J. of Inequal. & Appl., 2002, Vol. 7(3), pp. 309–330

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On Unitarily Invariant Norms of Matrix-Valued Linear Positive Operators

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(Received 25 April 2000; In final form 20 September 2000)

In this paper we prove several inequalities concerning invariant norms of matrices belonging to the range of some matrix-valued Linear Positive Operator (LPO). We provide a variational characterization of unitarily invariant norms in terms of bilinear forms and a kind of Cauchy-Schwarz inequality for matrix-valued LPOs. The latter inequality holds for matrix-valued LPOs acting on L^p spaces (*e.g.*, multi-level Toeplitz, Finite Elements matrices *etc.*) but it is still unclear if it is true in general. These tools turn out to be very effective in order to deduce inequalities concerning norms of multilevel Toeplitz matrices and of some related approximations in matrix algebras.

Keywords: Cauchy-Schwarz inequality; Unitarily invariant norms; Matrix-valued LPOs

AMS Subject Classifications: 14A60, 15A45, 15A69, 47B35

1. INTRODUCTION

In this paper we prove several inequalities and estimates concerning unitarily invariant (u.i.) norms of matrices that can be expressed as the range of matrix-valued Linear Positive Operators (LPOs) [12, 13] (the reader is referred to Chapter 4 of the beautiful book [1] for a detailed discussion of u.i. norms).

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ISSN: 1025-5834. Online ISSN: 1029-242X.
DOI: 10.1080/10255830290030417

Throughout, we denote $\mathcal{M}_n$ the vector space $\mathbf{C}^{n\times n}$ of $n \times n$ complex matrices. If $\mathcal{A}$ is a complex vector space of (complex valued) functions, closed under complex conjugancy and absolute value, then we say that

$$T : \mathcal{A} \to \mathcal{M}_n$$

is a matrix-valued operator. Furthermore, if T is linear and positive (*i.e.*, the matrix $T(f)$ is Hermitian and nonnegative definite, whenever the function $f \in \mathcal{A}$ is real-valued and nonnegative), then we say that T is a *matrix-valued LPO*. These kind of operators arise quite naturally in many applications, such multilevel Toeplitz matrices [20], discretizations (by finite differences or finite elements) of elliptic PDEs [15, 16], and some weighted Laplacian matrices associated to graphs [14, 5].

A matrix norm $\|\cdot\|$ is called *unitarily invariant* (u.i. for brevity) if $\|UAV\| = \|A\|$ holds for arbitrary A, whenever U and V are unitary matrices.

It is well known that u.i. norms are intimately connected with singular values; more precisely, a function $\|\cdot\|$ defined on $n \times n$ matrices is a unitarily invariant norm if, and only if, there exists a symmetric gauge function Φ on $\mathbf{R}^n$ such that

$$\|A\| = \Phi(\sigma_1(A), \dots, \sigma_n(A)),$$

where $\sigma_j(A)$ denotes the j-th singular value of A. If Φ is a symmetric gauge function, then $\|\cdot\|_\Phi$ will denote the corresponding u.i. norm.

We recall that a function Φ on $\mathbf{R}^n$ is called a *symmetric gauge* function if it satisfies the following properties:

(1) Φ is a norm.
(2) Φ is absolute, *i.e.*, $\Phi(x_1, \dots, x_n) = \Phi(|x_1|, \dots, |x_n|)$.
(3) Φ is symmetric, *i.e.*, $\Phi(x_1, \dots, x_n) = \Phi(x_{\pi_1}, \dots, x_{\pi_n})$ for every permutation $\pi: \{1, \dots, n\} \to \{1, \dots, n\}$.

In what follows, $[x_i]_{i=1}^n$, or simply $[x_i]$ if n is clear from the context, denotes the vector with entries x_i so that we may write $\Phi([x_i])$ instead of $\Phi(x_1, \dots, x_n)$. Moreover,

$$\langle x, y\rangle := \sum_{i=1}^{n} x_i \overline{y_i} \quad \text{if } x = [x_i] \in \mathbf{C}^n \quad \text{and} \quad y = [y_i] \in \mathbf{C}^n$$

denotes the Euclidean innner product.

Every symmetric gauge function Φ has the following properties (see [1]), which will be widely used in what follows:

$$\Phi([x_i]) \leq \Phi([y_i]) \quad \text{if } |x_i| \leq |y_i| \quad \text{for all i}, \tag{1}$$

and

$$\Phi([|x_i y_i|]) \leq \Phi([|x_i|^p])^{1/p} \Phi([|y_i|^q])^{1/q} \quad \text{if } p > 1 \quad \text{and} \quad 1/p + 1/q = 1. \tag{2}$$

Finally, if $P_1, \ldots, P_k$ are mutually orthogonal projections on $\mathbf{C}^n$ such that $P_1 \oplus \cdots \oplus P_k = I$, then

$$\left\| \sum_{i=1}^{k} P_i A P_i \right\| \leq \|A\| \tag{3}$$

holds for every matrix A of order n and every u.i. norm $\|\cdot\|$. The last inequality is known as the "pinching inequality" (see [1]).

Among u.i. norms, of particular interest are the so called Schatten p-norms $\|\cdot\|_p$ (see [1]), defined as follows:

$$\|A\|_\infty := \max_i\{\sigma_i\}, \quad \text{and} \quad \|A\|_p := \left(\sum_{i=1}^{n} (\sigma_i)^p\right)^{(1/p)} \quad \text{if } p \geq 1,$$

where $\{\sigma_i\}_{i=1}^n$ denote the singular values of A.

The paper is organized as follows. In Section 2, we prove a very general characterization of u.i. norms. Section 3 is devoted to prove a Cauchy-Schwarz inequality when $\mathcal{A} = L^p(\Omega, \mu)$ for some $p \geq 1$ and some measure space (Ω, μ) and to prove several inequalities for u.i. norms (in particular, Schatten norms) of matrices arising from matrix-valued LPOs. Finally, we discuss some examples and applications concerning u.i. norms of multilevel Toeplitz and Hankel matrices, and also matrix algebra approximations of Toeplitz matrices.

2. A VARIATIONAL CHARACTERIZATION OF u.i. NORMS

Our results concerning matrices from LPOs are based on the following variational characterization of u.i. norms.

THEOREM 2.1 *Let Φ be a symmetric gauge function on $\mathbf{R}^n$. Then for every matrix $A \in \mathcal{M}_n$ we have*

$$\|A\|_\Phi = sup\ \Phi([|\langle Au_i, v_i\rangle|]_{i=1}^n), \tag{4}$$

where the supremum is taken over all pairs of orthonormal basis $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$, i.e.,

$$u_i, v_i \in \mathbf{C}^n \quad and \quad \langle u_i, u_j\rangle = \langle v_i, v_j\rangle = \delta_{ij}, \quad 1 \le i,j \le n.$$

If, moreover, A is positive semidefinite, then we have

$$\|A\|_\Phi = sup\ \Phi([\langle Au_i, u_i\rangle]_{i=1}^n), \tag{5}$$

where the supremum is taken over all orthonormal basis $\{u_i\}_{i=1}^n$.

Proof Let U, V be two unitary matrices such that $V^*AU = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$ is the singular value decomposition of A. If $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ are, respectively, the columns of U and V, then we obtain

$$\|A\|_\Phi = \Phi([\sigma_i]) = \Phi([\langle Au_i, v_i\rangle]) = \Phi([|\langle Au_i, v_i\rangle|]),$$

and hence $\le$ holds in (4). On the other hand, if U (with columns $\{u_i\}_{i=1}^n$) and V (with columns $\{v_i\}_{i=1}^n$) are any two unitary matrices and P_i is the projection onto the i-th coordinate, then the singular values of $\sum_{i=1}^n P_i V^* AUP_i$ are $\{|\langle Au_i, v_i\rangle|\}$, and from the pinching inequality (3) we obtain

$$\Phi([|\langle Au_i, v_i\rangle|]) = \left\|\sum_{i=1}^n P_i V^* AUP_i\right\|_\Phi \le \|V^*AU\|_\Phi = \|A\|_\Phi.$$

Therefore, since $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ are arbitrary orthogonal systems, we obtain that also $\ge$ holds in (4), and this completes the proof of (4).

Finally, the proof of (5) is similar (it suffices to observe that the singular values of A coincide with its eigenvalues). ■

3. SOME INEQUALITIES FOR u.i. NORMS OF LPOs MATRICES

First we consider a kind of Cauchy-Schwarz inequality regarding matrices arising from a linear positive operator T defined over $\mathcal{A}$ and taking value in $\mathcal{M}_n$ where the linear space $\mathcal{A}$ is closed under conjugancy and absolute value (for other matrix versions of the Cauchy-Schwarz inequality refer to Chapter 9.5 of [1]): $\forall u, v \in \mathbf{C}^n$, $\forall f \in \mathcal{A}$

$$|\langle T(f)u, v\rangle|^2 \leq \langle T(|f|)u, u\rangle \langle T(|f|)v, v\rangle. \tag{6}$$

Remark 3.1 If the linear space $\mathcal{A}$ is not closed under absolute value, then relation (6) is generally false. Take $\mathcal{A} = \text{span}\{e^{\hat{\imath}\alpha x} : \alpha \in \{0, \pm 1\}\}$ and define $T : \mathcal{A} \to \mathcal{M}_2$ as the linear operator such that

$$T(1) = I_2, \quad T(e^{\hat{\imath}x}) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

It is trivial to check that T is a LPO. However, for $f = e^{\hat{\imath}x}$, $u = (0, 1)^T$, $v = (1, 0)^T$, we have $|f| = 1$ and $|\langle T(f)u, v\rangle|^2 = 4$ with $\langle T(|f|)u, u\rangle = \langle T(|f|)v, v\rangle = 1$ so that (6) does not hold.

In the following proposition we consider a parametric generalization of inequality (6) and we give some alternative formulations. By the way, it is shown that inequality (6) can be rewritten in other two equivalent ways.

PROPOSITION 3.1 *Let $T : \mathcal{A} \to \mathcal{M}_n$ be a LPO where $\mathcal{A}$ is a linear space closed under conjugancy and absolute value, and let γ be a positive constant. Then the following three statements are equivalent:*

(*a*) $\forall u, v \in \mathbf{C}^n$, $\forall f \in \mathcal{A}$

$$|\langle T(f)u, v\rangle|^2 \leq \gamma^2 \langle T(|f|)u, u\rangle \langle T(|f|)v, v\rangle \tag{7}$$

holds true;

(*b*) *the (nonlinear) operator $G : \mathcal{A} \to \mathcal{M}_{2n}$ defined as*

$$G(f) \equiv \begin{pmatrix} \gamma T(|f|) & T(f) \\ T(f) & \gamma T(|f|) \end{pmatrix} \tag{8}$$

is a positive operator i.e., for any $w \in \mathbf{C}^{2n}$ and for any $f \in \mathcal{A}$, $\langle G(f)w, w\rangle \geq 0$;

(*c*) $\forall u, v \in \mathbf{C}^n$, $\forall f \in \mathcal{A}$

$$|\langle T(f)u, v\rangle| \leq \frac{\gamma}{2}\langle T(|f|)u, u\rangle + \frac{\gamma}{2}\langle T(|f|)v, v\rangle. \tag{9}$$

Proof Note that (c) follows from (a) on taking square roots and using the arithmetic-mean – geometric-mean inequality.

Concerning (b), we observe that $G(f)$ is Hermitian since $T(\bar{f}) = T(f)^*$. Decomposing $w \in \mathbf{C}^{2n}$ as (u, v), we see that $G(f) \geq 0$ is equivalent to

$$\gamma\langle T(|f|)u, u\rangle + \gamma\langle T(|f|)v, v\rangle + 2\mathrm{Re}\langle T(f)u, v\rangle \geq 0 \quad \forall u, v \in \mathbf{C}^n, \tag{10}$$

hence (b) follows from (c) using $2\mathrm{Re}\, z \geq -2|z|$.

On the other hand, given $u, v \in \mathbf{C}^n$ we can find $\omega \in \mathbf{C}$ such that $|\omega| = 1$ and $2\mathrm{Re}\omega\langle T(f)u, v\rangle = -2|\langle T(f)u, v\rangle|$. Hence, if $G(f) \geq 0$ then rewriting (10) with ωu in place of u we obtain that (b) implies (c).

Finally, if (c) holds true, then rewriting (9) with u/t in place of u and tv in place of v, where $t > 0$ is a parameter, we obtain (a) on minimizing with respect to t the resulting right-hand side. ■

We observe that relation (7) with $\gamma = 1$ is exactly relation (6). In the following we will also consider the case where $\gamma = 2$. Indeed (7) with $\gamma = 2$ (and therefore its equivalent forms) holds for any linear positive operator for purely algebraic reasons.

Conversely we will prove (6) in the case where $\mathcal{A}$ is an L^p space for a certain $p \in [1, \infty)$ and T is a LPO from $\mathcal{A}$ to $\mathcal{M}_n$. In actuality the arguments in the proof are not purely algebraic but they have some essential analytic flavour as well.

THEOREM 3.1 *Let $T : \mathcal{A} \to \mathcal{M}_n$ be a LPO with $\mathcal{A}$ being closed under absolute value and conjugancy. Then inequality* (7) *holds true for $\gamma = 2$ and for any u,v and f.*

Proof The proof is organized in four steps.

Step 1 First we assume that f is real-valued and nonnegative. Therefore $T(f) = T(|f|)$ is nonnegative definite and (6) holds true by the classical Cauchy-Schwarz inequality in $\mathbf{C}^n$.

Step 2 If f is real-valued, then we can always write $f = f^+ - f^-$ where $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$ are nonnegative. We observe that $|f| = f^+ + f^-$, we recall the first step, and we invoke the monotonicity of the operator $T(\cdot)$. Therefore we have

$$
\begin{aligned}
|\langle T(f)u,v\rangle| &\le |\langle T(f^+)u,v\rangle| + |\langle T(f^-)u,v\rangle| \\
&\le_{\text{by Step 1}} \langle T(f^+)u,u\rangle^{1/2}\langle T(f^+)v,v\rangle^{1/2} + \\
&\quad + \langle T(f^-)u,u\rangle^{1/2}\langle T(f^-)v,v\rangle^{1/2} \\
&\le 2\langle T(|f|)u,u\rangle^{1/2}\langle T(|f|)v,v\rangle^{1/2}.
\end{aligned}
$$

Step 3 The proof in Step 2 can be improved and this will be useful to prove the general case. Let us suppose again that f is real-valued. Then

$$
\begin{aligned}
|\langle T(f)u,v\rangle|^2 &\le (|\langle T(f^+)u,v\rangle| + |\langle T(f^-)u,v\rangle|)^2 \\
&= (\langle |(\langle T(f^+)u,v\rangle|, |\langle T(f^-)u,v\rangle|), (1,1)\rangle)^2 \\
&\le_{\text{CS in } \mathbf{C}^2} 2(|\langle T(f^+)u,v\rangle|^2 + |\langle T(f^-)u,v\rangle|^2) \\
&\le_{\text{by Step 1}} 2(\langle T(f^+)u,u\rangle\langle T(f^+)v,v\rangle + \\
&\quad + \langle T(f^-)u,u\rangle\langle T(f^-)v,v\rangle) \\
&\le 2\langle T(|f|)u,u\rangle\langle T(|f|)v,v\rangle
\end{aligned}
$$

which is (7) with $\gamma = \sqrt{2}$.

Step 4 Suppose now that f is complex valued so that $f = \mathrm{Re}f + \hat{i}\mathrm{Im}f$.

Therefore we have

$$
\begin{aligned}
|\langle T(f)u,v\rangle|^2 &\le (|\langle T(\mathrm{Re}f)u,v\rangle| + |\langle T(\mathrm{Im}f)u,v\rangle|)^2 \\
&\le_{\text{by Step 3}} 2(\langle T(|\mathrm{Re}f|)u,u\rangle\langle T(|\mathrm{Re}f|)v,v\rangle + \\
&\quad + \langle T(|\mathrm{Im}f|)u,u\rangle\langle T(|\mathrm{Im}f|)v,v\rangle + \\
&\quad + \langle T(|\mathrm{Re}f|)u,u\rangle^{1/2}\langle T(|\mathrm{Re}f|)v,v\rangle)^{1/2} \cdot \\
&\quad \cdot \langle T(|\mathrm{Im}f|)u,u\rangle^{1/2}\langle T(|\mathrm{Im}f|)v,v\rangle^{1/2}) \\
&\le 2\langle T(|\mathrm{Re}f| + |\mathrm{Im}f|)u,u\rangle \\
&\quad \langle T(|\mathrm{Re}f| + |\mathrm{Im}f|)v,v\rangle \\
&\le_{(a)} 4\langle T(|f|)u,u\rangle\langle T(|f|)v,v\rangle
\end{aligned}
$$

where in the last inequality labeled with (a) we use the majorization $|\mathrm{Re}f| + |\mathrm{Im}f| \le \sqrt{2}|f|$. ■

Now we turn our attention to an important particular case, that is when the space of functions $\mathcal{A}$ is an L^p space over some measure space.

THEOREM 3.2 *Let $p \in [1, \infty)$, let n be a positive integer and let (Ω, μ) be a measure space with a σ-finite positive measure μ. If $T : L^p(\Omega, \mu) \to \mathcal{M}_n$ is a LPO, then*

(*a*) *T is continuous;*
(*b*) *inequality* (6) *holds true for any $u, v \in \mathbf{C}^n$ and any $f \in L^p$.*

Proof of Part (a) We first suppose that $n = 1$, *i.e.*, that T is a positive linear functional on L^p. In order to prove that T is bounded, it suffices to prove that for every sequence $\{u_k\}$ which is convergent in L^p, there exists a subsequence $\{u_{k_j}\}$ such that $T(u_{k_j})$ is bounded (see [11]). On the other hand, it is well known that for every sequence $\{u_k\}$ convergent in L^p, there exists $u \in L^p$ and a subsequence $\{u_{k_j}\}$ such that $|u_{k_j}| \leq u$ μ-a.e. in Ω. Therefore, from the positivity of T we obtain

$$|T(u_{k_j})| \leq T(|\mathrm{Re}\ u_{k_j}|) + T(|\mathrm{Im}\ u_{k_j}|) \leq 2T(|u|)$$

hence T is continuous.

In the general case, letting $T(f) =: \{T_{ij}(f)\}$ we see that T_{ii} is a positive linear functional on L^p, hence it is continuous. Finally, the continuity of T_{ij} when $i \neq j$ follows from

$$|T_{ij}(u)|^2 \leq T_{ii}(|u|)T_{jj}(|u|),$$

which is an easy consequence of positivity.

Proof of Part (b) First observe that $\mathcal{A}$ is closed under conjugancy and absolute value so that each term involved in (6) and in its equivalent rewritings is well defined. Each entry $(T(\cdot))_{i,j}$ is a continuous linear functional from $\mathcal{A}$ to $\mathbf{C}$ (by part (a)) and therefore by the Riesz representation Theorem there exists a function $u_{i,j} \in L^q(\Omega, \mu)$ such that $1/p + 1/q = 1$ and

$$(T(f))_{i,j} = \int_\Omega u_{i,j}(x) f(x) d\mu \tag{11}$$

Since $T(\cdot)$ is globally positive it follows that

$$\langle T(|f|)s, s\rangle = \int_\Omega \sum_{i,j=1}^n u_{i,j}(x) s_i \bar{s}_j |f(x)| d\mu \geq 0$$

$\forall f \in \mathcal{A}$, $\forall s \in \mathbf{C}^n$ and consequently, by standard measure theory arguments, the matrix $U(x) = (u_{i,j}(x))_{i,j=1}^n$ is nonnegative definite almost everywhere.

Finally let u and v belonging to $\mathbf{C}^n$. Therefore

$$
\begin{aligned}
|\langle T(f)u, v\rangle|^2 &= \left|\int_\Omega \sum_{i,j=1}^n u_{i,j}(x) u_i \bar{v}_j f(x) d\mu\right|^2 \\
&\leq \left|\int_\Omega \left|\sum_{i,j=1}^n u_{i,j}(x) u_i \bar{v}_j\right| |f(x)| d\mu\right|^2 \\
&= \left|\int_\Omega |\langle U(x)u, v\rangle| |f(x)| d\mu\right|^2 \\
&\leq_{(a1)} \left|\int_\Omega \sqrt{\langle U(x)u, u\rangle}\sqrt{\langle U(x)v, v\rangle} |f(x)| d\mu\right|^2 \\
&= \left|\int_\Omega \sqrt{\langle U(x)u, u\rangle |f(x)|}\sqrt{\langle U(x)v, v\rangle |f(x)|} d\mu\right|^2 \\
&\leq_{(a2)} \int_\Omega \langle U(x)u, u\rangle |f(x)| d\mu \cdot \int_\Omega \langle U(x)v, v\rangle |f(x)| d\mu \\
&= \langle T(|f|)u, u\rangle \langle T(|f|)v, v\rangle
\end{aligned}
$$

where the inequality labeled with $(a1)$ follows (for almost every x) from the Cauchy-Schwarz inequality in $\mathbf{C}^n$ since $\langle U(x)u, v\rangle = \langle R(x)u, R(x)v\rangle$ with $U(x) = R(x)^H R(x) = R(x)^2$ a.e. and where the inequality labeled with $(a2)$ is a consequence of the classical Cauchy-Schwarz inequality in L^2 since the function $\sqrt{\langle U(x)w, w\rangle |f(x)|}$ belongs to $L^2(\Omega, \mu)$. ■

Remark 3.2 If $T : \mathcal{A} \to \mathcal{M}_n$ is a continuous LPO with $\mathcal{A} = C(K)$, where K is some compact set in the Euclidean space, then inequality (6) holds true for any $u, v \in \mathbf{C}^n$ and any f. The argument is similar to the one of the preceding theorem (relying also on the Radon–Nikodym Theorem).

Remark 3.3 We point out that we were not able to construct a vector space of functions $\mathcal{A}$, closed under absolute value and complex conjugation, where the inequality (6) does not hold true. Note that the example presented in Remark 3.1 concerns a space $\mathcal{A}$ which is not closed under absolute value.

THEOREM 3.3 *Let $p \in [1, \infty)$, let n be a positive integer and (Ω, μ) be a measure space with a σ-finite measure μ. If $T : L^p(\Omega, \mu) \to \mathcal{M}_n$ is a LPO, then*

$$|||T(f)||| \leq |||T(|f|)||| \tag{12}$$

holds for every unitarily invariant norm $|||\cdot|||$ and every $f \in L^p$.

Proof Let $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ be two systems of unitary vectors. By Theorem 3.2 we know that (6) holds true. Therefore by (9) in Proposition 3.1 with $\gamma = 1$, we have for every i

$$\begin{aligned}\left|\int_\Omega f(x)\langle U(x)u_i, v_i\rangle d\mu\right| &\leq \frac{1}{2}\int_\Omega |f(x)|\langle U(x)u_i, u_i\rangle d\mu \\ &\quad + \frac{1}{2}\int_\Omega |f(x)|\langle U(x)v_i, v_i\rangle d\mu\end{aligned}$$

where the matrix $U(x) = (u_{s,t}(x))_{s,t=1}^n$ has entries defined as in (11). Let $\|\cdot\|$ be a u.i. norm, and let Φ be the associated symmetric gauge function. From the last inequality and the convexity of Φ we obtain

$$\begin{aligned}\Phi\left(\left[\left|\int_\Omega f\langle U(x)u_i, v_i\rangle d\mu\right|\right]_{i=1}^n\right) &\leq \Phi\left(\frac{1}{2}\left[\int_\Omega |f|\langle U(x)u_i, u_i\rangle d\mu\right]_{i=1}^n \right.\\ &\quad \left. + \frac{1}{2}\left[\int_\Omega |f|\langle U(x)v_i, v_i\rangle d\mu\right]_{i=1}^n\right) \\ &\leq \frac{1}{2}\Phi\left(\left[\int_\Omega |f|\langle U(x)u_i, u_i\rangle d\mu\right]_{i=1}^n\right) \\ &\quad + \frac{1}{2}\Phi\left(\left[\int_\Omega |f|\langle U(x)v_i, v_i\rangle d\mu\right]_{i=1}^n\right) \\ &\leq \|T(|f|)\|_\Phi,\end{aligned}$$

where the last inequality follows from (4). Finally, from the arbitrariness of $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ using (4) again yields $\|T(f)\|_\Phi \leq \|T(|f|)\|_\Phi$, which completes the proof. ■

COROLLARY 3.1 *Under the assumptions of Theorem 3.3, suppose that f, $F \in L^p(\Omega, \mu)$ are such that F is real valued and $F(x) \geq |f(x)|$ at μ-a.e. $x \in \Omega$. Then*

$$|||T(f)||| \leq |||T(F)||| \tag{13}$$

holds for every unitarily invariant norm $|||\cdot|||$.

Proof From $F \geq |f|$ and the positivity of T, it follows that $T(F) \geq T(|f|)$ (in the sense of the partial ordering of Hermitian matrices). From the Fan Dominance Theorem (see [1]) we obtain $|||T(|f|)||| \leq |||T(F)|||$ for every u.i. norm, hence (13) follows from (12). ■

THEOREM 3.4 *Let (Ω, μ) be a measure space with a σ-finite measure μ, and let $1 \leq r \leq p < +\infty$. If $T : L^r(\Omega, \mu) \cap L^1(\Omega, \mu) \to \mathcal{M}_n$ is a LPO, then for every $f \in L^r \cap L^1$ the following estimate holds, concerning the Schatten p-norm of $T(f)$:*

$$\|T(f)\|_p \leq \|T(|f|^r)\|_{(p/r)}^{(1/r)}. \tag{14}$$

Proof By Theorem 3.3, we can assume that f is real and nonnegative. Let $\{u_i\}_{i=1}^n$ be an orthonormal systems of vectors of $\mathbf{C}^n$. For every i, we have from Jensen's inequality

$$\left(\int_\Omega f(x)\langle U(x)u_i, u_i\rangle d\mu\right)^p \leq \left(\int_\Omega f(x)^r\langle U(x)u_i, u_i\rangle d\mu\right)^{(p/r)},$$

where the matrix $U(x) = (u_{s,t}(x))_{s,t=1}^n$ has entries defined as in (11). Summing over i and raising to the power $1/p$ yields

$$\left(\sum_{i=1}^n\left(\int_\Omega f(x)\langle U(x)u_i, u_i\rangle d\mu\right)^p\right)^{(1/p)} \leq \left(\sum_{i=1}^n\left(\int_\Omega f(x)^r\langle U(x)u_i, u_i\rangle d\mu\right)^{(p/r)}\right)^{(1/p)}.$$

Letting $\Phi_p([x_i]) := (\sum_{i=1}^n |x_i|^p)^{(1/p)}$ and $\Phi_{p/r}([x_i]) := (\sum_{i=1}^n |x_i|^{(p/r)})^{(r/p)}$, the last inequality can be written as

$$\Phi_p\left(\left[\int_\Omega f(x)\langle U(x)u_i, u_i\rangle d\mu\right]_{i=1}^n\right) \leq \Phi_{(p/r)}\left(\left[\int_\Omega f(x)^r\langle U(x)u_i, u_i\rangle d\mu\right]_{i=1}^n\right)^{(1/r)},$$

and the claim of the theorem follows from (5), since the orthonormal system $\{u_i\}$ was arbitrary. ■

THEOREM 3.5 *Let (Ω, μ) be a measure space with a σ-finite measure μ, and let $T : L^1(\Omega, \mu) \to \mathcal{M}_n$ be a LPO. If $f \in L^p(\Omega, \mu)$ and $g \in L^q(\Omega, \mu)$, where $1 \le p$, $q < \infty$ and $1/p + 1/q = 1$, then we have*

$$\|T(fg)\| \le \|T(|f|^p)\|^{(1/p)} \|T(|g|^q)\|^{(1/q)} \tag{15}$$

for every unitarily invariant norm $\|\cdot\|$. Moreover, if $f \in L^1(\Omega, \mu)$ and $g \in L^\infty(\Omega, \mu)$ then

$$\|T(fg)\| \le \|g\|_{L^\infty} \|T(|f|)\| \tag{16}$$

holds for every unitarily invariant norm $\|\cdot\|$.

Proof By Theorem 3.3, we can assume that f and g are real and non-negative. Let $\{u_i\}_{i=1}^n$ be an orthonormal system of vectors and let $U(x) = (u_{i,j}(x))_{i,j=1}^n$ be the matrix defined in (11). For every i, writing

$$\langle U(x)u_i, u_i\rangle = \langle U(x)u_i, u_i\rangle^{(1/p)} \langle U(x), u_i, u_i\rangle^{(1/q)}$$

we have from the Hölder inequality

$$\int_\Omega f(x)g(x)\langle U(x)u_i, u_i\rangle d\mu \le \left(\int_\Omega f(x)^p \langle U(x)u_i, u_i\rangle d\mu\right)^{(1/p)} \left(\int_\Omega g(x)^q \langle U(x)u_i, u_i\rangle d\mu\right)^{(1/q)}.$$

Let Φ be the symmetric gauge function associated with the u.i. norm $\|\cdot\|$. Using (1) and (2) we obtain from the previous inequality

$$\begin{aligned}
&\Phi\left(\left[\int_\Omega (fg)(x)\langle U(x)u_i, u_i\rangle d\mu\right]_{i=1}^n\right) \\
&\le \Phi\left(\left[\left(\int_\Omega f^p(x)\langle U(x)u_i, u_i\rangle d\mu\right)^{(1/p)} \left(\int_\Omega g^q(x)\langle U(x)u_i, u_i\rangle d\mu\right)^{(1/q)}\right]_{i=1}^n\right) \\
&\le \Phi\left(\left[\int_\Omega f^p(x)\langle U(x)u_i, u_i\rangle d\mu\right]_{i=1}^n\right)^{(1/p)} \Phi\left(\left[\int_\Omega g^q(x)\langle U(x)u_i, u_i\rangle d\mu\right]_{i=1}^n\right)^{(1/q)},
\end{aligned}$$

and (15) follows from (5), since the orthonormal system $\{u_i\}$ was arbitrary. Finally, (16) follows from Corollary 3.1, letting $\lambda := \|g\|_{L^\infty}$ and observing that

$$|f(x)g(x)| \leq \lambda|f(x)| \quad \text{and} \quad \|T(\lambda|f|)\| = \lambda\|T(|f|)\|.$$

■

Remark 3.4 An alternative proof of (15) is the following. By well known inequalities, we have for every $t > 0$

$$|f(x)g(x)| \leq \frac{t^p}{p}|f(x)|^p + \frac{t^{-q}}{q}|g(x)|^q,$$

and from Corollary 3.1 and the linearity and positivity of T we obtain

$$\|T(|fg|)\| \leq \frac{t^p}{p}\|T(|f|^p)\| + \frac{t^{-q}}{q}\|T(|g|^q)\| \quad \text{for all } t > 0.$$

Then (15) follows on choosing the value of t which minimizes the right-hand side.

Remark 3.5 All the inequalities proved in this section are sharp, as one can easily check letting $f(x) = 1$ and $g(x) = 1$.

4. EXAMPLES AND APPLICATIONS

In what follows we give an idea of the several contexts in which matrix-valued linear positive operators arise.

4.1. Multilevel Toeplitz Matrices

We first turn our attention to u.i. norms of (multilevel) Toeplitz matrices.

Let f be a complex-valued function of k variables, integrable on the k-cube $I_k := (0, 2\pi)^k$. Throughout, the symbol $⨍_{I_k}$ stands for $(2\pi)^{-k}\int_{I_k}$. The Fourier coefficients of f, given by

$$\hat{f}_j := ⨍_{I_k} f(x)e^{-\hat{\imath}\langle j,x\rangle}dx, \quad \hat{\imath}^2 = -1, \; j \in \mathbf{Z}^k, \tag{17}$$

are the entries of the k-level Toeplitz matrices generated by f. More precisely, if $n=(n_1,\ldots,n_k)$ is a k-index with positive entries, then $T_n(f)$ denotes the matrix of order $\hat{n}$ (throughout, we let $\hat{n} := \Pi_{i=1}^{k} n_i$) given by

$$T_n(f) = \sum_{|j_1| < n_1} \cdots \sum_{|j_k| < n_k} \hat{f}_{(j_1,\ldots,j_k)} J_{n_1}^{(j_1)} \otimes \cdots \otimes J_{n_k}^{(j_k)}. \tag{18}$$

In the above equation, $\otimes$ denotes tensor product, while $J_m^{(l)}$ denotes the matrix of order m whose (i,j) entry equals 1 if $j-i=l$ and equals zero otherwise. In other words, the $2m-1$ matrices $\{J_m^{(l)}\}$, $l=0, \pm 1,\ldots,\pm(m-1)$ are the canonical basis of the linear space of $m \times m$ Toeplitz matrices and the tensor notation emphasizes the k-level Toeplitz structure of $T_n(f)$. The reader is referred to [20, 2, 21, 19] for more details on multilevel Toeplitz matrices. Here we just recall the following elementary fact (see *e.g.* [9, 21]).

PROPOSITION 4.1 *Let $n=(n_1,\ldots,n_k)$ be a positive k-index with $k \geq 1$ and a let $T_n : L^1(I_k) \rightarrow \mathcal{M}_{\hat{n}}$ be defined as in* (18). *Then T_n is a matrix-valued LPO.*

In order to give one specific characterization of the results in Section 3 in the case of multilevel Toeplitz matrices, we identify $\hat{n}$ vectors and polynomials of degree n *via* a suitable isomorphism. Indeed, given $k \geq 1$ and a k-index $n=(n_1,\ldots,n_k)$ as above, we define

$$\mathcal{P}_n^k := \left\{ u : I_k \mapsto \mathbf{C} \mid u(x) = \sum_{j \in \mathcal{I}_n^k} a_j e^{\hat{\imath}\langle j,x\rangle} \right\},$$

where

$$\mathcal{I}_n^k := \{(j_1,\ldots,j_k) | j_i \in \mathbf{N} \text{ and } 0 \leq j_i < n_i \text{ for } i = 1,2,\ldots,k\}.$$

The set $\mathcal{P}_k^n$ is a vector space of trigonometric polynomials of dimension $\hat{n}$, and we endow it with the L^2 inner product

$$\langle u, v\rangle_{L^2} := \frac{1}{\!\!\!\!\!\!} \int_{I_k} u(x)\overline{v(x)}dx.$$

Given $u \in \mathcal{P}_k^n$, $u(x) := \sum_{j \in \mathcal{I}_n^k} a_j e^{\hat{\imath}\langle j,x\rangle}$, we can associate with it the vector $\tilde{u} \in \mathbf{C}^{\hat{n}}$ given by

$$\tilde{u} := \sum_{i_1=0}^{n_1-1} \cdots \sum_{i_k=1}^{n_k-1} a_{(i_1,\ldots,i_k)} e_{n_1}^{(i_1)} \otimes \cdots \otimes e_{n_k}^{(i_k)},$$

where $\{e_m^{(i)}\}_{i=0}^{m-1}$ is the canonical basis of $\mathbf{C}^m$. It is clear that the map $u \to \tilde{u}$ is a linear isomorphism. In fact, it is easy to check that

$$\langle u, v\rangle_{L^2} = \langle \tilde{u}, \tilde{v}\rangle, \quad \text{for all } u, v \in \mathcal{P}_k^n, \tag{19}$$

and we obtain a linear isometry between $\mathcal{P}_k^n$ and $\mathbf{C}^{\hat{n}}$. Therefore, we can drop the notation $\tilde{u}$, and we can regard u as a polynomial from $\mathcal{P}_k^n$ or as a vector from $\mathbf{C}^{\hat{n}}$, according to the necessity.

Observing that $\langle J_m^{(j)} e_m^{(h)}, e_m^{(i)}\rangle = \delta_{j,h-i}$ and using elementary properties of the tensor product, from (18) one obtains after straightforward computations

$$⨍_{I_k} f(x)u(x)\overline{v(x)}dx = \langle T_n(f)u, v\rangle \quad \text{for all } u, v \in \mathcal{P}_k^n \tag{20}$$

(in the right-hand side, u and v are meant as vectors from $\mathbf{C}^{\hat{n}}$). We remark that (19) can be obtained from (20) letting $f(x) = 1$, in such a way that $T_n(f) = I$.

Remark 4.1 From (20) one can see that, if $f(x)$ is real-valued and nonnegative, then $T_n(f)$ is positive semidefinite (also the converse is true, provided $T_n(f) \geq 0$ for every n). In this case, it is immediate to check that

$$\|T_n(f)\|_1 = \mathrm{tr}(T_n(f)) = \hat{n} ⨍_{I_k} f(x)dx. \tag{21}$$

The representation formula (20) is extremely useful. From it and from Theorem 2.1, we can deduce a variational characterization for any u.i. norm of any Toeplitz matrix.

COROLLARY 4.1 *Let $n = (n_1, \ldots, n_k)$ be a positive k-index, and let $f \in L^1(I_k)$. If $T_n(f)$ is the k-level Toeplitz matrix associated with f, then* *we have for every symmetric gauge function* Φ *on* $\boldsymbol{R}^{\hat{n}}$

$$\|T_n(f)\|_\Phi = sup\Phi\left(\left[\left|\fint_{I_k} f(x)u_i(x)\overline{v_i(x)}dx\right|\right]_{i=1}^{\hat{n}}\right), \tag{22}$$

where the supremum is taken over all pairs of orthonormal systems $\{u_i\}_{i=1}^{\hat{n}}$ *and* $\{v_i\}_{i=1}^{\hat{n}}$ *of trigonometric polynomials such that*

$$u_i, v_i \in \mathcal{P}_k^n, \quad \fint_{I_k} u_i(x)\overline{u_j(x)}dx = \fint_{I_k} v_i(x)\overline{v_j(x)}dx = \delta_{ij}, \quad 1 \le i, j \le \hat{n}. \tag{23}$$

Moreover, if f is real and nonnegative then we have

$$\|T_n(f)\|_\Phi = sup\Phi\left(\left[\fint_{I_k} f(x)\ |u_i(x)|^2 dx\right]_{i=1}^{\hat{n}}\right), \tag{24}$$

where the supremum is taken over all orthonormal systems $\{u_i\}_{i=1}^{\hat{n}}$ *of trigonometric polynomials satisfying* (23).

Proof Using (19) and (20), we obtain (22) applying Theorem 2.1 with $A = T_n(f)$. Similarly, in order to obtain (24) it suffices to observe that $T_n(f)$ is positive semidefinite if f is real and nonnegative (by Proposition 4.1). ■

In addition, a further consequence of the analysis in Section 3 is the following estimate, which improves (and generalizes to any Schatten p-norm) the estimate proven in Lemma 4 from [8].

COROLLARY 4.2 *Let $f \in L^p(I_k)$ for some p with $1 \le p \le \infty$. If $n = (n_1, \dots, n_k)$ is a positive k-index, then the following estimate holds concerning the Schatten norms*:

$$\|T_n(f)\|_p \le (\hat{n})^{(1/p)} \|f\|_{L^p}. \tag{25}$$

Proof The case where $p = +\infty$ is a well known fact concerning the spectral norm of Toeplitz matrices. If $p < +\infty$, then choosing $r = p$ we obtain from Theorem 3.4 with $T(\cdot) = T_n(\cdot)$

$$\|T_n(f)\|_p \le \|T_n(|f|^p)\|_1^{(1/p)} = (\mathrm{tr} T_n(|f|^p))^{(1/p)} = (\hat{n})^{(1/p)} \|f\|_{L^p},$$

where the last equalities follow from (21) with $|f|^p$ in place of f. ■

4.2. Hankel Matrices

A Hankel matrix is one whose entries are constant along any lower-left – upper-right diagonal. As with Toeplitz matrices, one can consider multilevel Hankel matrices, generated by a multivariate symbol f, integrable on the k-cube $I_k := (0, 2\pi)^k$. With the same notations of the previous subsection, let $\{\hat{f}_j\}$ denote the Fourier coefficients of f, according to (17). If $n = (n_1, \ldots, n_k)$ is a k-index with positive entries, then $H_n(f)$ denotes k-level Hankel matrix of order $\hat{n}$ generated by f, defined as

$$H_n(f) = \sum_{j_1=1}^{2n_1-1} \cdots \sum_{j_k=1}^{2n_k-1} \hat{f}_{(j_1,\ldots,j_k)} K_{n_1}^{(j_1)} \otimes \cdots \otimes K_{n_k}^{(j_k)}. \tag{26}$$

Here $K_m^{(l)}$ denotes the matrix of order m whose (i,j) entry equals 1 if $i+j = l+1$ and equals zero otherwise; the matrices $K_m^{(l)}$, $l = 1, \ldots, 2m-1$ are the natural basis of the linear space of Hankel matrices of order m. As with Toeplitz matrices, the tensor product $\otimes$ stresses the k-level block structure of the matrices we are considering.

It is well known that the operator H_n is not a LPO: to see this, take $k = 1, f(x) = 2(1 - \cos x)$ and $n = (n_1) = (1)$, and note that $H_1(f) = \hat{f}_1 = -1$ is negative despite that $f \geq 0$. However, in [8] it was proved that, given $f \in L^1(I_k)$, for every multiindex n there exists a function $g_n \in L^1(I_k)$ and a unitary matrix U_n such that

$$U_n H_n(f) = T_n(g_n) \quad \text{and} \quad |g_n(x)| = |f(x)| \quad \text{for every } x \in I_k. \tag{27}$$

In view of this fact, we obtain the following

COROLLARY 4.3 *Let $f \in L^p(I_k)$ for some p with $1 \leq p \leq \infty$. If $n = (n_1, \ldots, n_k)$ is a positive k-index, then the following estimate holds concerning the Schatten norms*:

$$\|H_n(f)\|_p \leq (\hat{n})^{(1/p)} \|f\|_{L^p}. \tag{28}$$

Proof Using (27), we see that $\|H_n(f)\| = \|T_n(g_n)\|$ for every u.i. norm, since U_n is unitary. Moreover, we have $\|f\|_{L^p} = \|g_n\|_{L^p}$, hence the estimate (28) is a consequence of the corresponding estimate for Toeplitz matrices, stated in Corollary 4.2. ∎

We stress that this corollary improves and generalizes the estimate obtained in Lemma 5 of [8].

Remark 4.2 From relations (25) and (28) we know that

$$\|T_n(f)\|_p, \|H_n(f)\|_p \leq (\hat{n})^{(1/p)} \|f\|_{L^p}$$

for any $f \in L^p(I_k)$. Nevertheless the Toeplitz and the Hankel case are substantially different. First of all inequality (25) is asymptotically tight since (see [18])

$$\lim_{n \to \infty} \|T_n(f)\|_p (\hat{n})^{-(1/p)} = \|f\|_{L^p}$$

and therefore for any nonzero f, $\|T_n(f)\|_p \leq (\hat{n})^{(1/p)} C$ cannot hold for any $n = (n_1, \ldots, n_k)$ if $C < \|f\|_{L^p}$. Conversely, for fixed n and $f \in L^p(I_k)$, setting $\tau(f,p,k,n) = \{g \in L^p(I_k) : \hat{g}_j = \hat{f}_j, e - n \leq j \leq n - e, e_i = 1\}$, we have $T_n(g) = T_n(f)$ for $g \in \tau(f,p,k,n)$ and then

$$\|T_n(f)\|_p \leq (\hat{n})^{(1/p)} \inf_{g \in \tau(f,p,k,n)} \|g\|_{L^p}$$

with a negligible improvement with respect to (25): for instance, for $\tilde{f}(x) = 2(1 - \cos(x))$ and $p = \infty$ we find

$$\|\tilde{f}\|_{L^\infty} - \inf_{g \in \tau(\tilde{f},p,k,n)} \|g\|_{L^\infty} = O(n^{-2}).$$

In the Hankel case the situation is quite different. For instance $\|T_n(\tilde{f})\|_p = 1$ which is much less than $(\hat{n})^{(1/p)} \|\tilde{f}\|_{L^p}$. However estimate (28) can be used in the opposite direction for obtaining information from the "discrete" to the "continuous". Setting $\sigma(f,p,k,n) = \{g \in L^p(I_k) : \hat{g}_j = \hat{f}_j, e \leq j \leq 2n - e, e_i = 1\}$, we have $H_n(g) = H_n(f)$ for $g \in \sigma(f,p,k,n)$. Then for the previous example $\tilde{f}(x)$ we find

$$\inf_{g \in \sigma(\tilde{f},p,k,n)} \|g\|_{L^\infty} = 1.$$

Along the same lines, consider α, β real numbers and the function $\tilde{f}(x) = \alpha e^{\hat{i}x} + \beta e^{2\hat{i}x}$. Then for any $g(x)$ of the form

$$g(x) = \tilde{f}(x) + \sum_{q=0}^{\infty} t_q e^{-q\hat{i}x} \in \bigcap_{n \geq 2} \sigma(\tilde{f},p,k,n)$$

we deduce the nontrivial estimate

$$\|g\|_{L^\infty} \geq \frac{|\alpha| + \sqrt{\alpha^2 + 4\beta^2}}{2} = \|H_2(\tilde{f})\|_\infty.$$

Moreover the minimum is realized by $\tilde{f}$ if and only if $\alpha = 0$ or $\beta = 0$.

In a general multilevel setting, given a positive k-index t, for $\tilde{f}(x) = \sum_{q=e}^{t} \alpha_q e^{-\hat{i}\langle j,x\rangle}$ and for $g(x) \in \bigcap_{n \geq t} \sigma(\tilde{f}, p, k, n)$ we have

$$\|g\|_{L^\infty} \geq \|H_t(\tilde{f})\|_\infty \geq \sqrt{\sum_{q=e}^{t} |\alpha_q|^2}$$

showing that the obtained estimate is better than the one deduced from the comparison with the L^2 norm (the estimates coincide, that is $\|H_t(\tilde{f})\|_\infty = \sqrt{\sum_{q=e}^{t} |\alpha_q|^2}$, if and only if all but one of the coefficients α_q are zero).

4.3. Multilevel Toeplitz Matrices and Optimal Matrix Algebras Approximation

Let U_n and V_n be two unitary complex $n \times n$ matrices and let us denote by $\mathcal{B}_n$ the linear space (commutative matrix algebra if $V_n = U_n$) of all the matrices defined as

$$\mathcal{B}_n = \{X = U_n \Lambda V_n^* : \Lambda \text{ is a diagonal complex matrix}\} \qquad (29)$$

where the symbol $*$ means transpose and conjugate.

Now suppose that a matrix A is given and let us consider the optimal approximation $\mathcal{P}_n(A)$ with respect to the Frobenius norm (the Schatten norm with $p = 2$) in the space $\mathcal{B}_n$. Then

$$\mathcal{P}_n(A) = \arg\min_{X \in \mathcal{B}_n} \|A - X\|_2$$

where the minimum exists and is unique since $\mathcal{B}_n$ is convex and $\|\cdot\|_2$ is a strictly convex norm.

It is well known that $\mathcal{P}_n(A)$ has the following explicit representation [6]

$$\mathcal{P}_n(A) = U_n \mathrm{diag}(U_n^* A V_n) V_n^*, \qquad (30)$$

where now diag(Y) denotes the diagonal matrix with the same entries as Y along the main diagonal. Formula (30) can be easily put in connection with the pinching inequality (3).

THEOREM 4.1 *Let* $\|\cdot\|$ *be a u.i. norm and let* $\mathcal{B}_n$, $\mathcal{P}_n(\cdot):\mathcal{M}_n \to \mathcal{B}_n$, *be as in* (29), (30) *respectively. Then for every matrix* $A \in \mathcal{M}_n$ *we have*

$$\|\mathcal{P}_n(A)\| \leq \|A\| \tag{31}$$

with equality holding if $A \in \mathcal{B}_n$.

Proof Since U_n and V_n are unitary, by relation (30) and by the pinching inequality (3), we have

$$\|\mathcal{P}_n(A)\| = \|U_n^* \mathcal{P}_n(A) V_n\| = \left\| \sum_{i=1}^{n} P_i U_n^* A V_n P_i \right\| \leq \|U_n^* A V_n\| = \|A\|.$$

If $A \in \mathcal{B}_n$, then $\mathcal{P}_n(A) = A$ and equality trivially holds. ■

Remark 4.3 Theorem 4.1 extends a result in [7, 4] where the relation (31) was proved in the special case of Schatten p-norms with $p = 2$ and $p = \infty$.

We recall that this kind of results is useful in a practical context of approximation of Toeplitz sequences by simpler and more appealing matrix structures (circulants, trigonometric algebras *etc.*) in order to solve Toeplitz linear systems in a computationally efficient way (see also [3, 17]).

Finally, in light of Proposition 4.1 and formula (30), we know that for any $n \in \mathbf{N}^k$, the operators T_n and $\mathcal{P}_n \circ T_n$ (with $V_n = U_n$) satisfy the assumption of Theorem 3.2 with $p = 1$ so that Theorems 3.3, 3.4 and 3.5 and Corollary 3.1 automatically hold.

The following final remark gives a insight on the wide range of applications of the results of this paper.

Remark 4.4 The list of the matrix-valued LPOs that arise in the applications is very long. We want just to mention the discretizations by Finite Elements or Finite Differences of some boundary value problems, the case of Laplacian graph matrices coming from optimization problems *etc.* In all these cases the vector $n=(n_1,\ldots,n_k)$ is related to a finesse parameter of the discretization process so that we have to deal with sequences of matrices of increasing order. The inequalities proved in the preceding sections can be used in order to analyze the behaviour of these structures for large n under some perturbations as it occurs and is of interest in a Numerical Analysis context.

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