset[\xi_j,\xi_{j+p+1}], \text{ with } \xi_{m_j}<\xi_{m_j+1}. \tag{103}$$

In other words, $\mathscr{Q}$ is a projector onto the spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$.

Proof. Let j with $1\le j\le n$ be fixed. By the linearity it suffices to prove that $\lambda_j(B_{i,p,\boldsymbol{\xi}})=\delta_{i,j}$ for $i=1,\ldots,n$. On the interval $[\xi_{m_j}^{+},\xi_{m_j+1}^{-}]$ the local support property implies that $\lambda_j(B_{i,p,\boldsymbol{\xi}})=0$ for $i\notin\{m_j-p,\ldots,m_j\}$. This follows because we use the left limit at ξ_{m_j+1} if necessary. Since $B_{i,p,\boldsymbol{\xi}}\in\mathbb{P}_p$ on this interval, we have

[5] This notation means that if $\lambda_j f$ uses the value of f or one its derivatives at ξ_{m_j} (or ξ_{m_j+1}) then this value is obtained by taking the one sided limit from the right (or left).

$$B_{i,p,\boldsymbol{\xi}}(x) = \mathscr{Q}(B_{i,p,\boldsymbol{\xi}})(x) = \sum_{k=m_j-p}^{m_j} \lambda_k(B_{i,p,\boldsymbol{\xi}}) B_{k,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_{m_j}, \xi_{m_j+1}),$$

and by local linear independence of the B-splines we obtain $\lambda_k(B_{i,p,\boldsymbol{\xi}}) = \delta_{k,i}$ for $k = m_j - p, \ldots, m_j$. In particular, it holds for $k = i$ since the condition (103) implies that $m_j - p \leq j \leq m_j$. □

Example 36. Let $p = 2$ and let $\boldsymbol{\xi}$ be a 3-open knot sequence. We consider the operator in Example 22 in the form

$$\mathscr{Q}_{2,\boldsymbol{\xi}} f(x) := \sum_{j=1}^{n} \big(\alpha_{2,0} f(\xi_{j+1}) + \alpha_{2,1} f(\xi^*_{j,2,\boldsymbol{\xi}}) + \alpha_{2,2} f(\xi_{j+2})\big) B_{j,2,\boldsymbol{\xi}}(x),$$

where $\xi^*_{j,2,\boldsymbol{\xi}} = (\xi_{j+2} + \xi_{j+1})/2$ is the j-th Greville point of degree 2. We know that if we choose $\alpha_{2,0} = \alpha_{2,2} = -1/2$ and $\alpha_{2,1} = 2$ then $\mathscr{Q}_{2,\boldsymbol{\xi}}$ reproduces $\mathbb{P}_2$, i.e., $l = 2$. Proposition 35 even implies that it is a projector on the spline space $\mathbb{S}_{2,\boldsymbol{\xi}}$. Moreover,

$$\Big| -\frac{1}{2} f(\xi_{j+1}) + 2 f(\xi^*_{j,2,\boldsymbol{\xi}}) - \frac{1}{2} f(\xi_{j+2}) \Big| \leq 3 \|f\|_{L_\infty([\xi_{j+1}, \xi_{j+2}])}.$$

It follows that $\mathscr{Q}_{2,\boldsymbol{\xi}}$ is a bounded local quasi-interpolant in the L_∞-norm with $l = 2$ and $C_{\mathscr{Q}} = 3$ and that $\nu_L = \nu_U = 0$ in (85). In this case Theorem 32 implies for any $f \in W^3_\infty([a,b])$,

$$\|f - \mathscr{Q}_{2,\boldsymbol{\xi}} f\|_{L_\infty([a,b])} \leq 4 \frac{5^3}{2!} h^3_{\boldsymbol{\xi}} \|D^3 f\|_{L_\infty([a,b])},$$

showing that the error is $O(h^3_{\boldsymbol{\xi}})$.

2.2.2 A general construction

We now describe a general recipe for constructing a wide class of local quasi-interpolants.

Recipe 37. *For fixed k, the value of $\lambda_k f$ is determined as follows:*

(i) Choose an interval $\hat{I}_k := [\xi_{m_{L,k}}, \xi_{m_{U,k}}] \subset [a,b]$ such that

$$(\xi_{m_{L,k}}, \xi_{m_{U,k}}) \cap (\xi_k, \xi_{k+p+1}) \neq \emptyset,$$

and $m_{U,k} - m_{L,k}$ is bounded independently of n.

(ii) Choose some linear approximation method $\mathscr{Q}_k$ which can be written in B-spline form as

$$\mathscr{Q}_k f(x) = \sum_{j=m_{L,k}-p}^{m_{U,k}-1} b_j B_{j,p,\boldsymbol{\xi}}(x) \text{ for } x \in (\xi_{m_{L,k}}, \xi_{m_{U,k}}),$$

and has the following local polynomial reproduction property

$$\mathscr{Q}_k g(x) = g(x) \text{ for all } g \in \mathbb{P}_l \text{ and } x \in (\xi_{m_{L,k}}, \xi_{m_{U,k}}), \tag{104}$$

for some fixed l with $0 \le l \le p$.

(iii) Set $\lambda_k f := b_k$.

Note that $\lambda_k f$ in (iii) is well defined because $\hat{I}_k$ intersects the interior of the support of $B_{k,p,\boldsymbol{\xi}}$ and therefore $m_{L,k} - p \le k \le m_{U,k} - 1$. Since the number of knot intervals in $\hat{I}_k$ is bounded independently of n, it is always possible to find an interval I_k that satisfies (85) containing $\hat{I}_k$. We now show that the local polynomial reproduction property (104) leads to global reproduction of $\mathbb{P}_l$ as required in Definition 29.

Proposition 38. *The spline approximation operator* $\mathscr{Q}f$ *determined by Recipe 37 has the property that* $\mathscr{Q}g(x) = g(x)$ *for all* $g \in \mathbb{P}_l$ *and* $x \in [a,b]$.

Proof. Given $g \in \mathbb{P}_l$, suppose that $g(x) = \sum_{j=1}^n c_j B_{j,p,\boldsymbol{\xi}}(x)$ for certain coefficients $(c_j)_{j=1}^n$ and $x \in [a,b]$. We must show that if $\mathscr{Q}g(x) = \sum_{j=1}^n \lambda_j g B_{j,p,\boldsymbol{\xi}}(x)$ then $\lambda_j g = c_j$. We note that $g(x) = \sum_{j=m_{L,k}-p}^{m_{U,k}-1} c_j B_{j,p,\boldsymbol{\xi}}(x)$ for $x \in (\xi_{m_{L,k}}, \xi_{m_{U,k}})$. Therefore, by (104) we have

$$\sum_{j=m_{L,k}-p}^{m_{U,k}-1} b_j B_{j,p,\boldsymbol{\xi}}(x) = \mathscr{Q}_k g(x) = g(x) = \sum_{j=m_{L,k}-p}^{m_{U,k}-1} c_j B_{j,p,\boldsymbol{\xi}}(x), \quad x \in (\xi_{m_{L,k}}, \xi_{m_{U,k}}),$$

so by local linear independence we have $b_j = c_j$ for $j = m_{L,k} - p, \dots, m_{U,k} - 1$, and in particular $b_k = c_k$. Since $\lambda_k g = b_k$ we have $\lambda_k g = c_k$, as required. $\square$

Example 39. The Schoenberg operator in Example 33 can be obtained by Recipe 37 as follows. First, choose $\hat{I}_k := [\xi_{m_k}, \xi_{m_k+1}]$ such that the interval $[\xi_{m_k}, \xi_{m_k+1})$ contains $\xi^*_{k,p,\boldsymbol{\xi}}$. Then, choose $\mathscr{Q}_k$ as the linear interpolant to f at the Greville point $\xi^*_{k,p,\boldsymbol{\xi}}$ and an additional point ξ in $\hat{I}_k$. This gives

$$\mathscr{Q}_k f(x) = \frac{x - \xi^*_{k,p,\boldsymbol{\xi}}}{\xi - \xi^*_{k,p,\boldsymbol{\xi}}} f(\xi) + \frac{\xi - x}{\xi - \xi^*_{k,p,\boldsymbol{\xi}}} f(\xi^*_{k,p,\boldsymbol{\xi}}) =: g_1(x).$$

By (48) and (49) we have for $x \in (\xi_{m_k}, \xi_{m_k+1})$,

$$g_1(x) = \sum_{j=m_k-p}^{m_k} b_j B_{j,p,\boldsymbol{\xi}}(x), \text{ where } b_j := g_1(\xi^*_{j,p,\boldsymbol{\xi}}).$$

Finally, set $\lambda_k f := g_1(\xi^*_{k,p,\boldsymbol{\xi}}) = f(\xi^*_{k,p,\boldsymbol{\xi}})$. This is indeed in agreement with (95).

With a suitable choice of $\mathscr{Q}_k$ we can even obtain that $\mathscr{Q}$ is a projector onto the spline space $\mathbb{S}_{p,\xi}$, i.e.,

$$\mathscr{Q}s = s \text{ for all } s \in \mathbb{S}_{p,\boldsymbol{\xi}}.$$

For this it is sufficient to replace the local polynomial reproduction property in (104) by the local spline reproduction property

$$\mathscr{Q}_k s(x) = s(x) \text{ for all } s \in \mathbb{S}_{p,\boldsymbol{\xi}} \text{ and } x \in (\xi_{m_{L,k}}, \xi_{m_{U,k}}). \tag{105}$$

Indeed, with the same line of arguments as in Proposition 38 it follows that the local spline reproduction implies the global spline reproduction as stated in the following proposition; see also [11].

Proposition 40. *The spline approximation operator $\mathscr{Q}f$ determined by Recipe 37 is a projector onto the spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$ provided that we replace* (104) *by* (105).

In view of Proposition 35, a simple way to obtain a local spline projector $\mathscr{Q}_k$ is to consider a local polynomial projector as in (104) with $l = p$ and $\hat{I}_k$ restricted to be a single knot interval.

2.3 Approximation power of splines

In this section we want to understand how well a function can be approximated by a spline. In order words, we want to investigate the **distance** between a general function f and the piecewise polynomial space $\mathbb{S}_p^{\boldsymbol{r}}(\Delta)$ defined in (41). ¿From Theorem 13 we know that $\mathbb{S}_p^{\boldsymbol{r}}(\Delta) = \mathbb{S}_{p,\boldsymbol{\xi}}$ for a suitable choice of the knot sequence $\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1}$. In particular, $\boldsymbol{\xi}$ can be chosen to be $(p+1)$-open. Therefore, without loss of generality, we consider the distance between a general function f and the spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$ of degree p over the $(p+1)$-open knot sequence $\boldsymbol{\xi}$. For a given $f \in L_q([\xi_{p+1}, \xi_{n+1}])$ with $1 \leq q \leq \infty$, we define

$$\operatorname{dist}_q(f, \mathbb{S}_{p,\boldsymbol{\xi}}) := \inf_{s \in \mathbb{S}_{p,\boldsymbol{\xi}}} \|f - s\|_{L_q([\xi_{p+1},\xi_{n+1}])}. \tag{106}$$

We are also interested in estimates for the distance between derivatives of f and derivative spline spaces. To this end, in this section we use the simplified notation $D^r s := D_+^r s$ for the derivatives of a spline $s \in \mathbb{S}_{p,\boldsymbol{\xi}}$ with the usual convention of left continuity at the right endpoint of the basic interval. Note that with such a notation we ensure that $D^r s(x)$ exists for all x. In the same spirit, we use the notation $D^r \mathbb{S}_{p,\boldsymbol{\xi}} := D_+^r \mathbb{S}_{p,\boldsymbol{\xi}}$ for the r-th derivative spline space. We recall from Section 1.2.2 that this derivative space is a piecewise polynomial space of degree $p - r$ with a certain smoothness, i.e.,

$$\mathbb{S}_{p-r}^{\boldsymbol{r}-r}(\Delta) = D^r \mathbb{S}_{p,\boldsymbol{\xi}},$$

where the partition Δ consists of the distinct break points in the knot sequence $\boldsymbol{\xi}$ and the smoothness $\boldsymbol{r}$ is related to the multiplicity of the knots, according to the rule in (34). This leads to the following more general definition of distance. For a given $f \in W_q^r([\xi_{p+1}, \xi_{n+1}])$ with $1 \leq q \leq \infty$ and $0 \leq r \leq p$, we define

$$\operatorname{dist}_q(D^r f, D^r \mathbb{S}_{p,\boldsymbol{\xi}}) := \inf_{s \in \mathbb{S}_{p,\boldsymbol{\xi}}} \|D^r(f - s)\|_{L_q([\xi_{p+1},\xi_{n+1}])}. \tag{107}$$

We will derive the following upper bound for $\operatorname{dist}_q(D^r f, D^r \mathbb{S}_{p,\boldsymbol{\xi}})$.

Theorem 41. *For any $0 \leq r \leq l \leq p$ and $f \in W_q^{l+1}([\xi_{p+1}, \xi_{n+1}])$ with $1 \leq q \leq \infty$ we have*

$$\operatorname{dist}_q(D^r f, D^r \mathbb{S}_{p,\boldsymbol{\xi}}) \leq K (h_{\boldsymbol{\xi}})^{l+1-r} \|D^{l+1} f\|_{L_q([\xi_{p+1},\xi_{n+1}])},$$

where $h_{\boldsymbol{\xi}} := \max_{p+1 \leq j \leq n}(\xi_{j+1} - \xi_j)$ and K is a constant depending only on p.

This will be shown by explicitly constructing a suitable spline quasi-interpolant which achieves this order of approximation; see Theorem 48. For $l = p$ the upper bound behaves like $(h_{\boldsymbol{\xi}})^{p+1-r}$ for sufficiently smooth f.

2.3.1 A spline quasi-interpolant

Given an integer $p \geq 0$ and a $(p+1)$-open knot sequence $\boldsymbol{\xi}$, we define a specific spline approximant of degree p over $\boldsymbol{\xi}$ to a given function f. Let $[\xi_{m_{j,p}}, \xi_{m_{j,p}+1}]$ be a knot interval of largest length in $[\xi_j, \xi_{j+p+1}]$ for any $j = 1, \dots, n$ and $h_{j,p,\boldsymbol{\xi}} := \xi_{m_{j,p}+1} - \xi_{m_{j,p}} > 0$. The spline approximant to f is constructed as

$$\mathscr{Q}_{p,\boldsymbol{\xi}} f(x) := \sum_{j=1}^{n} \mathscr{L}_{j,p,\boldsymbol{\xi}} f B_{j,p,\boldsymbol{\xi}}(x), \tag{108}$$

where

$$\mathscr{L}_{j,p,\boldsymbol{\xi}} f := \frac{1}{h_{j,p,\boldsymbol{\xi}}} \int_{\xi_{m_{j,p}}}^{\xi_{m_{j,p}+1}} \left(\sum_{i=0}^{p} \alpha_{j,i} \left(\frac{x - \xi_{m_{j,p}}}{h_{j,p,\boldsymbol{\xi}}} \right)^i \right) f(x)\, \mathrm{d}x, \tag{109}$$

and the coefficients $\alpha_{j,i}$, $i = 0, \dots, p$ are such that

$$\mathscr{L}_{j,p,\boldsymbol{\xi}} \left(\frac{x - \xi_{m_{j,p}}}{h_{j,p,\boldsymbol{\xi}}} \right)^i = c_{j,i,j}, \quad i = 0, \dots, p, \tag{110}$$

where

$$\left(\frac{x - \xi_{m_{j,p}}}{h_{j,p,\boldsymbol{\xi}}} \right)^i = \sum_{k=m_{j,p}-p}^{m_{j,p}} c_{j,i,k} B_{k,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_{m_{j,p}}, \xi_{m_{j,p}+1}), \quad i = 0, \dots, p. \tag{111}$$

In the next lemmas we collect some properties for the spline approximation (108).

Lemma 42. *The above spline approximation is well defined and reproduces polynomials, i.e., for any polynomial $g \in \mathbb{P}_p$ we have*

$$\mathscr{Q}_{p,\boldsymbol{\xi}} g(x) = g(x), \quad x \in [\xi_{p+1}, \xi_{n+1}]. \tag{112}$$

Moreover, it is a projector onto the spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$, i.e., for any spline $s \in \mathbb{S}_{p,\boldsymbol{\xi}}$ we have

$$\mathscr{Q}_{p,\boldsymbol{\xi}} s(x) = s(x), \quad x \in [\xi_{p+1}, \xi_{n+1}], \tag{113}$$

and, in particular,

$$s(x) = \sum_{j=1}^{n} (\mathscr{L}_{j,p,\boldsymbol{\xi}} s) B_{j,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_{p+1}, \xi_{n+1}]. \tag{114}$$

Proof. By applying $\mathscr{L}_{j,p,\boldsymbol{\xi}}$ to the polynomials $\left(\frac{x-\xi_{m_{j,p}}}{h_{j,p,\boldsymbol{\xi}}}\right)^r$, $r=0,\ldots,p$, the coefficients $\alpha_{j,i}$ are given by the solution of the linear system

$$H_{p+1}\boldsymbol{\alpha}_j = \boldsymbol{c}_j, \tag{115}$$

where $\boldsymbol{\alpha}_j := (\alpha_{j,0},\ldots,\alpha_{j,p})^T$, $\boldsymbol{c}_j := (c_{j,0,j},\ldots,c_{j,p,j})^T$, and H_{p+1} is a $(p+1)\times(p+1)$ matrix with elements

$$(H_{p+1})_{i+1,r+1} := \frac{1}{h_{j,p,\boldsymbol{\xi}}}\int_{\xi_{m_{j,p}}}^{\xi_{m_{j,p}+1}}\left(\frac{x-\xi_{m_{j,p}}}{h_{j,p,\boldsymbol{\xi}}}\right)^{r+i}\mathrm{d}x = \frac{1}{i+r+1}, \quad i,r=0,\ldots,p.$$

This is the well-known Hilbert matrix which is nonsingular and it follows that the spline approximation (108) is well defined. By Proposition 34 we deduce that (112) holds.

Since we only integrate over one subinterval when we define $\mathscr{L}_{j,p,\boldsymbol{\xi}}$, we conclude that it reproduces not only polynomials but also splines, and (113) follows from Proposition 35. □

Lemma 43. *For $p\geq 0$ and $1\leq q\leq\infty$ we have for any $f\in L_q([\xi_{m_{j,p}},\xi_{m_{j,p}+1}])$,*

$$|\mathscr{L}_{j,p,\boldsymbol{\xi}}f| \leq C h_{j,p,\boldsymbol{\xi}}^{-1/q}\|f\|_{L_q([\xi_{m_{j,p}},\xi_{m_{j,p}+1}])}, \quad j=1,\ldots,n, \tag{116}$$

where C is a constant depending only on p.

Proof. By (20), (10) and (13) we have

$$|c_{j,i,j}| = \frac{i!}{p!}\left|\frac{D^{p-i}\psi_{j,p,\boldsymbol{\xi}}(\xi_{m_{j,p}})}{h_{j,p,\boldsymbol{\xi}}^i}\right| \leq \left(\frac{\xi_{j+p+1}-\xi_j}{h_{j,p,\boldsymbol{\xi}}}\right)^i \leq (p+1)^i, \quad i=0,\ldots,p.$$

Here we used that $[\xi_{m_{j,p}},\xi_{m_{j,p}+1}]$ is a knot interval of largest length in $[\xi_j,\xi_{j+p+1}]$. Since $0\leq\frac{x-\xi_{m_{j,p}}}{h_{j,p,\boldsymbol{\xi}}}\leq 1$ for $x\in[\xi_{m_{j,p}},\xi_{m_{j,p}+1}]$, we get from (109),

$$\begin{aligned}|\mathscr{L}_{j,p,\boldsymbol{\xi}}f| &\leq (p+1)h_{j,p,\boldsymbol{\xi}}^{-1}\|\boldsymbol{\alpha}_j\|_\infty\|f\|_{L_1([\xi_{m_{j,p}},\xi_{m_{j,p}+1}])}\\ &\leq (p+1)h_{j,p,\boldsymbol{\xi}}^{-1}\|H_{p+1}^{-1}\|_\infty\|\boldsymbol{c}_j\|_\infty\|f\|_{L_1([\xi_{m_{j,p}},\xi_{m_{j,p}+1}])}.\end{aligned}$$

This gives $|\mathscr{L}_{j,p,\boldsymbol{\xi}}f|\leq Ch_{j,p,\boldsymbol{\xi}}^{-1}\|f\|_{L_1([\xi_{m_{j,p}},\xi_{m_{j,p}+1}])}$, where $C:=\|H_{p+1}^{-1}\|_\infty(p+1)^{p+1}$ only depends on p. By the Hölder inequality (78) we arrive at (116). □

We now give a bound for the derivative of $\mathscr{Q}_{p,\boldsymbol{\xi}}f$. For this we recall from (25) that

$$\Delta_{m,k} := \min_{m-k+1\leq i\leq m} h_{i,k}, \quad h_{i,k} := \xi_{i+k}-\xi_i, \quad k=1,\ldots,p,$$

and that $\Delta_{m,k}>0$ for all k.

Lemma 44. *For $0 \le r \le p$ and $1 \le q \le \infty$ we have for any $f \in L_q([\xi_{m-p}, \xi_{m+p+1}])$ with $p+1 \le m \le n$,*

$$\|D^r(\mathscr{Q}_{p,\boldsymbol{\xi}} f)\|_{L_q([\xi_m,\xi_{m+1}])} \le C \left(\prod_{k=p-r+1}^{p} \frac{1}{\Delta_{m,k}} \right) \|f\|_{L_q([\xi_{m-p},\xi_{m+p+1}])},$$

where $\Delta_{m,k}$ is defined in (25) and C is a constant depending only on p.

Proof. ¿From the quasi-interpolant definition (108), the local support property (35) and Lemma 43, we have for $x \in [\xi_m, \xi_{m+1})$,

$$\begin{aligned} |D^r(\mathscr{Q}_{p,\boldsymbol{\xi}} f)(x)| &= \left| \sum_{j=m-p}^{m} \mathscr{L}_{j,p,\boldsymbol{\xi}}(f) D^r B_{j,p,\boldsymbol{\xi}}(x) \right| \\ &\le \max_{m-p \le j \le m} |D^r B_{j,p,\boldsymbol{\xi}}(x)| \sum_{j=m-p}^{m} |\mathscr{L}_{j,p,\boldsymbol{\xi}}(f)| \\ &\le (p+1) \max_{m-p \le j \le m} |D^r B_{j,p,\boldsymbol{\xi}}(x)| \max_{m-p \le j \le m} h_{j,p,\boldsymbol{\xi}}^{-1/q} \|f\|_{L_q([\xi_{m-p},\xi_{m+p+1}])}. \end{aligned}$$

Since $[\xi_m, \xi_{m+1}] \subset [\xi_j, \xi_{j+p+1}]$ and $h_{j,p,\boldsymbol{\xi}}$ is the length of the largest knot interval in $[\xi_j, \xi_{j+p+1}]$ we have $\xi_{m+1} - \xi_m \le h_{j,p,\boldsymbol{\xi}}$ for $j = m-p, \dots, m$. Replacing $|D^r B_{j,p,\boldsymbol{\xi}}(x)|$ by the upper bound given in Proposition 10 and taking the L_q-norm complete the proof. □

The next lemma will complete the proof of Theorem 26 related to the condition number. Note that $[\xi_{p+1}, \xi_{n+1}] = [\xi_1, \xi_{n+p+1}]$ because the knot sequence $\boldsymbol{\xi}$ is open.

Lemma 45. *For any $p \ge 0$, there exists a positive constant K_p depending only on p such that for any vector $\boldsymbol{c} := (c_1, \dots, c_n)$ and for any $1 \le q \le \infty$ we have*

$$\|\boldsymbol{c}\|_q \le K_p \left\| \sum_{j=1}^{n} c_j N_{j,p,q,\boldsymbol{\xi}} \right\|_{L_q([\xi_{p+1},\xi_{n+1}])}, \tag{117}$$

where $N_{j,p,q,\boldsymbol{\xi}} := \gamma_{j,p,\boldsymbol{\xi}}^{-1/q} B_{j,p,\boldsymbol{\xi}}$ and $\gamma_{j,p,\boldsymbol{\xi}} := (\xi_{j+p+1} - \xi_j)/(p+1)$.

Proof. Let $s := \sum_{j=1}^{n} \gamma_{j,p,\boldsymbol{\xi}}^{-1/q} c_j B_{j,p,\boldsymbol{\xi}}$. Observe that (114) and (116) imply

$$|\gamma_{j,p,\boldsymbol{\xi}}^{-1/q} c_j| = |\mathscr{L}_{j,p,\boldsymbol{\xi}} s| \le C h_{j,p,\boldsymbol{\xi}}^{-1/q} \|s\|_{L_q([\xi_{m_{j,p}}, \xi_{m_{j,p}+1}])}.$$

Since $\gamma_{j,p,\boldsymbol{\xi}} / h_{j,p,\boldsymbol{\xi}} \le 1$ we obtain

$$|c_j| \le C \|s\|_{L_q([\xi_{m_{j,p}}, \xi_{m_{j,p}+1}])} \le C \|s\|_{L_q([\xi_j, \xi_{j+p+1}])}.$$

Raising both sides to the q-th power and summing over j gives

$$\sum_{j=1}^{n} |c_j|^q \le C^q \sum_{j=1}^{n} \int_{\xi_j}^{\xi_{j+p+1}} |s(x)|^q \, \mathrm{d}x \le (p+1) C^q \|s\|_{L_q([\xi_{p+1},\xi_{n+1}])}^q.$$

When taking the q-th roots we arrive at the lower inequality in (75) with $K_p = (p+1)C$ depending only on p. $\square$

2.3.2 Distance to a function

The quasi-interpolant $\mathscr{Q}_{p,\boldsymbol{\xi}} f$ described in the previous section can be used to obtain an upper bound for the distance between a given function f and the spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$ for $p \geq 0$, $n > p+1$ and $\boldsymbol{\xi} := \{\xi_j\}_{j=1}^{n+p+1}$, see Theorem 48. We recall that the knot sequence $\boldsymbol{\xi}$ is $(p+1)$-open. We start by giving a local and global upper bound for (the derivatives of) the difference between f and $\mathscr{Q}_{p,\boldsymbol{\xi}} f$.

Proposition 46. *Suppose $\xi_m < \xi_{m+1}$ for some $p+1 \leq m \leq n$, and let $f \in W_q^{l+1}([\xi_{m-p}, \xi_{m+p+1}])$ with $0 \leq l \leq p$ and $1 \leq q \leq \infty$. If $\mathscr{Q}_{p,\boldsymbol{\xi}} f$ is defined as in* (108) *then, for any $0 \leq r \leq l$,*

$$\|D^r(f - \mathscr{Q}_{p,\boldsymbol{\xi}} f)\|_{L_q([\xi_m,\xi_{m+1}])} \leq K_m (\xi_{m+p+1} - \xi_{m-p})^{l+1-r} \|D^{l+1} f\|_{L_q([\xi_{m-p},\xi_{m+p+1}])}.$$

Here,

$$K_m := 1 + C \prod_{k=p-r+1}^{p} \frac{\xi_{m+p+1} - \xi_{m-p}}{\Delta_{m,k}},$$

$\Delta_{m,k}$ is defined in (25) *and C is a constant depending only on p.*

Proof. ¿From Lemma 42 we know that $\mathscr{Q}_{p,\boldsymbol{\xi}}$ reproduces any polynomial in $\mathbb{P}_l$, and so the triangle inequality gives

$$\begin{aligned} &\|D^r(f - \mathscr{Q}_{p,\boldsymbol{\xi}} f)\|_{L_q([\xi_m,\xi_{m+1}])} \\ &\quad \leq \|D^r(f-g)\|_{L_q([\xi_m,\xi_{m+1}])} + \|D^r \mathscr{Q}_{p,\boldsymbol{\xi}}(f-g)\|_{L_q([\xi_m,\xi_{m+1}])}, \end{aligned}$$

for any $g \in \mathbb{P}_l$. Let us now set $g := \mathscr{T}_{\xi_m,l} f$, where $\mathscr{T}_{\xi_m,l} f$ is the Taylor polynomial of degree l defined in (79) with $a = \xi_m$, $b = \xi_{m+1}$. Then, Theorem 27 implies

$$\|D^r(f-g)\|_{L_q([\xi_m,\xi_{m+1}])} \leq (\xi_{m+1} - \xi_m)^{l+1-r} \|D^{l+1} f\|_{L_q([\xi_m,\xi_{m+1}])}.$$

On the other hand, since $f - g \in C([\xi_{m-p}, \xi_{m+p+1}])$, it follows from Lemma 44 that

$$\|D^r \mathscr{Q}_{p,\boldsymbol{\xi}}(f-g)\|_{L_q([\xi_m,\xi_{m+1}])} \leq C \left(\prod_{k=p-r+1}^{p} \frac{1}{\Delta_{m,k}} \right) \|f-g\|_{L_q([\xi_{m-p},\xi_{m+p+1}])},$$

where C is a constant depending only on p. Combining the above three inequalities gives the result. $\square$

We know that the ratio $\frac{\xi_{m+p+1}-\xi_{m-p}}{\Delta_{m,k}}$ is well defined because $\Delta_{m,k} > 0$. For a uniform knot sequence

$$\frac{\xi_{m+p+1} - \xi_{m-p}}{\Delta_{m,k}} = \frac{2p+1}{k}.$$

For a general knot sequence it is related to the "local mesh ratio", i.e., the ratio between the lengths of the largest and smallest knot intervals in a neighborhood of ξ_m.

The local error bound in Proposition 46 can be turned into a global one as in the following proposition.

Proposition 47. *Let $f \in W_q^{l+1}([\xi_{p+1}, \xi_{n+1}])$ with $0 \le l \le p$ and $1 \le q \le \infty$. If $\mathscr{Q}_{p,\boldsymbol{\xi}} f$ is defined as in* (108) *then, for any $0 \le r \le l$,*

$$\|D^r(f - \mathscr{Q}_{p,\boldsymbol{\xi}} f)\|_{L_q([\xi_{p+1},\xi_{n+1}])} \le K h_{\boldsymbol{\xi}}^{l+1-r} \|D^{l+1} f\|_{L_q([\xi_{p+1},\xi_{n+1}])}, \qquad (118)$$

where $h_{\boldsymbol{\xi}} := \max_{p+1 \le j \le n}(\xi_{j+1} - \xi_j)$, and

$$K := (2p+1)^{l+2-r}\left[1 + C \max_{p+1 \le m \le n} \prod_{k=p-r+1}^{p} \frac{\xi_{m+p+1} - \xi_{m-p}}{\Delta_{m,k}}\right],$$

where $\Delta_{m,k}$ is defined in (25) *and C is a constant depending only on p.*

Proof. For $q = \infty$ the result follows immediately from Proposition 46 by taking into account that $\boldsymbol{\xi}$ can be assumed to be a $(p+1)$-open knot sequence. We now assume $1 \le q < \infty$. Since

$$\max_{p+1 \le m \le n}(\xi_{m+p+1} - \xi_{m-p}) \le (2p+1) h_{\boldsymbol{\xi}},$$

the result follows from Lemma 31 and the local error bound in Proposition 46. □

The expression K in the upper bound in Proposition 47 depends on the position of the knots for $r > 0$. However, for any knot sequence $\boldsymbol{\xi}$, it is possible to construct a coarser knot sequence $\boldsymbol{\xi}^\sharp$ such that the corresponding K only depends on p. This can be obtained by a clever thinning process. The idea of thinning out a knot sequence to get a quasi-uniform sequence is credited to [19]. Since $\boldsymbol{\xi}^\sharp$ is a subsequence of $\boldsymbol{\xi}$, we have that $\mathbb{S}_{p,\boldsymbol{\xi}^\sharp}$ is a subspace of $\mathbb{S}_{p,\boldsymbol{\xi}}$. In particular, for any $f \in L_\infty([\xi_{p+1}, \xi_{n+1}])$ the spline approximation

$$s_p := \mathscr{Q}_{p,\boldsymbol{\xi}^\sharp} f$$

as defined in (108) belongs to the spline space $\mathbb{S}_{p,\boldsymbol{\xi}}$. This spline quasi-interpolant leads to the following important result.

Theorem 48. *Let $f \in W_q^{l+1}([\xi_{p+1}, \xi_{n+1}])$ with $1 \le q \le \infty$ and $0 \le l \le p$. Then, there exists $s_p \in \mathbb{S}_{p,\boldsymbol{\xi}}$ such that*

$$\|D^r(f - s_p)\|_{L_q([\xi_{p+1},\xi_{n+1}])} \le K h_{\boldsymbol{\xi}}^{l+1-r} \|D^{l+1} f\|_{L_q([\xi_{p+1},\xi_{n+1}])}, \quad 0 \le r \le l, \qquad (119)$$

where $h_{\boldsymbol{\xi}} := \max_{p+1 \le j \le n}(\xi_{j+1} - \xi_j)$ and K is a constant depending only on p.

The constant K in Theorem 48 grows exponentially with p. However, this dependency on p can be removed in some cases, see [1, Theorem 2] for details. Theorem 48 immediately leads to the distance result in Theorem 41.

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