On the persistence of periodic Lagrangian tori for symplectic twist maps

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(joint work with M.-C. Arnaud and J.E. Massetti)

GLADS22 Conference:

"Global and local aspects in dynamical systems: from exponentially small phenomena to Instability"

in honour of Tere's
(still) 60^{th} birthday

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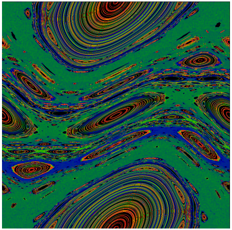
Introduction: Rigidity versus Fragility

In the study of Hamiltonian systems an important role is played by integrable dynamics.

Integrability appears to be a very fragile property since it is not expected to persist under generic, yet small, perturbations.

Understanding which of features of these systems break or are preserved in the passage from the integrable regime to non-integrable one is a very natural question, related to several interesting problems in dynamics.

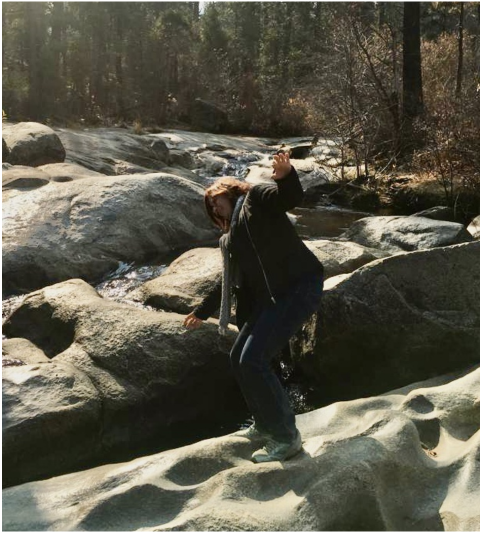


Phase portrait of the standard map
(Credits: Wikipedia)

In this talk we consider one-parameter families of a twist maps and investigate:

- The persistence and the properties of Lagrangian tori foliated by periodic points (very fragile objects).
- The rigidity of integrable twist maps: to which extent it is possible to deform in a non-trivial way an integrable twist map, preserving some of its features.

The struggle for persistence of an invariant Lagrangian torus



Starring Tere
Sequoia & Kings Canyon National Park (US), 2018.

Symplectic twist maps of the $2d$-dimensional annulus

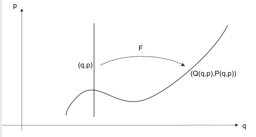
Definition (Symplectic twist maps)

A symplectic twist map of the $2d$-dimensional annulus, $d \geqslant 1$, is a C^1 diffeomorphism $f : \mathbb{T}^d \times \mathbb{R}^d \circlearrowleft$ that admits a lift

$$\begin{aligned} F : \mathbb{R}^d \times \mathbb{R}^d &\longrightarrow \mathbb{R}^d \times \mathbb{R}^d \\ (q, p) &\longmapsto (Q(q,p), P(q,p)) \end{aligned}$$

satisfying:

- $F(q + m, p) = F(q, p) + (m, 0) \quad \forall m \in \mathbb{Z}^d$;
- (Twist condition) the map $(q, p) \mapsto (q, Q(q, p))$ is a diffeomorphism of $\mathbb{R}^{2d}$;



- (Exact symplecticity) There exists a generating function, namely $S : \mathbb{R}^{2d} \to \mathbb{R}$ such that
  - $PdQ - pdq = dS(q, Q) \qquad (\Longleftrightarrow$ it preserves $\omega = dq \wedge dp)$.
  - $S(q + m, Q + m) = S(q, Q) \quad \forall m \in \mathbb{Z}^d$.

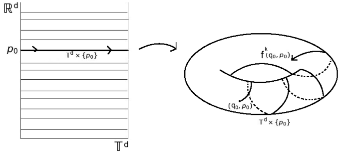
Example: Completely integrable twist maps

Let us consider a completely integrable symplectic twist map

$$\boxed{f(q,p) := (q + \nabla\ell(p), p) \qquad q \in \mathbb{T}^d,\ p \in \mathbb{R}^d}$$

where $\ell : \mathbb{R}^d \to \mathbb{R}$ is C^2 and $\nabla\ell : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1 diffeomorphism.

- The phase space is completely foliated by invariant tori $\mathcal{T}_{p_0} := \mathbb{T}^d \times \{p_0\}$, with $p_0 \in \mathbb{R}^d$, on which the dynamics is a translation by $\nabla\ell(p_0)$.



- Let $r(p_0) := \operatorname{rank}\{\nu \in \mathbb{Z}^d : \langle \nu, \nabla\ell(p_0)\rangle = 0\} \in \{0, \ldots, d\}$; the closure of each orbit on $\mathcal{T}_{p_0}$ is a $(d - r(p_0))$-dimensional torus. In particular:
  - If $r(p_0) = 0$ (non-resonant), then every orbit is dense on $\mathcal{T}_{p_0}$;
  - if $r(p_0) = d - 1$ (maximally resonant), every orbit on $\mathcal{T}_{p_0}$ is periodic.
- The generating function is $S(q,Q) := h(Q - q)$, where h is such that $(\nabla\ell)^{-1} = \nabla h$ (Fenchel-Legendre transform).

Periodic and completely periodic tori

Let $F : \mathbb{R}^d \times \mathbb{R}^d \circlearrowleft$ be a lift of a symplectic twist map $f : \mathbb{T}^d \times \mathbb{R}^d \circlearrowleft$.

Definition (Periodic and completely periodic graphs)

Let $\mathcal{L} := \mathrm{graph}(\gamma)$, where $\gamma : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a $\mathbb{Z}^d$-periodic and continuous, and let $(m, n) \in \mathbb{Z}^d \times \mathbb{N}^*$ with m and n coprime.

- $\mathcal{L}$ is a (m, n)-periodic graph of F, if

$$F^n(q, \gamma(q)) = (q + m, \gamma(q)) \qquad \forall\, q \in \mathbb{R}^d.$$

- $\mathcal{L}$ is a (m, n)-completely periodic graph of F, if it is (m, n)-periodic and invariant by F.

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- $\mathcal{L}$ is as regular as F is (twist condition + Implicit function theorem).

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- $\mathcal{L}$ is as regular as F is (twist condition + Implicit function theorem).
- For positive symplectic twist maps:
  $\mathcal{L}$ Lagrangian + (m, n)-periodic $\Longrightarrow$ invariant (hence, completely periodic)
  (using action-minimizing properties).

Some non-degeneracy (stronger twist) conditions

Let $F : \mathbb{R}^d \times \mathbb{R}^d \circlearrowleft$ be a lift of a symplectic twist map $f : \mathbb{T}^d \times \mathbb{R}^d \circlearrowleft$.

- f is said to be positive if F admits a C^2 generating function $S : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $\alpha > 0$ such that

$$\boxed{\partial^2_{q,Q} S(q,Q)(v,v) \leqslant -\alpha \|v\|^2 \quad \forall\, q, Q, v \in \mathbb{R}^d.}$$

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For a completely integrable map $f(q,p) := (q + \nabla \ell(p), p)$, recall that $S(q,Q) := h(Q-q)$, where h is such that $(\nabla \ell)^{-1} = \nabla h$. Hence:

- f is positive $\iff \exists\, \alpha > 0$ such that

$$D^2 \ell(p)(v,v) \geqslant \alpha \|v\|^2 \qquad \forall\, p, v \in \mathbb{R}^d.$$

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Family of maps obtained by perturbing by a potential

Let

- $f : \mathbb{T}^d \times \mathbb{R}^d \circlearrowleft$ be symplectic twist map,
- $F : \mathbb{R}^d \times \mathbb{R}^d \circlearrowleft$ be a lift of f,
- $S : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ its generating function.

Given a potential $G \in C^2(\mathbb{T}^d)$, one can obtain a 1-parameter family of symplectic twist maps in the following way:

For every $\varepsilon \in \mathbb{R}$, we consider the maps $f_\varepsilon : \mathbb{T}^d \times \mathbb{R}^d \circlearrowleft$ with generating functions

$$\boxed{S_\varepsilon(q, Q) := S(q, Q) + \varepsilon G(q).}$$

We denote by $\{F_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ a continuous family of lifts of the family $\{f_\varepsilon\}_{\varepsilon\in\mathbb{R}}$.

- This kind of perturbations are also called perturbations in the sense of Mañé.
- These perturbations do not alter the property of being (strongly) positive.

Persistence of Lagrangian periodic tori

Theorem 1 [Arnaud, Massetti, S. (2022)]

Let $f : \mathbb{T}^d \times \mathbb{R}^d \circlearrowleft$ be symplectic twist map and let $F : \mathbb{R}^d \times \mathbb{R}^d \circlearrowleft$ be a lift of f and $S : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ its generating function.

Let $G \in C^2(\mathbb{T}^d)$ and $\{f_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ be 1-parameter family of maps with generating functions $S_\varepsilon(q, Q) := S(q, Q) + \varepsilon G(q)$. Denote by $\{F_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ a continuous family of lifts of $\{f_\varepsilon\}_{\varepsilon\in\mathbb{R}}$.

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Then, for every $(m, n) \in \mathbb{Z}^d \times \mathbb{N}^*$, with m and n coprime, the set

$$\Sigma_{m,n} := \{\varepsilon \in \mathbb{R} : \ F_\varepsilon \text{ has a Lagrangian } (m, n)\text{-periodic graph}\} = \begin{cases} \mathbb{R} \\ \text{isolated points.} \end{cases}$$

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If, in addition, $\|\partial^2_{qq} S\|_\infty$ and $\|\partial^2_{Q,Q} S\|_\infty$ are bounded + G is non-constant, then:

$\Sigma_{m,n}$ consists of at most finitely many points.

Some comments to Theorem 1

- The proof can be adapted to the case in which F (resp., G) admits just a holomorphic extension to a strip Σ_σ^{2d}, where $\Sigma_\sigma^{2d} := \{z \in \mathbb{C}^{2d} : \|\mathrm{Im}\, z\| < \sigma\}$ for some $\sigma > 0$.

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- One can deduce the following rigidity result:

  (Hypotheses of Theorem 1, including the boundedness of $\|\partial^2_{q,q} S\|_\infty$ and $\|\partial^2_{Q,Q} S\|_\infty$)

  If for some $(m, n) \in \mathbb{Z}^d \times \mathbb{N}^*$, with m and n coprime, the maps $\{f_\varepsilon\}_{\varepsilon \in \mathbb{R}}$ admit a Lagrangian (m, n)-periodic torus for infinitely many values of $\varepsilon \in \mathbb{R}$, then, G must be constant.

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- Compare with related results for 2-dimensional Birkhoff billiards concerning the existence of rational periodic caustics (Kaloshin, Koudjinan, Ke Zhang).

  Although Theorem 1 cannot be applied to billiards (due to the nature of the perturbation), the same techniques could work and prove (in progress):

  Given an one-parameter analytic family of Birkhoff billiards, either rational periodic caustics for a fixed rotation number exist for all members of the family, or it does appear for at most an isolated set of values of the parameter.

Rigidity of completely integrable maps

Theorem 2 [Arnaud, Massetti, S. (2022)]

Let $f : \mathbb{T}^d \times \mathbb{R}^d \circlearrowleft$ be completely integrable symplectic twist map of the form

$$f(q, p) = (q + \nabla \ell(p), p),$$

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Then, G must be identically constant.

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- For a given $\varepsilon \in \mathbb{R}$, the assumption on F_ε in (*) is satisfied if there exists an open set $\mathcal{A} \subset \mathbb{R}^d$ such that F_ε has a Lagrangian (m, n)- periodic graph for any $\frac{m}{n} \in \mathcal{A} \cap \mathbb{Q}^d$ (weakly rational integrability).

  In fact, one can choose $q_1, \ldots, q_d \in \mathbb{Q}^d$ linearly independent, such that the half-lines $\sigma_{q_i}(t) = tq_i$, for $t \geqslant 0$, intersect $\mathcal{A}$; for every $j = 1, \ldots, d$, choose $0 < a_j < b_j$ such that $\sigma_{q_j}((a_j, b_j)) \subset \mathcal{A}$, and let $I_j := (a_j, b_j)$.

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- We do not ask analiticity of G. We shall see that (*) implies that G must be a trigonometric polynomial (hence Theorem 1 applies).
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- In the case $d = 1$, Suris (1989) exhibited examples of generalized standard maps that are integrable (i.e., they have an integral of motion) for all values of the parameter for which they are defined.

Proof of Theorem 2 (Part 1/2)

Lemma 1

(Under the assumptions of Theorem 2) Let $(m, n) \in \mathbb{Z}^d \times \mathbb{N}^*$, with m and n coprime, and assume that there exist infinitely many values of $\varepsilon \in \mathbb{R}$, accumulating to 0, for which F_ε has an (m, n)-completely periodic Lagrangian graph. Then, $\forall\, \nu \in \mathbb{Z}^d \backslash \{0\}$ such that $\langle \nu, \frac{m}{n} \rangle \in \mathbb{Z}$, we have the ν-th Fourier coefficient $\widehat{G}(\nu) = 0$.

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- Let $\{\varepsilon_k\}_{k \geqslant 1}$ be the values of ε, accumulating to 0, whose existence is assumed in the statement.
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  Using a standard Melnikov argument, let us focus on their Lagrangian action:

$$\mathcal{A}^{\varepsilon_k}_{(m.n)}(q) \quad := \quad \sum_{j=0}^{n-1} S_{\varepsilon_k}(q_j^{\varepsilon_k}, q_{j+1}^{\varepsilon_k}) = \sum_{j=0}^{n-1} S(q_j^{\varepsilon_k}, q_{j+1}^{\varepsilon_k}) + \varepsilon_k G(q_j^{\varepsilon_k}) \equiv \text{ constant w.r.t. q}$$

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\end{aligned}$$

Proof of Theorem 2 (Part 2/2)

- $\sum_{j=0}^{n-1} G(q + \frac{j}{n}) \equiv$ constant w.r.t. q

  Integrating it against $e^{-2\pi i \langle \nu, q \rangle}$ ($\nu \neq 0$), we obtain:

$$0 = \sum_{j=0}^{n-1} \int_{\mathbb{T}^d} G(q + j\frac{m}{n}) e^{-2\pi i \langle \nu, q \rangle} \, dq = \sum_{j=0}^{n-1} \int_{\mathbb{T}^d} G(u) e^{-2\pi i \langle \nu, u \rangle} \, du = n \, \widehat{G}(\nu). \quad \square$$

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Let $q_1,\ldots,q_d \in \mathbb{Q}^d$ be linearly independent vectors over $\mathbb{R}$ and let $0 < a_i < b_i$, $i = 1,\ldots,d$. Then, the set

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Lemma 1 + Lemma 2 + Assumption (*) $\implies$ G is a trigonometric polynomial.

Hence, Theorem 2 follows from Theorem 1. $\square$

Proof of Theorem 1 (Part 1/6)

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Fix $(m, n) \in \mathbb{Z}^d \times \mathbb{N}^*$, with m and n coprime, and let $\{F_\varepsilon\}_{\varepsilon \in \mathbb{K}}$ be a continuous family of lifts of $\{f_\varepsilon\}_{\varepsilon \in \mathbb{K}}$, as in the assumptions.

Introduce the following sets (π_1 denotes the projection on the first component):

- The set of radially transformed points:

$$\mathcal{R}_{(m,n)}(\mathbb{K}) := \{(\varepsilon, q, p) \in \mathbb{K} \times \mathbb{K}^d \times \mathbb{K}^d : \ \pi_1 \circ F_\varepsilon^n(q, p) = q + m\}.$$

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Lemma 3

(i) $\mathcal{R}^*_{(m,n)}(\mathbb{K})$ is a $(d+1)$-dimensional submanifold of $\mathbb{K} \times \mathbb{K}^d \times \mathbb{K}^d$, which is as regular as F is and it locally coincides with the graph of a function $\Gamma_{m,n} : \mathcal{V}_{(m,n)} \subset \mathbb{K} \times \mathbb{K}^d \to \mathbb{K}^d$ defined for (ε, q) in some open set $\mathcal{V}_{(m,n)} \subset \mathbb{K} \times \mathbb{K}^d$.

(ii) $\mathcal{J}^*_{(m,n)}(\mathbb{R})$ is an open subset of $\mathbb{R}$.

Proof of Theorem 1 (Part 2/6)

Similarly:

- The set of periodic points:

$$\mathcal{P}_{(m,n)}(\mathbb{K}) := \{(\varepsilon, q, p) \in \mathbb{K} \times \mathbb{K}^d \times \mathbb{K}^d : \; F_\varepsilon^n(q,p) = q + m\}$$

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Lemma 4

(i) $\Sigma_{(m,n)} \equiv \mathcal{I}^*_{(m,n)}(\mathbb{R})$ and it is closed.

(ii) For every $\varepsilon \in \mathcal{I}^*_{(m,n)}(\mathbb{R})$, F_ε has exactly one Lagrangian (m,n)-completely periodic graph, denoted $\mathrm{graph}(\gamma_\varepsilon)$.
The map $(\varepsilon, q) \in \mathcal{I}^*_{(m,n)}(\mathbb{R}) \times \mathbb{R}^d \mapsto \gamma_\varepsilon(q)$ is as regular as the map $(\varepsilon, q, p) \mapsto F_\varepsilon(q,p)$ is (in Whitney's sense), and $\mathbb{Z}^d$-periodic in the q-variable.

Proof of Theorem 1 (Part 3/6)

Lemma 5

The set $\Sigma_{m,n}$ has empty interior or is the whole $\mathbb{R}$.

Idea of the proof:

Let $\Sigma_{m,n} \neq \mathbb{R}$ with a connected component A that is not a single point. Then:

$$\Gamma := \{(\varepsilon, q, \gamma_\varepsilon(q)) \,:\, \varepsilon \in A,\, q \in \mathbb{R}^d\} \subseteq \mathcal{R}^*_{(m,n)}(\mathbb{C})$$

is connected and let $\mathcal{V}$ the connected component of $\mathcal{R}^*_{(m,n)}(\mathbb{C})$ that contains it.

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$$\begin{array}{rcl} \Delta : \mathcal{V} & \longrightarrow & \mathbb{C}^d \\ (\varepsilon, q, p) & \longmapsto & \pi_2 \circ F^n_\varepsilon(q, p) - p, \end{array}$$

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- The map Δ is holomorphic and vanishes on $\Gamma \implies \Delta$ vanishes on $\mathcal{V}$
  (The real dimension of Γ is $d+1$ and the complex dimension of $\mathcal{V}$ is $d+1$).

Proof of Theorem 1 (Part 4/6)

Define

$$\begin{array}{rcl} \chi : \mathcal{V} & \longrightarrow & \mathbb{C}_{2d}[X] \\ (\varepsilon, q, p) & \longmapsto & \det\big(X\,\mathbb{I}_{2d} - DF_\varepsilon^n(q,p)\big), \end{array}$$

where $\mathbb{C}_{2d}[X]$ is the set of complex polynomials with degree $\leqslant 2d$ and $\mathbb{I}_{2d}$ denotes the $2d$-identity matrix.

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  For $\varepsilon \in A$, the graph of γ_ε is analytic, Lagrangian and F_ε^n restricted to it coincides with the map$(q,p) \mapsto (q+m,p)$. Since F_ε^n is symplectic, then at every point of $\mathrm{graph}(\gamma_\varepsilon)$ all the eigenvalues of DF_ε^n must be equal to 1.

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Let I be the connected component of $\mathcal{J}^*_{(m,n)}(\mathbb{R})$ that contains β and consider the connected subset of $\mathcal{R}^*_{(m,n)}(\mathbb{C})$

$$\mathcal{U} := \bigcup_{\varepsilon \in I} \{\varepsilon\} \times \mathrm{graph}(\eta_\varepsilon) \subseteq \mathcal{V}.$$

Proof of Theorem 1 (Part 5/6)

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If $\mathcal{L} = \mathrm{graph}(\gamma)$ is a (m, n)-periodic graph of F such that for all $q \in \mathbb{R}^d$

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- The graphs of η_ε are also invariant.

  Let $f : \mathbb{T}^d \times \mathbb{R}^d \circlearrowleft$ be a positive twist map and let $F : \mathbb{R}^d \times \mathbb{R}^d \circlearrowleft$ be a lift of f. Every Lipschitz Lagrangian (m, n)-periodic graph of F is invariant.

  (It follows from the fact that each orbit starting on this graph is action-minimizing + a graph property à la Mather).

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  If $\mathcal{L} = \mathrm{graph}(\gamma)$ is a (m,n)-periodic graph of F such that for all $q \in \mathbb{R}^d$

  $$\det\Big(\partial_p(\pi_1 \circ F^n)(q,\gamma(q))\Big) \neq 0 \quad \text{and} \quad \det\Big(X\,\mathbb{I}_{2d} - DF^n(q,\gamma(q))\Big) = (X-1)^{2d},$$

  then $\mathcal{L}$ is C^1 and Lagrangian.

- The graphs of η_ε are also invariant.

  Let $f : \mathbb{T}^d \times \mathbb{R}^d \circlearrowleft$ be a positive twist map and let $F : \mathbb{R}^d \times \mathbb{R}^d \circlearrowleft$ be a lift of f. Every Lipschitz Lagrangian (m,n)-periodic graph of F is invariant.

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$\Longrightarrow$ $I \subseteq \Sigma_{n,m}$ and $\beta \in I^{\circ}$, which is contradicts the fact that A is a connected component of $\Sigma_{n,m}$. Therefore: $\beta = \sup A = +\infty$.

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Similarly one can deduce that $\inf A = -\infty$ and therefore $A = \mathbb{R}$, which contradicts our assumption. $\square$

Proof of Theorem 1 (Part 6/6)

Lemma 6

If $\Sigma_{m,n}$ has an accumulation point $\bar{\varepsilon}$, then $\exists\, \delta > 0$ such that $(\bar{\varepsilon} - \delta, \bar{\varepsilon} + \delta) \subset \Sigma_{m,n}$.

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There exists $\Lambda > 0$ such that for all $\varepsilon \in \mathbb{R}$ such that $|\varepsilon| \geqslant \Lambda$, f_ε does not admit any C^1 Lagrangian invariant graph.

See also: - Mather (1984), in the case of the standard map in dimension 1;
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Lemma 7 $\Longrightarrow$ $\Sigma_{m,n}$ is bounded and therefore it is at most finite. $\square$

Per molts anys Tere!

(...and more fun & maths to come!)