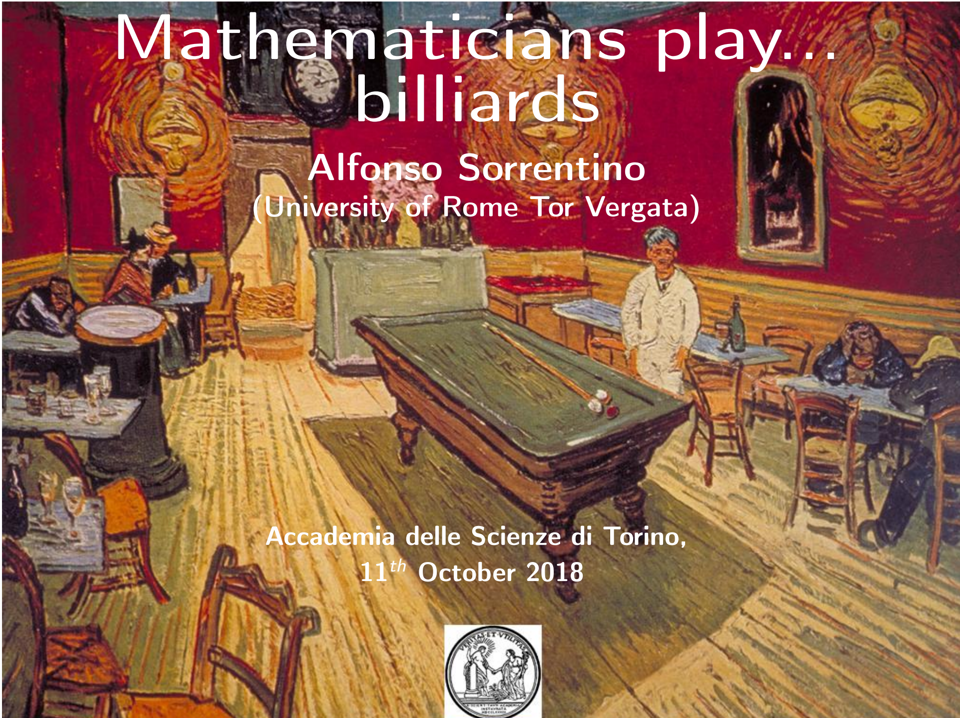

Mathematicians play... billiards

Alfonso Sorrentino
(University of Rome Tor Vergata)

Accademia delle Scienze di Torino,
11^{th} October 2018

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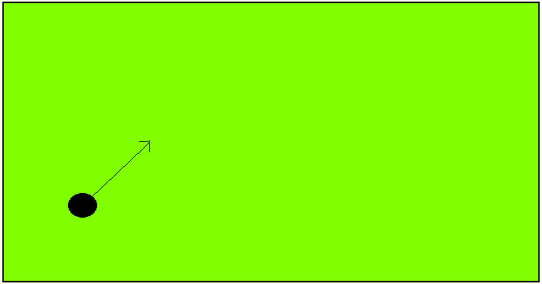

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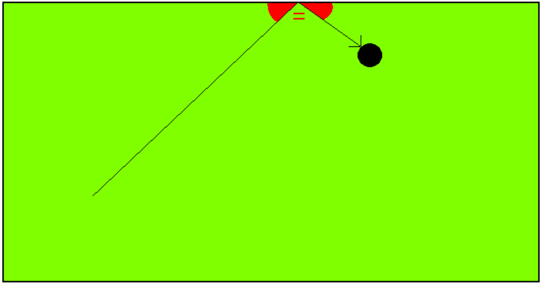

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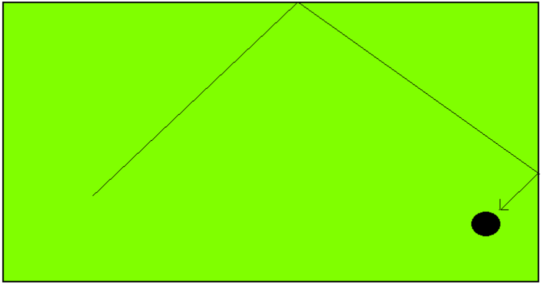

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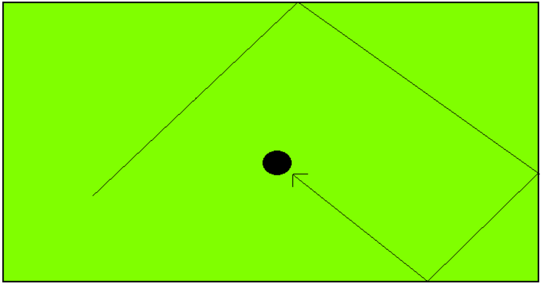

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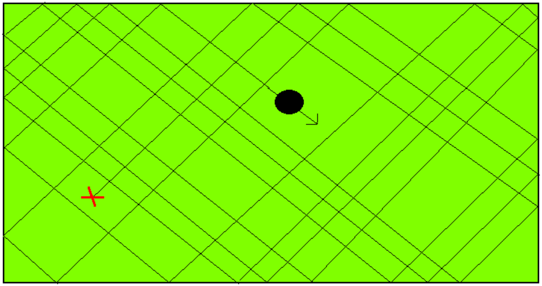

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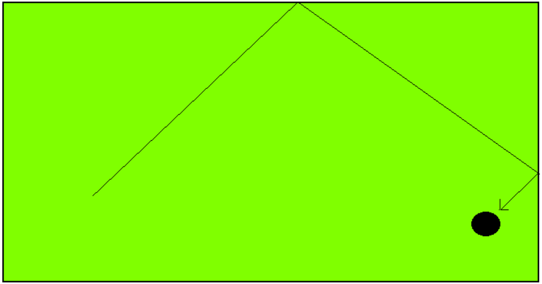
Observation: Between two consecutive bounces, the ball moves along a segment with constant velocity (nothing interesting happens!).
It suffices to know the points where the ball hits the boundary to reconstruct the whole dynamics!



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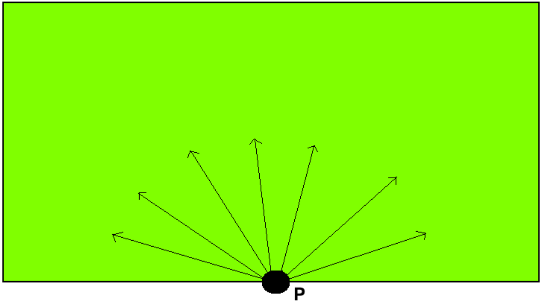

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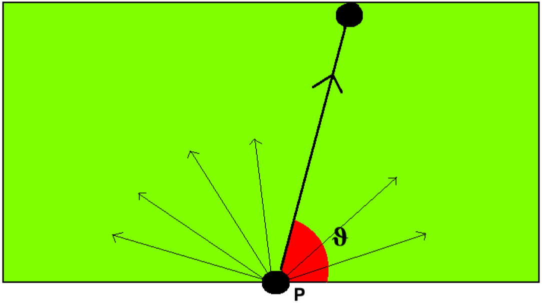

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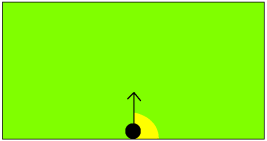

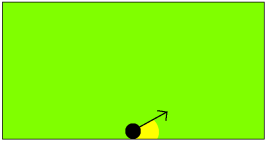

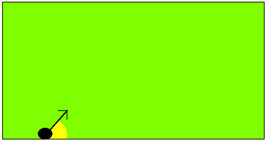

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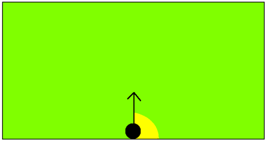
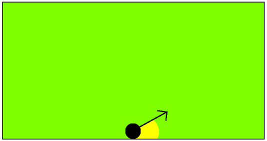
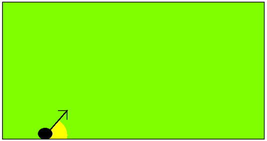
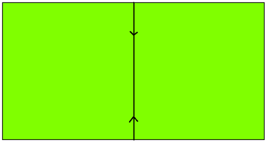






Periodic orbit

Number of bounces (period) $= 2$



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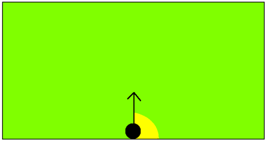

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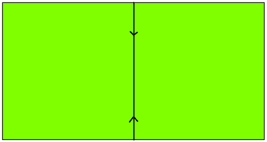

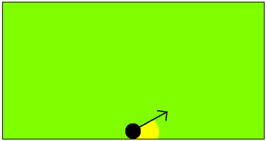

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Number of bounces (period)

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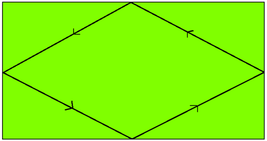

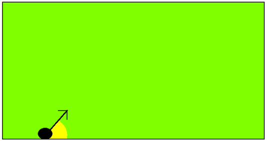

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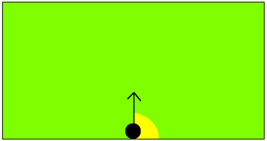

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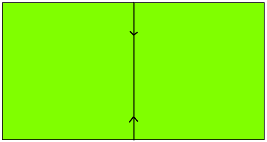

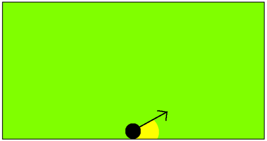

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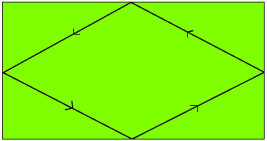
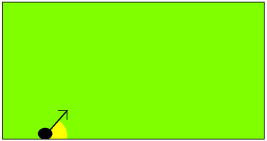
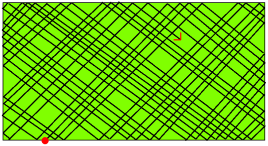




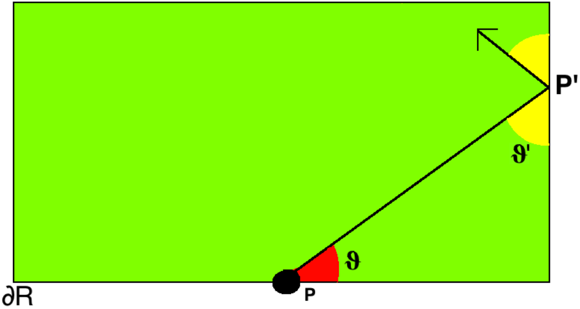
Non-periodic orbit



The Billiard Map

The billiard map is a map that to each initial pair (P, ϑ) associates the point at which the ball will hit the boundary next and the corresponding angle of incidence:

$$\begin{aligned} B : \partial R \times (0, \pi) &\longrightarrow \partial R \times (0, \pi) \\ (P, \vartheta) &\longrightarrow (P', \vartheta') \end{aligned}$$



A Mathematical Billiard is an example of Dynamical System

What is a Dynamical System?

It is a system whose state evolves in time.
Goal: To study and describe its evolution in time.

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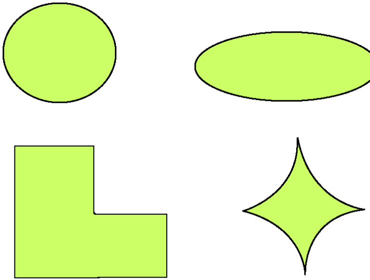
- Evolution: the law/map that allows one to deduce the "next" state, by knowing the current one (e.g., for the billiard, the map B).
- Time: it can be **continuous** (at every time we want to know the state of the ball) or **discrete** (we want to know the state of the ball only when it hits the boundary).

Why do we consider only rectangular billiards?

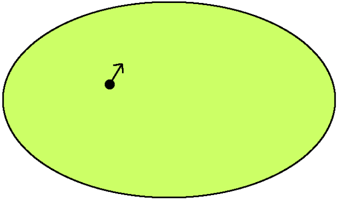
The dynamics inside a billiard is completely determined by its geometry (*i.e.*, its shape)!

One could choose billiard tables with different shapes:



One could also assume that the domain lies inside a Riemannian metric rather than the Euclidean plane.

Dynamics inside a general (Euclidean) table



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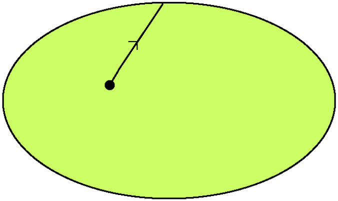

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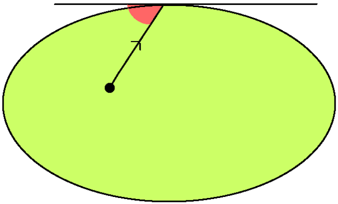

Reflection law: One considers the angle formed with the tangent line

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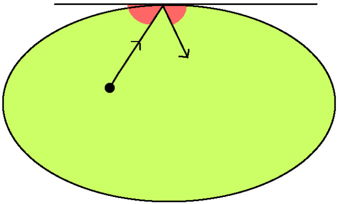
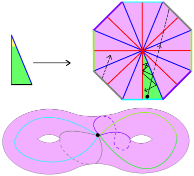


Reflection law: One considers the angle formed with the tangent line

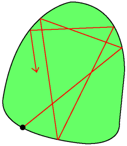
$$\text{angle of incidence} = \text{angle of reflection}$$

In the case of a table lying in a Riemannian manifold, the ball moves along geodesics instead of straight lines.

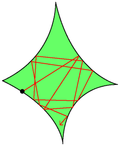
The study of the dynamics of billiards is a very active area of research. Dynamical behaviours and properties are strongly related to the shape of the table.



Polygonal billiards:
- Related to the study of the geodesic flow on a translation surface (with singular points);
- Teichmüller theory.



(Strictly) Convex Billiards:
- Birkhoff billiards (G. Birkhoff, 1927: paradigmatic example of Hamiltonian systems).
- The billiard map is a twist map.
- Coexistence of regular (KAM, Aubry-Mather) and chaotic dynamics.



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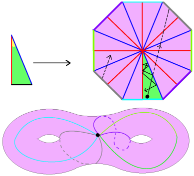

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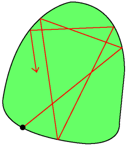

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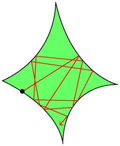

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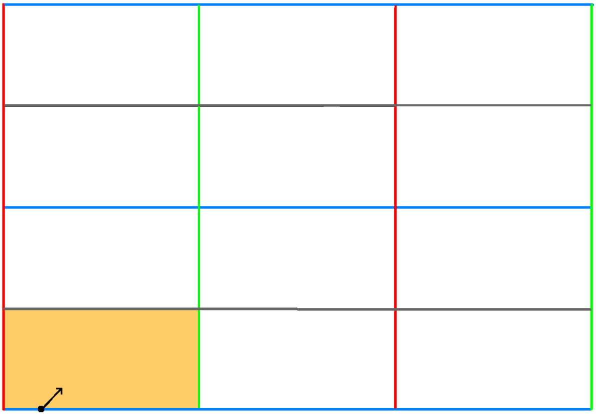
IDEA: Instead of reflecting the ball... Let us reflect the table!

"Though this be madness, yet there is method in 't" (Hamlet, Act II, Scene II)



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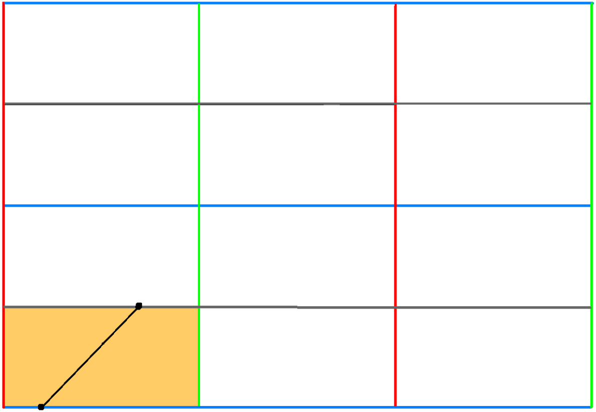

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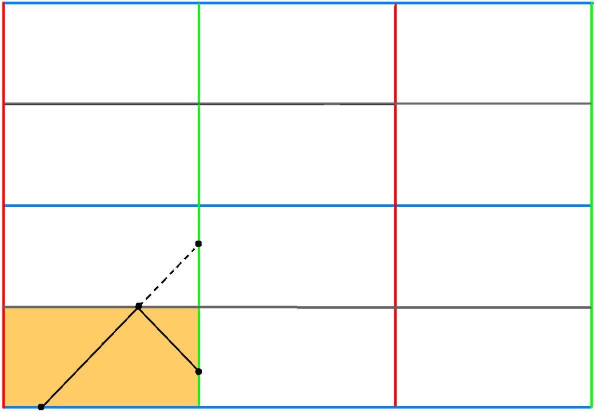

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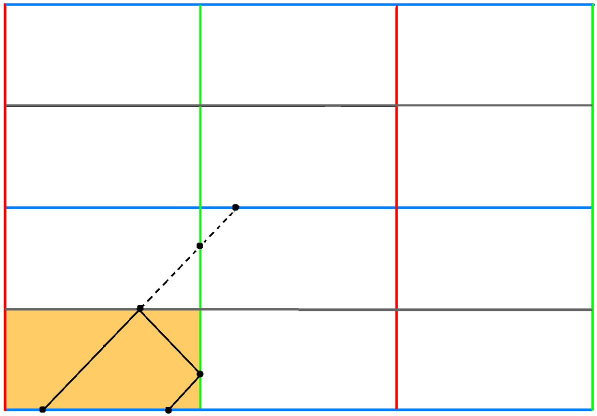

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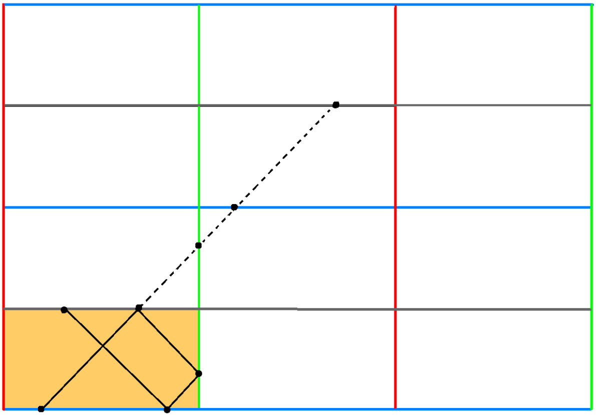

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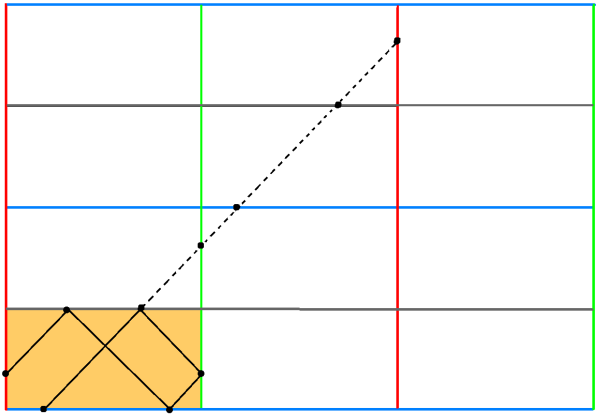

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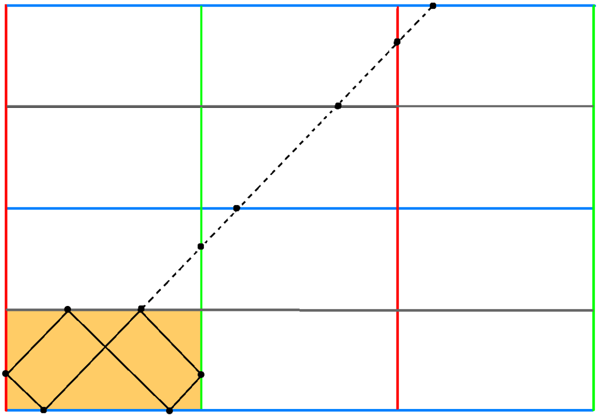

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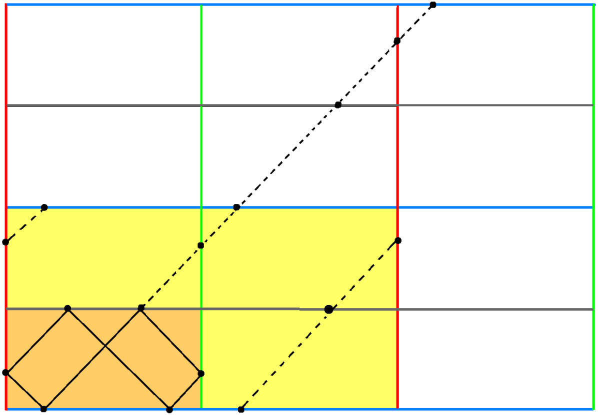

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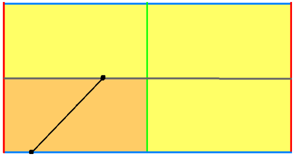

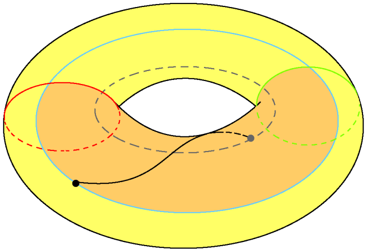

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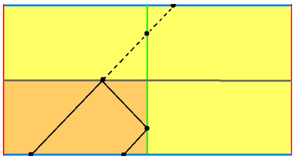

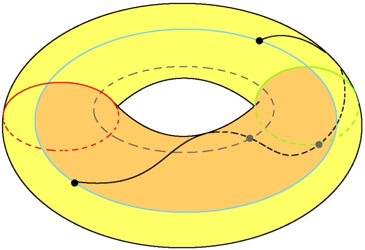

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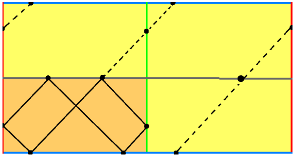

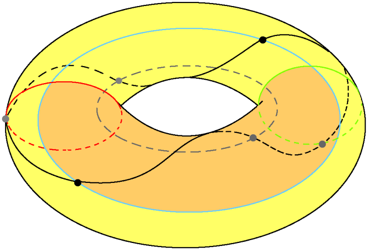

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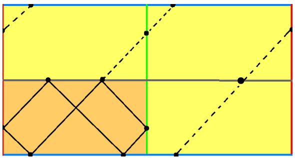
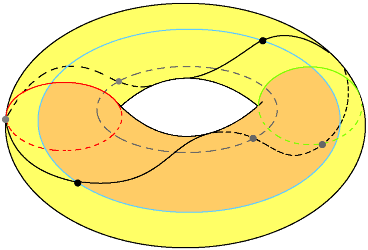




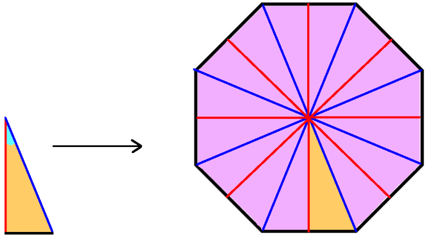
Billiard in a rectangle $\longrightarrow$ Geodesic (linear) flow on the Torus

Periodic orbits $\longrightarrow$ Closed Geodesics

From a polygonal billiard to a surface

IDEA(Katok, Zemlyakov): the same reasoning can be applied to other polygons whose angles are of the form $\frac{p_i}{q_i}\pi$, $p_i, q_i \in \mathbb{N}$ (Rational Billiards).

Example (Right triangle with one angle of $\frac{\pi}{8}$):



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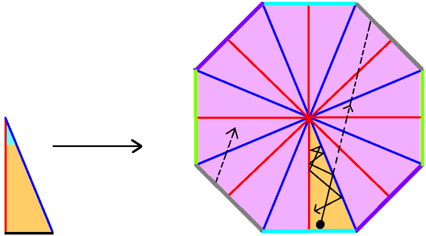

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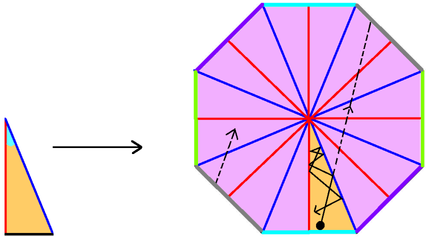
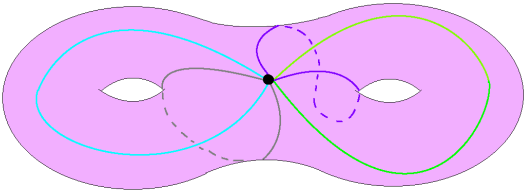
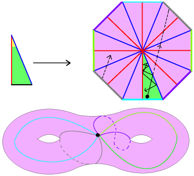
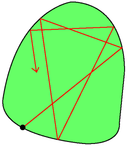




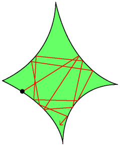
The study of the dynamics of billiards is a very active area of research. Dynamical behaviours and properties are strongly related to the shape of the table.



Polygonal billiards:
- Related to the study of the geodesic flow on a translation surface (with singular points);
- Teichmüller theory.



(Strictly) Convex Billiards:
- Birkhoff billiards (G. Birkhoff, 1927: paradigmatic example of Hamiltonian systems).
- The billiard map is a twist map.
- Coexistence of regular (KAM, Aubry-Mather) and chaotic dynamics.



Concave Billiards (or dispersive):
- Nearby Orbits tend to move apart (exponentially).
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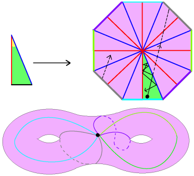

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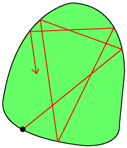

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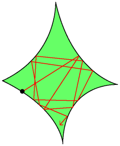

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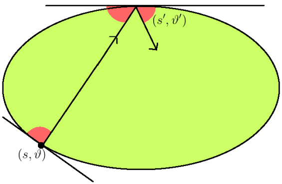
Birkhoff Billiards

Let Ω be a strictly convex domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ (fix an orientation). Let ϑ the shooting angle (w.r.t. the positive tangent to $\partial\Omega$). The Billiard map is:

$$\begin{array}{rcl} B : \partial\Omega \times (0,\pi) & \longrightarrow & \partial\Omega \times (0,\pi) \\ (s,\vartheta) & \longmapsto & (s',\vartheta'). \end{array}$$



This simple model has been first proposed by G.D. Birkhoff (1927) as a mathematical playground where "*the formal side, usually so formidable in dynamics, almost completely disappears and only the interesting qualitative questions need to be considered*".

George D. Birkhoff (1884-1944)

Dynamics $\longleftrightarrow$ Geometry

Study of Dynamics: understand the properties of orbits (periodicity, symmetries, chaos, etc...)

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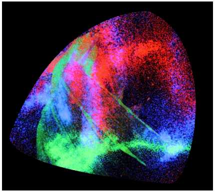
While the dependence of the dynamics on the geometry of the domain is well perceptible, an intriguing challenge is:

To what extent dynamical information can be used
to reconstruct the shape of the domain.



This apparently naïve question is at the core of different intriguing conjectures, among the most difficult to tackle in the study of dynamical systems!

Example I: Circular billiard

Digression: A Mad Tea-Party

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Charles Lutwidge Dodgson (1832-1898)
(better known as Lewis Carroll).

'But I don't want to go among mad people', Alice remarked. 'Oh, you can't help that', said the Cat: 'we're all mad here. You're mad.' 'How do you know I'm mad?', said Alice. 'You must be', said the Cat, 'or you wouldn't have come here.'

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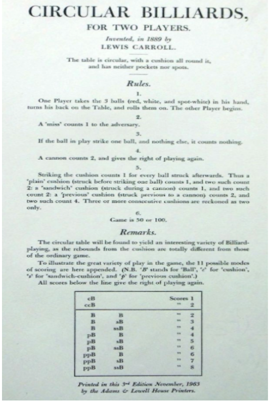


CIRCULAR BILLIARDS,

FOR TWO PLAYERS.

Invented, in 1889 by
LEWIS CARROLL.

The table is circular, with a cushion all round it, and has neither pockets nor spots.

Rules.

1.

One Player takes the 3 balls (red, white, and spot-white) in his hand, turns his back on the Table, and rolls them on. The other Player begins.

2.

A 'miss' counts 1 to the adversary.

3.

If the ball in play strike one ball, and nothing else, it counts nothing.

4.

A cannon counts 2, and gives the right of playing again.

5.

Striking the cushion counts 1 for every ball struck afterwards. Thus a 'plain' cushion (struck before striking one ball) counts 1, and two such count 2: a 'sandwich' cushion (struck during a cannon) counts 1, and two such count 2: a 'previous' cushion (struck previous to a cannon) counts 2, and two such count 4. Three or more consecutive cushions are reckoned as two only.

6.

Game is 50 or 100.

Remarks.

The circular table will be found to yield an interesting variety of Billiard-playing, as the rebounds from the cushion are totally different from those of the ordinary game.

To illustrate the great variety of play in the game, the 11 possible modes of scoring are here appended. (N.B. '*B*' stands for 'Ball', '*c*' for 'cushion', '*s*' for 'sandwich-cushion', and '*p*' for 'previous cushion'.)

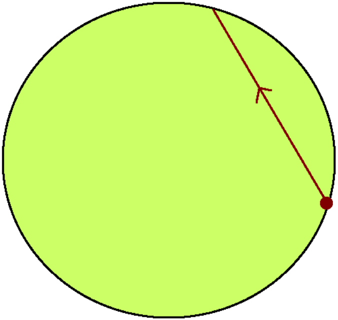
All scores below the line give the right of playing again.

cB		Scores	1
ccB		"	2
B	B	"	2
B	sB	"	3
B	ssB	"	4
pB	B	"	4
pB	sB	"	5
pB	ssB	"	6
ppB	B	"	6
ppB	sB	"	7
ppB	ssB	"	8

Printed in this 3rd Edition November, 1963 by the Adams & Lowell House Printers.

Lewis Carroll thought of playing billiards on a circular table in 1889 and first published its rules the following year (and a circular billiard table was actually made for him!)

Example I: Circular billiard



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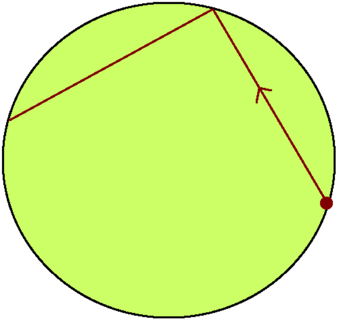

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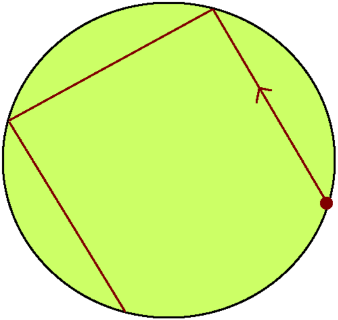

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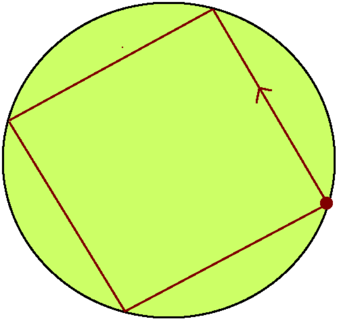

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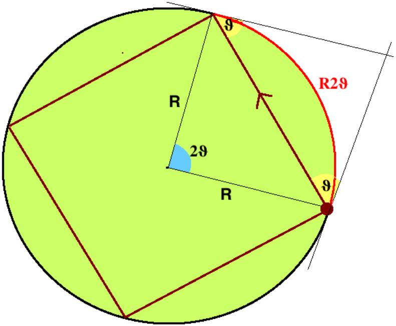
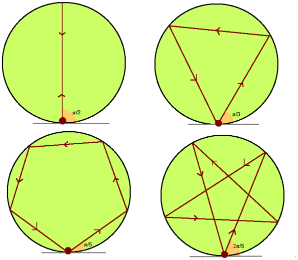


The angle remains constant at each bounce: it is an Integral of motion.

Example I: Circular billiard

If ϑ is a rational multiple of π, then the orbit is periodic.



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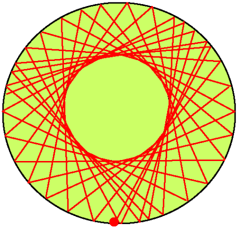

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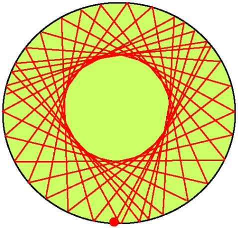

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The trajectory is always tangent to a circle (this is an example of caustic).

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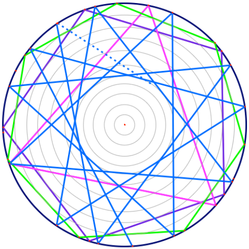

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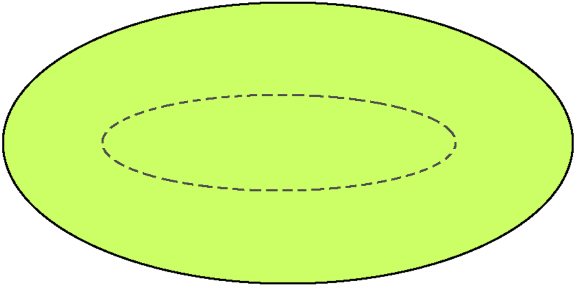
The disc is foliated by convex caustics $\longrightarrow$ Integrable billiard.

Caustics

A convex caustic is a closed C^1 curve in the interior of Ω, bounding itself a strictly convex domain, with the property that each trajectory that is tangent to it stays tangent after each reflection.



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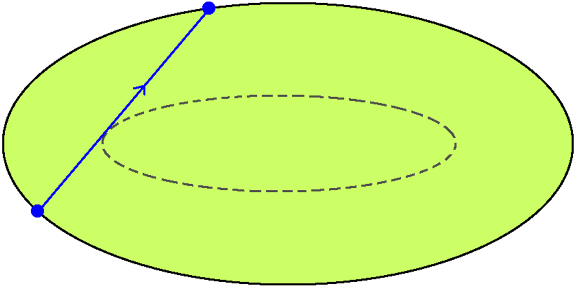

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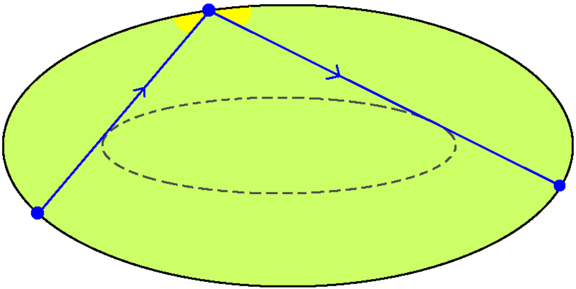

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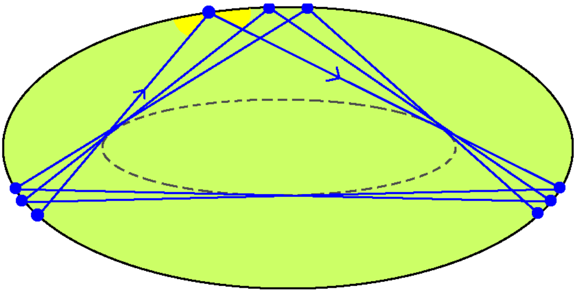

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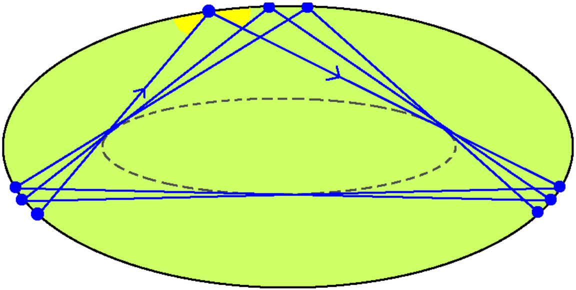
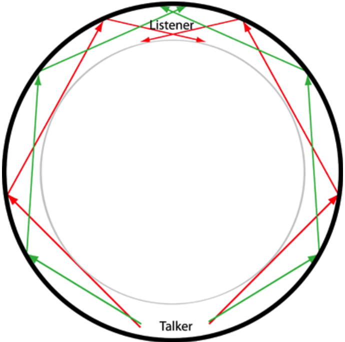


To a convex caustic in Ω corresponds an invariant circle for the billiard map.

(The converse is not entirely true: invariant curves give rise to caustics, but they might not be convex, nor differentiable).

Caustics and Whispering Galleries

Whispering Gallery in St. Paul Cathedral in London (Lord Rayleigh, 1878 ca.)



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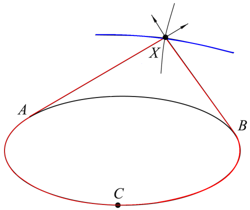

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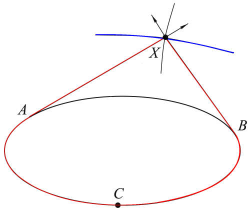

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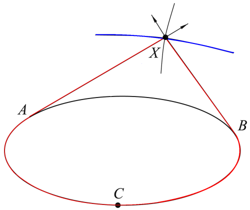

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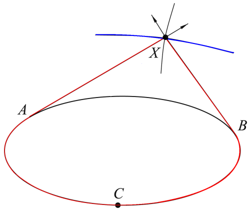

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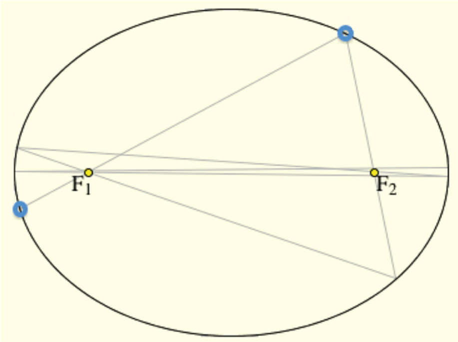
- Do there exist other examples of billiards admitting a foliation by caustics?

Example II: Elliptic billiard

Curiosity: The New York Times (1st July 1964) ran a full-page ad for Elliptipool, played on an elliptical table with a single pocket at one of the two foci. The ad said that on the following day the game would be demonstrated at Stern's department store by movie stars Paul Newman and Joanne Woodward.

Example II: Elliptic billiard



If the trajectory passes through one of the foci, then it always passes through them, alternatively.

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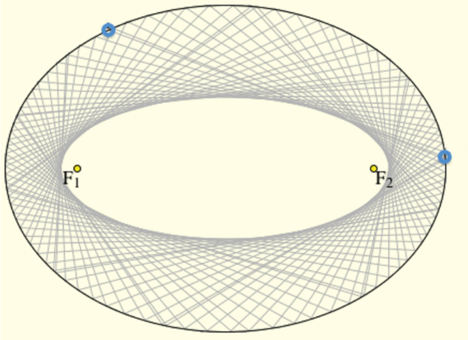

If the trajectory does not intersect the segment between the foci, then it never does and it is tangent to a confocal ellipse (a convex caustic).

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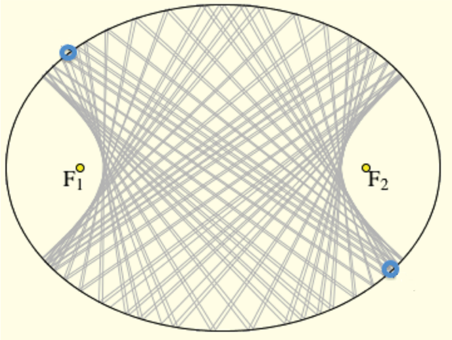

If the trajectory intersects the segment between the foci, then it always does and it is tangent to a confocal hyperbola (a non-convex caustic).

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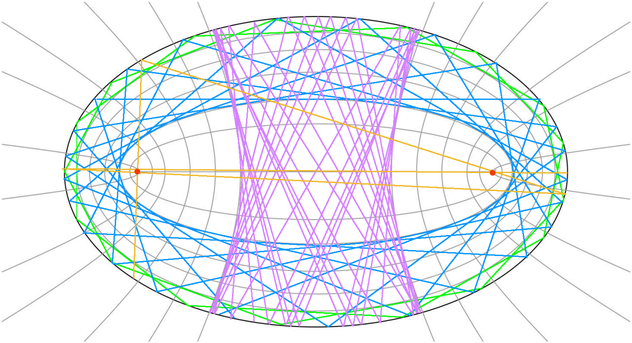

The elliptic billiard is also foliated by convex caustics (with the exception of the segment between the two foci).

Birkhoff conjecture

Conjecture (Birkhoff-Poritsky)

The only integrable billiard maps correspond to billiards inside ellipses.

Although some vague indications of this question can be found in Birkhoff's works (1920's-30's), its first appearance was in a paper by Poritsky (1950), who was a National Research Fellow in Mathematics at Harvard University, presumably under the supervision of Birkhoff.

It quickly became one of the most famous - and hard to tackle - questions in dynamical systems.

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J. Mather (1982): non-existence of caustics if the curvature of the boundary vanishes at (at least) one point.

M. Bialy (1993): If the phase space of the billiard map is completely foliated by continuous invariant curves which are not null-homotopic, then it is a circular billiard.

Perturbative Birkhoff conjecture

One could restrict the analysis to what happens for domains that are sufficiently close to ellipses.

Birkhoff Conjecture (Perturbative version)

A smooth strictly convex domain that is sufficiently close (w.r.t. some topology) to an ellipse and whose corresponding billiard map is integrable, is necessarily an ellipse.

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In a recent work in collaboration with Vadim Kaloshin (Univ. of Maryland, USA), we proved that this version of the conjecture holds TRUE!

V. Kaloshin, A. Sorrentino,
"On the local Birkhoff conjecture for convex billiards"
Annals of Mathematics 188 (1): 315–380, 2018.

(Advertisement: For more details, come tomorrow to the Analysis Seminar, Palazzo Campana, Sala S, 9:30 am)

Thank you
for your attention

Guido Fubini-Ghiron
(1879-1943)

... and for the prize!