On the Homogenization of the Hamilton-Jacobi Equation

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SAPIENZA UNIVERSITÀ DI ROMA

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This equation can be easily generalized on a general manifold M and in this case the Hamiltonian H will be defined on the cotangent bundle T^*M and $u : M \times \mathbb{R} \longrightarrow \mathbb{R}$.

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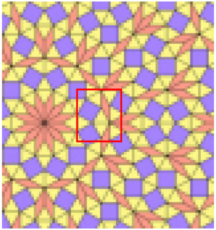

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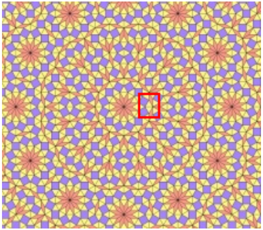

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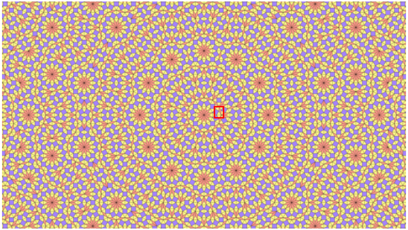

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Problem: Consider faster and faster oscillations of the x-variable and study the associated HJ equations:

$$(\mathrm{HJ}_\varepsilon): \quad \begin{cases} \partial_t u^\varepsilon(x,t) + H(\frac{x}{\varepsilon}, \partial_x u^\varepsilon(x,t)) = 0 & x \in \mathbb{R}^n, t > 0 \\ u^\varepsilon(x,0) = f_\varepsilon(x) \end{cases}$$

where $\varepsilon > 0$ and $f_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$ is some initial datum.

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Theorem (Lions, Papanicolaou & Varadhan, 1987)

Let $f_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$ be Lipschitz and assume that $f_\varepsilon \stackrel{\varepsilon \to 0^+}{\longrightarrow} \bar{f}$ uniformly. Then, as $\varepsilon \to 0^+$, the unique viscosity solution u^ε of $(\mathrm{HJ}_\varepsilon)$ converges locally uniformly to a function $\bar{u} : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$, which solves

$$(\overline{\mathrm{HJ}}) : \quad \boxed{\begin{cases} \partial_t \bar{u}(x,t) + \overline{H}(\partial_x \bar{u}(x,t)) = 0 & x \in \mathbb{R}^n, t > 0 \\ \bar{u}(x,0) = \bar{f}(x), \end{cases}}$$

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- $\overline{H}$ depends only on H.
- As one expects, $\overline{H}$ is independent of x (due to the limit process).
- $\overline{H}$ is in general not differentiable.
- $\overline{H}$ is convex, but not necessarily strictly convex.

Since $\overline{H}$ is convex, let us consider its Legendre-Fenchel transform:

$$\begin{array}{rcl} \overline{L}:\mathbb{R}^n & \longrightarrow & \mathbb{R} \\ v & \longmapsto & \sup\limits_{p\in\mathbb{R}^n}\left(p\cdot v-\overline{H}(p)\right) \end{array}$$

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- $\overline{L}$ is called the effective Lagrangian (it is also convex and not necessarily differentiable).
- Representation formula for $\bar{u}$:

$$\bar{u}(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ \bar{f}(y) + t\overline{L}\left(\frac{x-y}{t} \right) \right\} \quad x \in \mathbb{R}^n, t > 0.$$

This follows from the fact that, although $\overline{H}$ is not differentiable, characteristic lines of $(\overline{HJ})$ are straight lines.

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A first generalization of [LPV] to non Euclidean setting has been proved in:
[CIS] - G. Contreras, R. Iturriaga and A. Siconolfi, "*Homogenization on arbitrary manifolds*", Calc. Var. & PDE Vol. 52 (1-2): 237-252, 2015.

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then $v^\varepsilon(x,t) = u^\varepsilon(\varepsilon x, t)$ is a solution to

$$(\widetilde{\mathrm{HJ}}_\varepsilon): \quad \boxed{\begin{cases} \partial_t v^\varepsilon(x,t) + H(x, \frac{1}{\varepsilon}\partial_x v^\varepsilon(x,t)) = 0 & x \in \mathbb{R}^n, t > 0 \\ v^\varepsilon(x,0) = f_\varepsilon(\varepsilon x) =: \tilde{f}_\varepsilon(x). \end{cases}}$$

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If we denote by d_{euc} the Euclidean metric on $\mathbb{R}^n$, then $(\widetilde{\mathrm{HJ}}_\varepsilon)$ can be interpreted as the Hamilton-Jacobi equation (HJ) associated to H on the rescaled metric space $(\mathbb{R}^n, \varepsilon d_{\mathrm{euc}})$.

Rescale the metric, not the space!

The Effective Hamiltonian and the Cell-Problem

In [LPV], $\overline{H} : \mathbb{R}^n \longrightarrow \mathbb{R}$ was obtained by means of the cell problem (or stationary ergodic HJ), namely: for a fixed $c \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ one search for solutions of the following equation

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- Problem: A-priori there is no relation between $\dim M$ and $\dim H^1(M;\mathbb{R})$!

The Effective Hamiltonian and Aubry-Mather theory

Notation: Let $L : TM \to \mathbb{R}$ be the Tonelli Lagrangian associated to H, let $\mathfrak{M}_L$ be the set of its invariant probability measures and let A_L denote the Lagrangian action on curves associated to L.

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- It coincides with Mather's α-function (Mather, 1991):

$$\overline{H}(c) = -\min_{\mu \in \mathfrak{M}_L} \int_{TM} (L(x,v) - \langle \eta_c(x), v \rangle)\, d\mu =: \alpha(c).$$

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- $\overline{H}$ coincides with the Symplectic Homogenization introduced by Viterbo in 2009 (and also by Monzner, Vichery, Zapolsky, 2012).

How to generalize?

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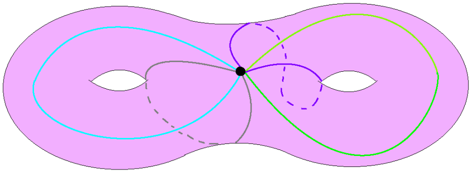
- We should work in a non-compact metric space, otherwise the rescaling process becomes trivial!
- The effective Hamiltonian is $\overline{H} : H^1(M; \mathbb{R}) \longrightarrow \mathbb{R}$. But in general M and $H^1(M; \mathbb{R})$ may have drastically different dimensions (e.g., for a surface Σ_g of genus g, $H^1(\Sigma_g; \mathbb{R}) \simeq \mathbb{R}^{2g}$!)



In particular: how to define convergence of functions on M to a function on $H^1(M; \mathbb{R})$?

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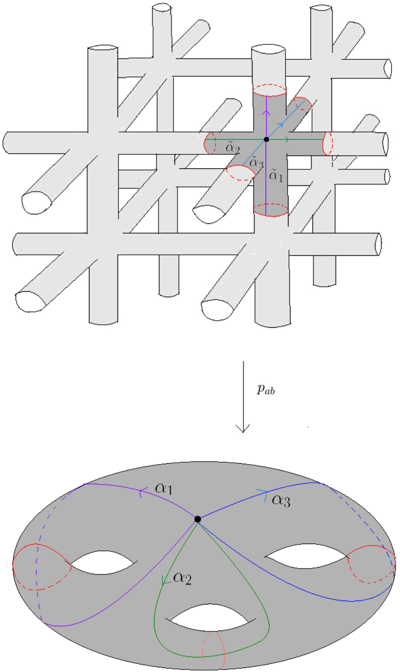
Example:

Let us consider a surface Σ_3 of genus 3 and consider a cover space whose group of Deck transformations is isomorphic to $\mathbb{Z}^3$.



Remark: This is a free abelian cover, but not the maximal one (since $b_1(\Sigma_3) = 6$).

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- The advantage of this cover is that it has a $\mathbb{Z}^{b_1(M)}$-periodic structure given by the action of the group of Deck transformations.
- Heuristically, the rescaled metric space $(\widetilde{M}, \varepsilon\tilde{d})$ has a $\varepsilon\mathbb{Z}^{b_1(M)}$-structure; hence, as $\varepsilon \to 0^+$, it is reasonable to expect that it "converges" to $\mathbb{R}^{b_1(M)}$ with some metric d_∞.

Theorem (Contreras, Iturriaga & Siconolfi)

Let $f_\varepsilon : \widetilde{M} \longrightarrow \mathbb{R}$ and $\bar{f} : H_1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ be continuous functions, such that $\bar{f}$ has at most linear growth and f_ε converges uniformly to $\bar{f}$ as $\varepsilon \to 0^+$.
Then, the viscosity solution $u^\varepsilon : \widetilde{M} \times [0,+\infty) \longrightarrow \mathbb{R}$ to

$$\begin{cases} \partial_t u^\varepsilon(x,t) + H(x, \frac{1}{\varepsilon}\partial_x u^\varepsilon(x,t)) = 0 & x \in \widetilde{M}, t > 0 \\ u^\varepsilon(x,0) = f_\varepsilon(x), \end{cases}$$

converges locally uniformly to the viscosity solution $\bar{u} : H_1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ to

$$\begin{cases} \partial_t \bar{u}(x,t) + \overline{H}(\partial_x \bar{u}(x,t)) = 0 & x \in H_1(M;\mathbb{R}),\ t > 0 \\ \bar{u}(x,0) = \bar{f}(x), \end{cases}$$

where $\overline{H} : H^1(M;\mathbb{R}) \to \mathbb{R}$ is the effective Hamiltonian (or Mather's α function).
Moreover,

$$\bar{u}(x,t) = \inf_{y \in H_1(M;\mathbb{R})} \left\{ \bar{f}(y) + t\overline{L}\left(\frac{x-y}{t}\right)\right\} \quad x \in H_1(M;\mathbb{R}),\ t > 0,$$

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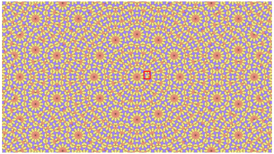
- This seems to be the right setting to obtain $\overline{H} : H^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ (Mather's α function) as the effective Hamiltonian.
  Would it be possible to obtain a different one, in spite of the analogy with LPV's case?

A Different Point of View

Lifting to a cover is equivalent to give a periodicity to the Hamiltonian H and it determines the structure of the homogenized problem.



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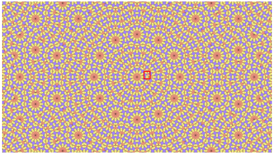

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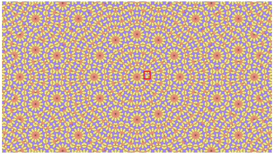

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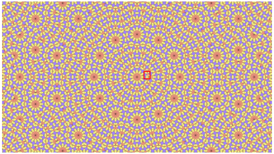

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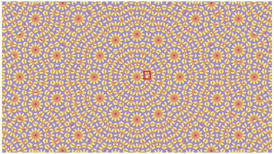

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Idea/Problem: Let us consider Hamiltonians on non compact manifolds which are invariant under the action of a discrete group.

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Is it possible to prove an homogenization result for HJ in this setting?

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- This fact makes more evident the leading rôle of Γ (periodicity) in the homogenization process and not of the fundamental domain X/Γ.

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- For non-compact metric spaces, a more useful notion is the one of pointed Gromov-Hausdorff (pGH) convergence.

Rescaling (and Convergence) of Metric Spaces

The rescaling process consists in considering HJ on the rescaled metric space $(X, \varepsilon d)$ as $\varepsilon \to 0^+$. Does it have a limit? In which sense?

Gromov-Hausdorff (GH) "distance": Let $\widetilde{X}_1 := (X_1, d_1)$ and $X_2 := (\widetilde{X}_2, d_2)$ be metric spaces. We say that $d_{\mathrm{GH}}(\widetilde{X}_1, \widetilde{X}_2) < r$ if there exist a metric space (Z, d) and two subspaces $Z_1, Z_2 \subset Z$ isometric (respectively) to $\widetilde{X}_1$ and $\widetilde{X}_2$, such that their Hausdorff distance in (Z, d) is $d_H(Z_1, Z_2) < r$.

[Recall that $d_H(A, B) = \inf\{r > 0 : \mathcal{N}_r(A) \supset B \,\mathrm{and}\, \mathcal{N}_r(B) \supset A\}$, where $\mathcal{N}_r(\cdot)$ denotes the open neighborhood of size r]

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Roughly: $(X_n, d_n, x_n) \to (X, d, x_0)$ if balls of radius $r > 0$ and centers at x_n (in X_n) converge (in the GH distance) to the ball of radius r and center at x_0 (in X).

Asymptotic Cone of Metric Spaces

Let (X, d) be a metric space. If there exists any limit (in the (pGH) sense) of $(X, \varepsilon d)$ as ε goes to 0^+, then this is called an asymptotic cone of (X, d).

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The Group Γ as a Metric Space

Any orbit of Γ represents a metric space embedded in (X, d) and at finite GH distance (because the action is cocompact).

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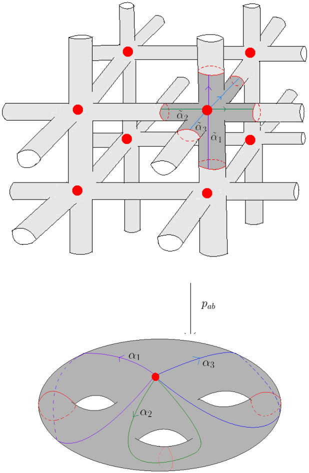

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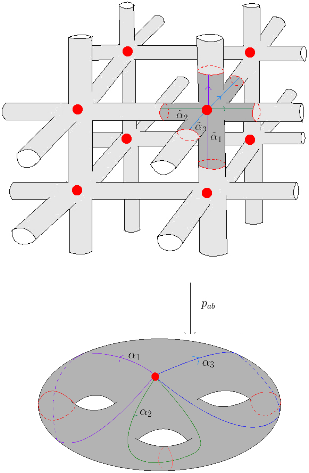

Two possible metrics in Γ:

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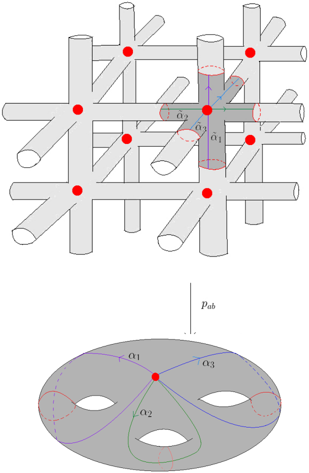

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$$\|\gamma\|_S := \min\{n \in \mathbb{N} : \gamma \in S^n\}$$

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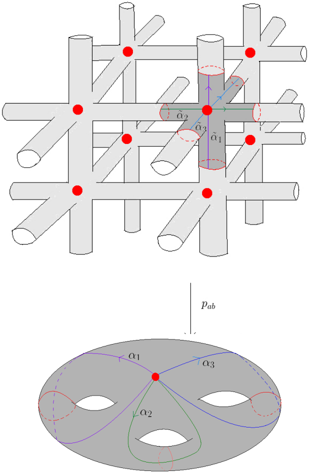

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Idea: We study the asymptotic cone of Γ as a metric space.

Asymptotic Cone of a Group

Let Γ be a finitely generated group with a metric d_Γ (one of the metric introduced before).

- Γ abelian: $\Gamma \simeq \mathbb{Z}^k \oplus \Gamma_0$, where $k = \mathrm{rank}\,\Gamma$ and Γ_0 is the torsion subgroup (a finite group). Then, the asymptotic cone is $G_\infty \simeq \mathbb{R}^k$ and the asymptotic distance is related to the stable norm, i.e., the unique norm $\|\cdot\|_\infty$ such that for each $\gamma \in \mathbb{Z}^k$:
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then the asymptotic cone exists and it is unique (Gromov, 1981).

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  Gromov proved that Γ has polynomial growth if and only if it is virtually nilpotent, i.e., it contains a nilpotent subgroup of finite index.
- Polynomial growth is the optimal condition to ensure both existence and uniqueness of the asymptotic cone.

Nilpotent Groups

A finitely generated group Γ is said to be nilpotent if the lower central series ends after finitely many steps:

$$\Gamma^{(1)} := \Gamma \geq \Gamma^{(2)} := [\Gamma^{(1)}, \Gamma] \geq \ldots \geq \Gamma^{(i+1)} := [\Gamma^{(i)}, \Gamma] \geq \ldots \geq \Gamma^{(r)} > \Gamma^{(r+1)} = \{e\},$$

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- Abelian groups ($r = 1$);
- Quaternionic group Q_8 ($r = 2$) $\leftarrow$ (smallest non-abelian example)
- Heisenberg group $\mathbb{H}_{2n+1}(\mathbb{Z})$ ($r = 2$):

$$\mathbb{H}_{2n+1}(\mathbb{Z}) = \langle a_1, b_1, \ldots, a_n, b_n, t : [a_i, b_i] = t \ \forall\, i = 1, \ldots, n \text{ and all others brackets } = 0\rangle.$$

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Nilpotent groups have polynomial growth (Wolf, 1968). In particular, Bass (1972) proved that the rate is

$$K = \sum_{k=1}^{r} k \cdot \mathrm{rank} \left(\Gamma^{(k)} / \Gamma^{(k+1)} \right),$$

also called the homogeneous dimension of Γ.

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  Remark: $d = \dim G = \sum_{k=1}^{r} \operatorname{rank} \left(\Gamma^{(k)}/\Gamma^{(k+1)}\right)$.

Stratified Lie Algebra and Dilations

A Lie Algebra $\mathfrak{g}$ is called a stratified algebra if it admits a stratification, i.e., there exist vector subspaces $V_1, \ldots, V_r \subset \mathfrak{g}$ (called strata) such that

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- We can define dilations: for each $\lambda \in \mathbb{R}$ we define $\delta_\lambda : \mathfrak{g} \longrightarrow \mathfrak{g}$ which is an algebra automorphism defined linearly by imposing $\delta_\lambda(v) = \lambda^i v$ for every $v \in V_i$, with $i = 1, \dots, r$. Using the exponential map, we can define the associated group automorphisms $\delta_\lambda : G \longrightarrow G$.

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If Γ is a finitely generated nilpotent group, not only the asymptotic cone (G_∞, d_∞) exists and is unique, but it also enjoys many interesting properties (Pansu, 1983) $\longrightarrow$ G_∞ it is related to the Malcev closure of Γ.

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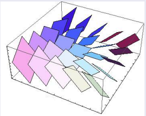
- d_∞ is a Carnot-Carathéodory distance and $d_\infty(\delta_\lambda(\bar{x}), \delta_\lambda(\bar{y})) = \lambda d_\infty(\bar{x}, \bar{y})$ for all $\bar{x}, \bar{y} \in G_\infty$ and $\lambda > 0$.

Let $\Delta \subset TG_\infty$ the horizontal distribution induced by V_1. There exists a norm $\|\cdot\|_\infty$ on Δ (obtained similarly to what done for the stable norm) and d_∞ is the sub-Riemannian distance induced by $(G_\infty, \Delta, \|\cdot\|_\infty)$, i.e., for $\bar{x}, \bar{y} \in G_\infty$ we define:



$$d_\infty(\bar{x}, \bar{y}) = \inf \left\{ \int_0^T \|\dot{\gamma}(t)\|_\infty : \underbrace{\gamma : [0, T] \to G_\infty \text{is horizontal and}} \gamma(0) = \bar{x},\ \gamma(T) = \bar{y} \right\}$$

Main Theorem (Part 1/3)

Let $H : T^*X \longrightarrow \mathbb{R}$ be a Γ-invariant Tonelli Hamiltonian and let $L : TX \longrightarrow \mathbb{R}$ be the associated Γ-invariant Tonelli Lagrangian.

For $\varepsilon > 0$, let X_ε denote the rescaled metric spaces $(X, d_\varepsilon := \varepsilon d)$ and consider the rescaled Hamilton-Jacobi equation:

$$(\widetilde{\mathrm{HJ}}_\varepsilon) \begin{cases} \partial_t u^\varepsilon(x,t) + H(x, \frac{1}{\varepsilon}\partial_x u^\varepsilon(x,t)) = 0 & x \in X_\varepsilon,\ t > 0 \\ u^\varepsilon(x,0) = f_\varepsilon(x), \end{cases}$$

where $f_\varepsilon : X_\varepsilon \longrightarrow \mathbb{R}$ are equiLipschitz with respect to the metrics d_ε and, as ε goes to zero, they converge uniformly on compact sets to a function $\bar{f} : G_\infty \longrightarrow \mathbb{R}$ with at most linear growth.

Then:

Main Theorem (Part 1/3)

Let $H : T^*X \longrightarrow \mathbb{R}$ be a Γ-invariant Tonelli Hamiltonian and let $L : TX \longrightarrow \mathbb{R}$ be the associated Γ-invariant Tonelli Lagrangian.

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Then:

1. The rescaled solutions (for $x \in X_\varepsilon$ and $T > 0$)

$$u^\varepsilon(x,T) = \inf\left\{ f_\varepsilon(\gamma(0)) + \int_0^T L(\gamma(t), \varepsilon\dot{\gamma}(t))\, dt \;\middle|\; \gamma \in C^1([0,T], X_\varepsilon),\ \gamma(T) = x \right\}$$

   converge uniformly on compact sets of $G_\infty \times (0, +\infty)$ to a function $\bar{u} : G_\infty \times (0, +\infty) \to \mathbb{R}$.

Main Theorem (Part 2/3)

2. For $\bar{x} \in G_\infty$ and $T > 0$:

$$\bar{u}(\bar{x}, T) = \inf_{\bar{y} \in G_\infty} \left\{ \bar{f}(\bar{y}) + T\overline{L}\left(\delta_{1/T}(\bar{y}^{-1}\bar{x}) \right) \right\},$$

where $\overline{L} : G_\infty \longrightarrow \mathbb{R}$ depends only on the Hamiltonian H, it is superlinear, i.e.,

$$\forall\, A > 0 \qquad \exists\, B = B(A) \geq 0 \;: \qquad \overline{L}(\bar{x}) \geq A\, d_\infty(e, \bar{x}) - B \qquad \forall\, \bar{x} \in G_\infty$$

and convex, namely

$$\overline{L}\Big(\delta_\lambda(\bar{x}) \cdot \delta_{1-\lambda}(\bar{y})\Big) \leq \lambda \overline{L}(\bar{x}) + (1-\lambda)\overline{L}(\bar{y}) \qquad \forall\, \lambda \in (0,1) \quad \text{and} \quad \forall\, \bar{x}, \bar{y} \in G_\infty.$$

We shall call this function Generalized Mather's β-function.

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We shall call this function Generalized Mather's β-function.

3. For each $\bar{x} \in G_\infty$

$$\overline{L}(\bar{x}) := \inf_{\sigma \in \mathcal{H}_{\bar{x}}} \int_0^1 \beta(\bar{\pi}(\dot{\sigma}(s)))\, ds,$$

where $\mathcal{H}_{\bar{x}}$ denotes the set of absolutely continuous horizontal curves $\sigma : [0,1] \longrightarrow G_\infty$ connecting e to $\bar{x}$, $\beta : H_1(X/\Gamma; \mathbb{R}) \longrightarrow \mathbb{R}$ is Mather's β-function associated to the Lagrangian L projected on $T(X/\Gamma)$, and $\bar{\pi} : \mathfrak{g}_\infty \longrightarrow \frac{\mathfrak{g}_\infty}{[\mathfrak{g}_\infty, \mathfrak{g}_\infty]} \hookrightarrow H_1(X/\Gamma; \mathbb{R})$.

Main Theorem (Part 3/3)

- $\bar{u}$ is the unique viscosity solution to the following problem:

$$\begin{cases} \partial_t \bar{u}(\bar{x}, t) + \overline{H}(\nabla_{\mathcal{H}} \bar{u}(\bar{x}, t)) = 0 & (\bar{x}, t) \in G_\infty \times (0, \infty) \\ \bar{u}(\bar{x}, 0) = \bar{f}(\bar{x}) & \bar{x} \in G_\infty, \end{cases}$$

where $\nabla_{\mathcal{H}} \bar{u}(\bar{x}, t)$ denotes the horizontal gradient of $\bar{u}(\cdot, t)$ (with respect to the $\bar{x}$-component) and $\overline{H} : \left(\frac{\mathfrak{g}_\infty}{[\mathfrak{g}_\infty, \mathfrak{g}_\infty]}\right)^* \longrightarrow \mathbb{R}$ is the convex conjugate of β restricted to the subspace $\bar{\pi}(\mathfrak{g}_\infty) \subseteq H_1(X/\Gamma; \mathbb{R})$.

Some Ideas on the Proof: Rescaling Maps

Note: For simplifying the presentation, let us assume that G_∞ coincides with the Malcev closure of Γ.

Fix $x_0 \in X$ and choose a fundamental domain Ω of the action of Γ in G_∞, such that $x_0 \in \Omega$.

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- Rescaling maps: For each $\varepsilon > 0$ we define a map $h_\varepsilon : G_\infty \longrightarrow \Gamma$ in the following way. If $\bar{x} \in G_\infty$, then $h_\varepsilon(\bar{x}) = \gamma$ such that $\delta_{\frac{1}{\varepsilon}}(\bar{x}) \in \gamma \cdot \Omega$.

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  Properties:

  1. For each $R > 0$, there exists $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, such that h_ε maps the ball $B_R^{d_\infty}(e)$ into $B_{R+\theta(\varepsilon)}^{d_\varepsilon}(x_0)$.

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  2. $\delta_\varepsilon(h_\varepsilon(\bar{x})) \longrightarrow \bar{x}$ in G_∞ and if $\bar{x} \neq \bar{y}$ then $\delta_\varepsilon(h_\varepsilon(\bar{x})h_\varepsilon(\bar{y})^{-1}) \longrightarrow \bar{x}\bar{y}^{-1}$ in G_∞.

Some Ideas on the Proof: Notion of Convergence

- Convergence: Let $F_\varepsilon : X_\varepsilon \to \mathbb{R}$ and $\overline{F} : G_\infty \to \mathbb{R}$. We say that:

  1. $F_\varepsilon \to F$ poinwise as $\varepsilon \to 0^+$, if for each $\bar{x} \in G_\infty$ we have

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  2. $F_\varepsilon \to F$ locally uniformly as $\varepsilon \to 0^+$, if for each $R > 0$ we have

  $$\lim_{\varepsilon \to 0^+} \sup_{B_R^{d_\infty}(e)} \left| F_\varepsilon(h_\varepsilon(\bar{x}) \cdot x_0) - \overline{F}(\bar{x}) \right| = 0.$$

Note: If F_ε are equicontinuous, also this notion is independent of x_0.

Some Ideas on the Proof: Rescaled Mañé potential

Recall the representation formula for the solution to $(\widetilde{\mathrm{HJ}}_\varepsilon)$:

$$u^\varepsilon(x, T) = \inf \left\{ f_\varepsilon(\gamma(0)) + \int_0^T L(\gamma(t), \varepsilon\dot\gamma(t))\, dt \;\middle|\; \gamma \in C^1([0, T], X_\varepsilon),\, \gamma(T) = x \right\}$$

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$$u^\varepsilon(x,T) = \inf\left\{ f_\varepsilon(\gamma(0)) + \underbrace{\int_0^T L(\gamma(t),\varepsilon\dot\gamma(t))\,dt}_{\text{Mañé potential}} \,\Big|\, \gamma \in C^1([0,T],X_\varepsilon),\ \gamma(T)=x \right\}$$

We define the Mañé potential associated to $L_\varepsilon(x,v) = L(x,\varepsilon v)$ as (let $x,y \in X$ and $T>0$):

$$\begin{aligned}
\Phi^\varepsilon(x,y,T) &:= \inf\left\{ \int_0^T L(\gamma(t),\varepsilon\dot\gamma(t))\,dt \,\Big|\, \gamma \in C^1([0,T],X_\varepsilon),\ \gamma(0)=x,\ \gamma(T)=y \right\} \\
&= \inf\left\{ \int_0^{T/\varepsilon} L(\gamma(t),\dot\gamma(t))\,dt \,\Big|\, \gamma \in C^1([0,T/\varepsilon],X),\ \gamma(0)=x,\ \gamma(T/\varepsilon)=y \right\}.
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Note: We are assuming that the Mañé critical value of $L : T(X/\Gamma) \to \mathbb{R}$ is zero.

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Some Ideas on the Proof: Limit of Rescaled Mañé potential

Let us define the following function

$$\begin{aligned} F_\varepsilon : G_\infty \times (0, +\infty) &\longrightarrow \mathbb{R} \\ (\bar{x}, T) &\longmapsto \Phi^\varepsilon(x_0, h_\varepsilon(\bar{x}) \cdot x_0, T). \end{aligned}$$

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Rescaling property: $F_\varepsilon(\bar{x}, S) = \frac{S}{T} F_{\frac{T}{S}\varepsilon}(\delta_{\frac{T}{S}}(\bar{x}), T)$ for all $\bar{x} \in G_\infty$ and $S, T > 0$.

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Proposition

For all $\bar{x} \in G_\infty$ and $T > 0$

$$\exists \lim_{\varepsilon \to 0^+} F_\varepsilon(\bar{x}, T) =: \hat{\beta}(\bar{x}, T).$$

- Clearly, $\hat{\beta}(\bar{x}, S) = \frac{S}{T} \hat{\beta}(\delta_{\frac{T}{S}}(\bar{x}), T)$ for all $\bar{x} \in G_\infty$ and $S, T > 0$.

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- The effective Lagrangian is defined as $\overline{L}(\bar{x}) := \hat{\beta}(\bar{x}, 1)$. It has an explicit expression in terms of Mather's β function (see statement of the theorem).

Some Ideas on the Proof: Convergence

Then one can show that $u\varepsilon \to \bar{u}$ locally uniformaly in $G_\infty \times (0, +\infty)$ as ε goes to zero, where

$$\bar{u}(\bar{x}, T) = \inf_{\bar{y} \in G_\infty} \left\{ \bar{f}(\bar{y}) + T\overline{L}\left(\delta_{1/T}(\bar{y}^{-1}\bar{x}) \right) \right\}.$$

Moreover, from this Hopf-like representation formula & a result by Balogh, Calogero and Pini (2014), we can deduce that $\bar{u}$ solves a Hamilton-Jacobi equation associated to some specific effective Hamiltonian $\overline{H}$ (see the statement of the theorem):

$$\begin{cases} \partial_t \bar{u}(\bar{x}, t) + \overline{H}(\nabla_{\mathcal{H}} \bar{u}(\bar{x}, t)) = 0 & (\bar{x}, t) \in G_\infty \times (0, \infty) \\ \bar{u}(\bar{x}, 0) = \bar{f}(\bar{x}) & \bar{x} \in G_\infty, \end{cases}$$

Thank you for your attention!