# Hamilton-Jacobi Equations on Networks


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Figure courtesy Fabio Fagnani.
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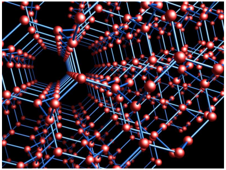
# Introduction

Over the last years there has been an increasing interest in the study of the Hamilton-Jacobi (HJ) equation on networks and related problems.

These problems:

- involve a number of subtle theoretical issues;
- have a great impact in the applications in various fields, e.g., to data transmission, traffic management problems, *etc*...



While locally, *i.e.*, on each branch (arc) of the network, the study reduces to the analysis of 1-dimensional problems, the main difficulties arise in:

- matching together information converging at the juncture of two or more arcs;
- relating the local analysis at a juncture with the global structure (topology) of the network.

# Plan of the Talk

In this talk I shall discuss some works in collaboration with Antonio Siconolfi (Sapienza, Università di Roma).

I. Global results for Eikonal HJ equations on networks

- Solutions:
  - existence of a (unique) critical value for which global solutions exist,
  - determination of a uniqueness set (Aubry set),
  - Hopf-Lax type representation formulae, *etc*...
- Critical case:
  - properties, regularity, existence of $C^1$ critical subsolutions, *etc*...
- Supercritical case:
  - properties, representation formulae for maximal subsolutions, *etc*...
  - Existence and uniqueness of solutions on subsets of the network, continuously extending admissible data on the complement.

II. Homogenization on topological crystals (in progress)

# Main ideas

The main rationale consists in neatly distinguishing between:

1) The local problem on the arcs:

- (classical) 1-dimensional viscosity techniques.

2) The global matching on the network:

- we associate to the network an abstract graph, encoding all of the information on the complexity of the network;
- we relate the problem to a discrete functional equation on the graph, to be studied by means of techniques *à la* weak KAM.

3) Combine the global analysis (on the abstract graph) with the local one:

- define the Aubry set and provide its PDE characterization;
- establish regularity properties for critical subsolutions and solutions;
- uniqueness results and representation formulae.

# Advantages and novelties

- Global analysis that goes beyond what happens at a single juncture.
- The Network is assumed to be finite and connected
  $\longrightarrow$ multiple arcs between two vertices and loops are allowed.
- Hamiltonians are assumed continuous, quasi-convex and coercive.
  $\longrightarrow$ No compatibility conditions at the vertices are required.
- We prove uniqueness and comparison principles in a simple way
  $\longrightarrow$ completely bypassing the difficulties involved in the Crandall-Lions doubling variable method, in favor of a more direct analysis of a discrete equation.
- We identify an intrinsic boundary (Aubry set) on which admissible traces can be assigned to get unique global solutions
  $\longrightarrow$ Formulating boundary problems on the network and determining 'natural' subsets on which to assign boundary data is a subtle issue, yet not well settled in the literature.

# Comparison with previous literature

Large amount of literature related to differential equations on networks, or others non-regular geometric structures (ramified/stratified spaces), in various contexts: hyperbolic problems, traffic flows, evolutionary equations, (regional) control problems, Hamilton-Jacobi equations, etc...

Some references closer related to our work:

- Schieborn-Camilli (2013):
  - PDE approach;
  - Eikonal equation in the supercritical case;
  - restrictions on the topology of the network;
  - they require a-priori existence of a regular strict subsolution;
  - continuity of Hamiltonians at vertices (and, accordingly, mixed conditions on the test functions at vertices).

Other related contributions by: Achdou, Cutrí, Marchi, Oudet and Tchou.

# Comparison with previous literature

- Imbert-Monneau (2013, 2016):
    - rather different point of view and techniques from ours;
    - local analysis at a juncture;
    - they use the doubling variable method by introducing an extra parameter (flux limiter), a companion equation (junction condition) and by using special vertex test functions.

  Other related contributions in collaborations with Galise and Zidani.

- Lions-Souganidis (2016):
    - one dimensional junction-type problems for non convex discounted HJ equations;
    - we adopt the same notion of solution.

- Discrete weak KAM and Aubry Mather theories:
    - Bernard-Buffoni (2006-2007): optimal transport maps.
    - Zavidovique (2010-2012): more systematic development.

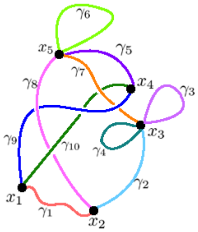
  Other related contributions by: Gomes (2005), Iturriaga-Sánchez Morgado (2017), Su-Thieullen (2016).

# The Network

An embedded network is a compact subset $\Gamma$ in $(\mathbb{R}^N, d_{\mathrm{eucl}})$, or in any Riemannian manifold $(M, g)$, of the form

$$\Gamma = \bigcup_{\gamma \in \mathcal{E}} \gamma \subset \mathbb{R}^N,$$

where $\mathcal{E}$ is a finite collection of arcs, *i.e.*, simple $C^1$ regular (oriented) curves, which are disjoint, except at the end-points (called vertices). We denote the set of vertices by $\mathcal{V}$.




Observe that $\Gamma$ inherits:

- a metric $d_\Gamma$ from the ambient space, hence a topology; we assume that $\Gamma$ is path-connected.
- a differential structure (vertices are special points).

# The Network

We introduce the following maps:

- A fixed-point-free involution $\overline{\phantom{x}} : \mathcal{E} \longrightarrow \mathcal{E}$ that to each arc $\gamma \in \mathcal{E}$ associates the arc $\overline{\gamma} \in \mathcal{E}$, *i.e.*, the same arc with opposite orientation(reversed arc).
- The map $\mathrm{o} : \mathcal{E} \longrightarrow \mathcal{V}$ which associates to each oriented arc $\gamma \in \mathcal{E}$ its initial vertex $\mathrm{o}(\gamma) \in \mathcal{V}$ (origin).
- The map $\mathrm{t} : \mathcal{E} \longrightarrow \mathcal{V}$ which associates to each oriented arc $\gamma \in \mathcal{E}$ its final vertex $\mathrm{t}(\gamma) \in \mathcal{V}$ (end).

In particular, for each $\gamma \in \mathcal{E}$:

$$\mathrm{t}(\gamma) = \mathrm{o}(\overline{\gamma}) \quad \text{and} \quad \mathrm{t}(\overline{\gamma}) = \mathrm{o}(\overline{\overline{\gamma}}) = \mathrm{o}(\gamma).$$

It follows from the connectedness assumption on $\Gamma$, that the maps $\mathrm{o}$ and $\mathrm{t}$ are surjective.

# Hamiltonians on the network

A Hamiltonian on a network $\Gamma$ is a function $\mathcal{H} : T^*\Gamma \longrightarrow \mathbb{R}$. For each $\gamma \in \mathcal{E}$, let us denote by $H_\gamma$ the restriction on the Lagrangian on $T^*\gamma$ (vertices included); we ask each $H_\gamma$ to satisfy the following conditions:

- $H_\gamma$ is continuous on $T^*\gamma$;
- $H_\gamma$ is coercive in each fiber $T^*_x\gamma$, where $x \in \gamma$;
- $H_\gamma$ is quasi-convex in each fiber, namely the set $\{H_\gamma \leq a\} \cap T^*_x\gamma$ is convex (if nonempty) for every $a \in \mathbb{R}$ and $x \in \gamma$.

$+$ Extra condition related to their critical values (see next slide).

Hamiltonians corresponding to geometrically different arcs are unrelated, even for arcs with some vertex in common.
No continuity or compatibility conditions at common vertices!

# Critical values for $H_\gamma$

We set for any $\gamma \in \mathcal{E}$

$$\begin{aligned} a_\gamma &:= \max_{x\in\gamma} \min_{T^*_x\gamma} H_\gamma \\ c_\gamma &:= \min\{a \in \mathbb{R} : \; H_\gamma(x, du) = a \text{ admits periodic subsolutions}\}. \end{aligned}$$

By periodic subsolution, we mean subsolution to the equation in $\gamma$ taking the same value at the endpoints.

The definition of $c_\gamma$ is well-posed and $a_\gamma \leq c_\gamma$ for any $\gamma \in \mathcal{E}$.

We define

$$a_0 := \max\left\{ \max_{\gamma\in\mathcal{E}\setminus\mathcal{E}^*} a_\gamma, \max_{\gamma\in\mathcal{E}^*} c_\gamma \right\},$$

where $\mathcal{E}^* \subset \mathcal{E}$ denotes the subset of arcs which are loops:

$$\mathcal{E}^* := \{\gamma \in \mathcal{E} : \; \mathrm{o}(\gamma) = \mathrm{t}(\gamma)\}.$$

# Extra condition on the Hamiltonians

We require a further condition:

- Given any $\gamma \in \mathcal{E}$ with $a_\gamma = a_0$, the map $x \in \gamma \longmapsto \min_{T_x^*\gamma} H_\gamma$ is constant.

Notice that this condition is automatically satisfied if the $H_\gamma$'s are independent of the state variable.

The main rôle of this condition is to ensure uniqueness of solutions to the Dirichlet problem associated to the equation $H_\gamma = a_\gamma$ for any value assigned at $\mathrm{o}(\gamma)$, at least for the $\gamma$'s with $a_\gamma = a_0$.

Note: The uniqueness property for such kind of problems holds in general when the equation $H_\gamma = a$ admits a strict subsolution (for example when $a > a_\gamma$), which is not the case at level $a_\gamma$.

# The Eikonal HJ equation on networks

We consider the equation

$$\mathcal{H}(x, du) = a \quad \text{on } \Gamma. \tag{$\mathcal{H}Ja$}$$

This notation synthetically indicates the family (for $\gamma$ varying in $\mathcal{E}$) of Hamilton–Jacobi equations $H_\gamma(x, du) = a$ on $\gamma \setminus \{\mathrm{o}(\gamma), \mathrm{t}(\gamma)\}$.

On a single arc, these equations possess infinitely many (viscosity) solutions, depending on the boundary data at $\mathrm{o}(\gamma)$ and $\mathrm{t}(\gamma)$.
We need to introduce suitable viscosity tests on the vertices so to:

- select a unique solution on any arc;
- match these (local) solutions in a continuous way at vertices.

Two basic properties are needed (true under our assumptions on $H_\gamma$'s ):
- existence and uniqueness of solutions on any arc, coupled with suitable Dirichlet boundary conditions at $\mathrm{o}(\gamma)$ and $\mathrm{t}(\gamma)$;
- characterization of the maximal (sub)solution with a given datum at $\mathrm{o}(\gamma)$.

# Notion of (sub)solution in our setting

## Definition of subsolution

We say that $u : \Gamma \longrightarrow \mathbb{R}$ is subsolution to $(\mathcal{H}Ja)$ if
i) it is continuous on $\Gamma$;
ii) it is (viscosity) subsolution on each $\gamma \setminus \{\mathrm{o}(\gamma), \mathrm{t}(\gamma)\}$, for any $\gamma \in \mathcal{E}$.

Given a continuous function $w$ on $\gamma$, we say that a $C^1$ function $\varphi$ is a constrained subtangent to $w$ at $\mathrm{t}(\gamma)$ if $w = \varphi$ at $\mathrm{t}(\gamma)$ and $w \geq \varphi$ in a sufficiently small neighborhood of $\mathrm{t}(\gamma)$ in $\gamma$ (*cfr.* Soner, 1986).

## Definition of solution

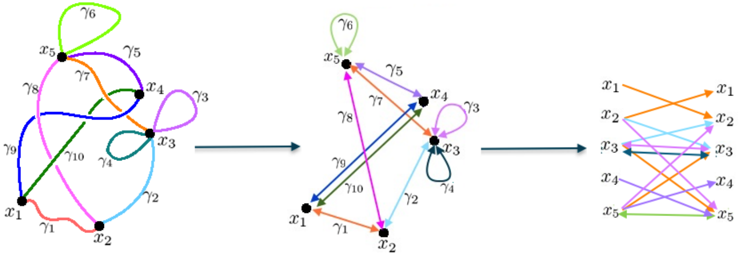
We say that $u : \Gamma \longrightarrow \mathbb{R}$ is solution to $(\mathcal{H}Ja)$ if
i) it is continuous on $\Gamma$;
ii) it is a (viscosity) solution on each $\gamma \setminus \{\mathrm{o}(\gamma), \mathrm{t}(\gamma)\}$, for any $\gamma \in \mathcal{E}$;
iii) (state constraint boundary conditions) for every vertex $x$ there is an arc $\gamma$ with $\mathrm{t}(\gamma) = x$ such that any constrained $C^1$ subtangent $\varphi$ to $u\big|\gamma$ at $\mathrm{t}(\gamma)$ satisfies $H_\gamma(x, d\varphi(x)) \geq a$.

# Some remarks

- In the definition of subsolution no conditions are required on vertices. These assumptions are minimal. The validity of this approach is supported by the fact that the notion of solutions can be recovered in terms of maximal subsolution attaining a specific value at a given point (vertex or internal).
- In the definition of solution there are no mixing conditions between equations on different arcs incident at the same vertex.
- The (unique) place where the global topology of $\Gamma$ plays a rôle is iii).
- The constraint boundary condition at $\mathrm{t}(\gamma)$ selects the maximal solution taking a given value at $\mathrm{o}(\gamma)$. In a sense, it leaves a degree of freedom at $\mathrm{o}(\gamma)$, which can be exploited to get solutions to the HJ equations on any arc, that match continuously.
- If $\gamma$ is a loop, we must have in addition $u(\mathrm{o}(\gamma)) = u(\mathrm{t}(\gamma))$, *i.e.*, periodicity. This explains why for $\gamma \in \mathcal{E}^*$ we must consider the value $c_\gamma$.
- If the network is augmented by changing the status of a finite number of intermediate points of arcs in $\Gamma$, which become new vertices, then the notion of solution is not affected.

# From the network to the abstract graph

The main novelty of our method is to put in relation the HJ equation on the network to a discrete functional equation on the underlying abstract graph $\Gamma = (\mathcal{E}, \mathcal{V})$, where $\mathcal{E}$ is the (abstract) set of arcs and $\mathcal{V}$ the (abstract) set of vertices.




For any $\gamma \in \mathcal{E}$, the relevant information to transfer is $\sigma_a(\gamma) := \hat{u}_\gamma(\mathrm{t}(\gamma))$, where $\hat{u}_\gamma$ is the maximal subsolution on $\gamma$, vanishing at $\mathrm{o}(\gamma)$.

In some sense, we are considering the difference at endpoints of maximal subsolution(s) in $\gamma$ (intrinsic semidistance on $\gamma$).

# The discrete functional equation

Let $a \geq a_0$. Let us start by observing the following admisssibility condition:

- there exists a subsolution on $\gamma$ attaining the values $\alpha$ and $\beta$ at, respectively, $\mathrm{o}(\gamma)$ and $\mathrm{t}(\gamma)$, if and only if $-\sigma_a(\overline{\gamma}) \leq \beta - \alpha \leq \sigma_a(\gamma)$.

If $u : \Gamma \to \mathbb{R}$ is a subsolution to $(\mathcal{H}Ja)$, then

$$u(x) \leq \min_{\gamma \in \mathcal{E}, \mathrm{o}(\gamma) = x} \left( u(\mathrm{t}(\gamma)) + \sigma_a(\overline{\gamma}) \right) \qquad \forall\, x \in \mathcal{V}.$$

If $u : \Gamma \to \mathbb{R}$ is a solution to $(\mathcal{H}Ja)$, then equality holds at each $x \in \mathcal{V}$.

We introduce the following discrete functional equation:

$$\boxed{u(x) = \min_{\gamma \in \mathcal{E}, \mathrm{o}(\gamma) = x} \left( u(\mathrm{t}(\gamma)) + \sigma_a(\overline{\gamma}) \right)} \qquad \forall\, x \in \mathcal{V}. \qquad (\mathcal{D}FEa)$$

Note: Equality is required only at (at least) one arc for each vertex!
Moreover, the formulation of the discrete problem takes somehow into account the backward character of viscosity solutions.

# Relation between $(\mathcal{H}Ja)$ and $(\mathcal{D}FEa)$

Let $a \geq a_0$. Then:

- Any solution to $(\mathcal{D}FEa)$ can be (uniquely) extended to a solution of $(\mathcal{H}Ja)$ in $\Gamma$. Conversely, the trace on $\mathcal{V}$ of any solution to $(\mathcal{H}Ja)$ in $\Gamma$ is solution to $(\mathcal{D}FEa)$.
- Any subsolution to $(\mathcal{D}FEa)$ can be ( uniquely) extended to a subsolution of $(\mathcal{H}Ja)$ in $\Gamma$. Conversely, the trace on $\mathcal{V}$ of any subsolution of $(\mathcal{H}Ja)$ in $\Gamma$ is subsolution to $(\mathcal{D}FEa)$.
- Similar results can be stated for subsets of $\Gamma$ and, consequently, of $\mathcal{V}$.

**Theorem**

There exists a unique $c$ such that $\mathcal{D}FEc$ admits solutions.

- $c \geq a_0$ is called Mañé critical value.
- It can be characterized in terms of the existence of vanishing cycles for the intrinsic semidistance $S_a(\cdot,\cdot)$ associated to $\sigma_a$. These cycles play an important role in the construction of solutions and in the definition of the (discrete) Aubry set.

# Summary of our main results I

I. GLOBAL SOLUTIONS

(i) (Existence) There exists a unique value $c = c(\mathcal{H}) \geq a_0$ – called Mañé critical value – for which the equation $\mathcal{H}(x, du) = c$ admits global solutions. In particular, these solutions are Lipschitz continuous on $\Gamma$.

(ii) (Uniqueness) There exists a *uniqueness set* $\mathcal{A}_\Gamma = \mathcal{A}_\Gamma(\mathcal{H}) \subseteq \mathcal{V}$ called the (projected) Aubry set of $\mathcal{H}$, such that the following holds. Given any *admissible trace* $g$ on $\mathcal{A}_\Gamma$, i.e., a function $g : \mathcal{A}_\Gamma \longrightarrow \mathbb{R}$ such that for every $x, y \in \mathcal{A}_\Gamma$

$$g(x) - g(y) \leq S_c(y, x),$$

($S_c(x, y)$ denotes the intrinsic semidistance corresponding to the energy value $c$) there exists a unique global solution $u \in C(\Gamma, \mathbb{R})$ to $\mathcal{H}(x, du) = c$ agreeing with $g$ on $\mathcal{A}_\Gamma$. Conversely, for any solution $u$ to $\mathcal{H}(x, du) = c$, the function $g = u_{|\mathcal{A}_\Gamma}$ gives rise to an admissible trace on $\mathcal{A}_\Gamma$.

(iii) (Hopf–Lax type representation formulae) Explicit representation formulae are provided both for global solutions and for solutions on subsets of $\Gamma$.

# Summary of our main results II

II. Subsolutions

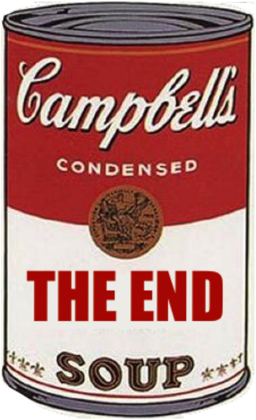
(i) (Maximal subsolutions) For $a \geq c$, $y \in \Gamma$, the maximal subsolution to $(\mathcal{H}Ja)$ taking an assigned value at $y$ is solution in $\Gamma \setminus \{y\}$.

(ii) (PDE characterization of the Aubry set) Let $\mathcal{A}^*_\Gamma = \mathcal{A}^*_\Gamma(\mathcal{H}) \subset \Gamma$ be the Aubry set (on the network). The maximal subsolution to $(\mathcal{H}Jc)$ taking a given value at a point $y \in \Gamma$ is a critical solution on the whole network if and only if $y \in \mathcal{A}^*_\Gamma$.

(iii) (Regularity of critical subsolutions) Any subsolution $v : \Gamma \to \mathbb{R}$ to $\mathcal{H}(x, du) = c$ is of class $C^1(\Gamma \setminus \mathcal{V})$ and they all possess the same differential on $\mathcal{A}^*_\Gamma \setminus \mathcal{V}$.

(iv) (Existence of $C^1$ critical subsolutions) Given an admissible trace $g : \mathcal{V} \longrightarrow \mathbb{R}$ there exists a critical subsolution $v$ on $\Gamma$, with $v = g$ on $\mathcal{V}$, which is of class $C^1$ on $\Gamma \setminus \mathcal{V}$.
In addition, there exists a critical subsolution $v$ of class $C^1(\Gamma \setminus \mathcal{V})$ which is strict outside $\mathcal{A}^*_\Gamma$.

(v) (Hopf–Lax type representation formulae) Explicit representation formulae are provided both for critical and supercritical subsolutions.




**Thank you for your attention.**