The Hamilton-Jacobi Equation on Networks:

From Weak KAM and Aubry-Mather Theories to Homogenization

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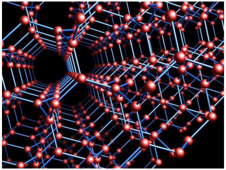
Introduction

Over the last years there has been an increasing interest in the study of the Hamilton-Jacobi (HJ) equation on networks and related problems.

These problems:

- involve a number of subtle theoretical issues;
- have a great impact in the applications in various fields, e.g., to data transmission, traffic management problems, *etc*...



While locally, *i.e.*, on each branch (arc) of the network, the study reduces to the analysis of 1-dimensional problems, the main difficulties arise in:

- matching together information converging at the juncture of two or more arcs;
- relating the local analysis at a juncture with the global structure (topology) of the network.

Plan of the Talk:

In this talk I shall discuss some works in collaboration with Antonio Siconolfi (Sapienza, Università di Roma).

I. Global PDE results for (Eikonal) HJ equations on networks (weak KAM)
(A. Siconolfi, A.S., *Analysis & PDE*, 2018)

- Solutions:
  - existence of a (unique) critical value for which global solutions exist,
  - determination of a uniqueness set (Aubry set),
  - Hopf-Lax type representation formulae, *etc*...
- Critical case:
  - properties, regularity, existence of C^1 critical subsolutions, *etc*...
- Supercritical case:
  - properties, representation formulae for maximal subsolutions, *etc*...
  - Existence and uniqueness of solutions on subsets of the network, continuously extending admissible data on the complement.

II. Action-minimizing properties on networks (Aubry-Mather theory)

(A. Siconolfi, A.S., Preprint 2020)

- Action-minimizing measures:
  - Definition, existence and properties of action-minimizing measures (Mather measures).
  - Definition of Mather sets and its structural properties (graph property).
  - Definition and properties of Mather's α and β functions (effective Hamiltonian and Lagrangian).
- Connection with weak KAM theory.

III. Homogenization on periodic networks (Work in progress)

- (Gromov-Hausdorff) Limit of periodic networks (topologycal crystals);
- Limit problem and homogenization.

Main ideas

The main rationale consists in neatly distinguishing between:

1) The local problem on the arcs:

- (classical) 1-dimensional viscosity or variational techniques.

2) The global matching on the network:

- we associate to the network an abstract graph, encoding all of the information on the complexity of the network;
- we relate the problems to a discrete problems on the graph, to be studied by means of techniques inspired by weak KAM and Aubry-Mather theories.

3) Combine:

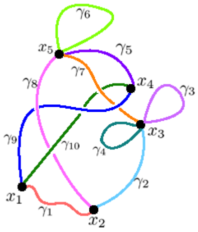
- the global analysis (on the abstract graph)
- the local analysis on the arcs of the network.

The Network

An embedded network is a compact subset Γ in $(\mathbb{R}^N, d_{\mathrm{eucl}})$, or in any Riemannian manifold (M, g), of the form

$$\Gamma = \bigcup_{\gamma \in \mathcal{E}} \gamma \subset \mathbb{R}^N,$$

where $\mathcal{E}$ is a finite collection of arcs, *i.e.*, simple C^1 regular (oriented) curves, that are disjoint, except at the end-points (called vertices). We denote the set of vertices by $\mathcal{V}$.



Observe that Γ inherits:

- a metric d_Γ from the ambient space, hence a topology; we assume that Γ is path-connected.
- a differential structure (vertices are special points).

The Network

We introduce the following maps:

- A fixed-point-free involution $^{-} : \mathcal{E} \longrightarrow \mathcal{E}$ that to each arc $\gamma \in \mathcal{E}$ associates the arc $\overline{\gamma} \in \mathcal{E}$, *i.e.*, the same arc with opposite orientation(reversed arc).
- The map $\mathrm{o} : \mathcal{E} \longrightarrow \mathcal{V}$ which associates to each oriented arc $\gamma \in \mathcal{E}$ its initial vertex $\mathrm{o}(\gamma) \in \mathcal{V}$ (origin).
- The map $\mathrm{t} : \mathcal{E} \longrightarrow \mathcal{V}$ which associates to each oriented arc $\gamma \in \mathcal{E}$ its final vertex $\mathrm{t}(\gamma) \in \mathcal{V}$ (end).

In particular, for each $\gamma \in \mathcal{E}$:

$$\mathrm{t}(\gamma) = \mathrm{o}(\overline{\gamma}) \quad \text{and} \quad \mathrm{t}(\overline{\gamma}) = \mathrm{o}(\overline{\overline{\gamma}}) = \mathrm{o}(\gamma).$$

It follows from the connectedness assumption on Γ, that the maps o and t are surjective.

Hamiltonians on the network

A Hamiltonian on a network Γ is a function $\mathcal{H} : T^*\Gamma \longrightarrow \mathbb{R}$. For each $\gamma \in \mathcal{E}$, let us denote by H_γ the restriction on the Hamiltonian on $T^*\gamma$ (vertices included); we ask each H_γ to satisfy the following conditions:

- H_γ is continuous on $T^*\gamma$;
- H_γ is coercive in each fiber $T^*_x\gamma$, where $x \in \gamma$;
- H_γ is quasi-convex in each fiber, namely the set $\{H_\gamma \leq a\} \cap T^*_x\gamma$ is convex (if nonempty) for every $a \in \mathbb{R}$ and $x \in \gamma$.
- (compatibity condition) $H_\gamma(x,p) = H_{\overline{\gamma}}(x,-p)$ for every $(x,p) \in T^*\gamma$.

$+$ Extra condition related to their critical values (see next slide).

Hamiltonians corresponding to geometrically different arcs are unrelated, even for arcs with some vertex in common.
No continuity or compatibility conditions at common vertices!

Critical values for H_γ

We set for any $\gamma \in \mathcal{E}$

$$
\begin{aligned}
a_\gamma &:= \max_{x\in\gamma} \min_{T_x^*\gamma} H_\gamma \qquad \big(\text{From compatibility conditions: } a_\gamma = a_{\overline{\gamma}}\big) \\
c_\gamma &:= \min\{a \in \mathbb{R} : \ H_\gamma(x, du) = a \text{ admits periodic subsolutions}\}.
\end{aligned}
$$

By periodic subsolution, we mean subsolution to the equation in γ taking the same value at the endpoints.

The definition of c_γ is well-posed and $a_\gamma \leq c_\gamma$ for any $\gamma \in \mathcal{E}$.

We define

$$a_0 := \max\left\{ \max_{\gamma\in\mathcal{E}\setminus\mathcal{E}^*} a_\gamma, \max_{\gamma\in\mathcal{E}^*} c_\gamma \right\},$$

where $\mathcal{E}^* \subset \mathcal{E}$ denotes the subset of arcs which are loops:

$$\mathcal{E}^* := \{\gamma \in \mathcal{E} : \ \mathrm{o}(\gamma) = \mathrm{t}(\gamma)\}.$$

Extra condition on the Hamiltonians

We require a further condition:

- Given any $\gamma \in \mathcal{E}$ with $a_\gamma = a_0$, the map $x \in \gamma \longmapsto \min_{T_x^*\gamma} H_\gamma$ is constant.

Notice that this condition is automatically satisfied if the H_γ's are independent of the state variable.

The main rôle of this condition is to ensure uniqueness of solutions to the Dirichlet problem associated to the equation $H_\gamma = a_\gamma$ for any value assigned at $\mathrm{o}(\gamma)$, at least for the γ's with $a_\gamma = a_0$.

Note: The uniqueness property for such kind of problems holds in general when the equation $H_\gamma = a$ admits a strict subsolution (for example when $a > a_\gamma$), which is not the case at level a_γ. This condition ensures that the family of subsolutions to $H_\gamma = a_\gamma$ reduces to a singleton up to a constant.

The Eikonal HJ equation on networks

We consider the equation

$$\mathcal{H}(x, du) = a \quad \text{on } \Gamma. \tag{$\mathcal{H}Ja$}$$

This notation synthetically indicates the family (for γ varying in $\mathcal{E}$) of Hamilton–Jacobi equations $H_\gamma(x, du) = a$ on $\gamma \setminus \{\mathrm{o}(\gamma), \mathrm{t}(\gamma)\}$.

On a single arc, these equations possess infinitely many (viscosity) solutions, depending on the boundary data at $\mathrm{o}(\gamma)$ and $\mathrm{t}(\gamma)$.
We need to introduce suitable viscosity tests on the vertices so to:

- select a unique solution on any arc;
- match these (local) solutions in a continuous way at vertices.

Two basic properties are needed (true under our assumptions on H_γ's):
- existence and uniqueness of solutions on any arc, coupled with suitable Dirichlet boundary conditions at $\mathrm{o}(\gamma)$ and $\mathrm{t}(\gamma)$;
- characterization of the maximal (sub)solution with a given datum at $\mathrm{o}(\gamma)$.

Notion of (sub)solution in our setting

Definition of subsolution

We say that $u : \Gamma \longrightarrow \mathbb{R}$ is subsolution to $(\mathcal{H}Ja)$ if
i) it is continuous on Γ;
ii) it is (viscosity) subsolution on each $\gamma \setminus \{\mathrm{o}(\gamma), \mathrm{t}(\gamma)\}$, for any $\gamma \in \mathcal{E}$.

Given a continuous function w on γ, we say that a C^1 function φ is a constrained subtangent to w at $\mathrm{t}(\gamma)$ if $w = \varphi$ at $\mathrm{t}(\gamma)$ and $w \geq \varphi$ in a sufficiently small open neighborhood of $\mathrm{t}(\gamma)$ (*cfr.* Soner, 1986).

Definition of solution

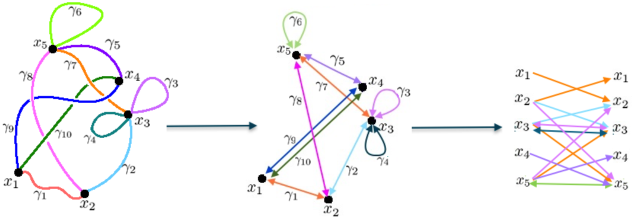
We say that $u : \Gamma \longrightarrow \mathbb{R}$ is solution to $(\mathcal{H}Ja)$ if
i) it is continuous on Γ;
ii) it is a (viscosity) solution on each $\gamma \setminus \{\mathrm{o}(\gamma), \mathrm{t}(\gamma)\}$, for any $\gamma \in \mathcal{E}$;
iii) (state constraint boundary conditions) for every vertex x there is an arc γ with $\mathrm{t}(\gamma) = x$ such that any constrained C^1 subtangent φ to $u\big|\gamma$ at $\mathrm{t}(\gamma)$ satisfies $H_\gamma(x, d\varphi(x)) \geq a$.

Some remarks

- In the definition of subsolution no conditions are required on vertices. These assumptions are minimal. The validity of this approach is supported by the fact that the notion of solutions can be recovered in terms of maximal subsolution attaining a specific value at a given point (vertex or internal).
- In the definition of solution there are no mixing conditions between equations on different arcs incident at the same vertex.
- The (unique) place where the global topology of Γ plays a rôle is iii).
- The constraint boundary condition at $\mathrm{t}(\gamma)$ selects the maximal solution taking a given value at $\mathrm{o}(\gamma)$. In a sense, it leaves a degree of freedom at $\mathrm{o}(\gamma)$, which can be exploited to get solutions to the HJ equations on any arc, that match continuously.
- If γ is a loop, we must have in addition $u(\mathrm{o}(\gamma)) = u(\mathrm{t}(\gamma))$, *i.e.*, periodicity. This explains why for $\gamma \in \mathcal{E}^*$ we must consider the value c_γ.
- If the network is augmented by changing the status of a finite number of intermediate points of arcs in Γ, which become new vertices, then the notion of solution is not affected.

From the network to the abstract graph

The main novelty of our method is to put in relation the HJ equation on the network to a discrete functional equation on the underlying abstract graph $\Gamma = (\mathcal{E}, \mathcal{V})$, where $\mathcal{E}$ is the (abstract) set of arcs and $\mathcal{V}$ the (abstract) set of vertices.



- When referring to the abstract graph, we think of elements of $\mathcal{E}$ as immaterial edges (we use the same notation).
- We say that $\xi = (\gamma_1, \dots, \gamma_M)$ is a path linking two vertices $x, y \in \mathcal{V}$ if
  - $\gamma_i \in \mathcal{E}$ for every $i = 1, \dots, M$,
  - $\mathrm{o}(\gamma_1) = x$ and $\mathrm{o}(\gamma_M) = y$,
  - $\mathrm{t}(\gamma_i) = \mathrm{o}(\gamma_{i+1})$ for every $i = 1, \dots, M - 1$.

From the network to the abstract graph

The subsequent step is to transfer the Hamilton-Jacobi equation from Γ to the abstract graph, where it will take the form of a discrete functional equation.

For any $\gamma \in \mathcal{E}$ and $a \geq a_\gamma$, the relevant information to transfer is

$$\sigma_a(\gamma) := \int_\gamma \sigma^+_{a,\gamma}(x)\, dx$$

where $\sigma^+_{a,\gamma}(x) = \max\{p : \; H_\gamma(x,p) = a\}$.

- $\sigma_a(\gamma)$ is the value at $\mathrm{t}(\gamma)$ of the maximal subsolution to $H_\gamma(x, du) = a$ on γ, vanishing at $\mathrm{o}(\gamma)$.
- This object can be used to define semi-distances on the abstract graph Γ. If $x, y \in \mathcal{V}$ and $a \geq a_0$:

$$S_a(x,y) := \inf\{\sigma_a(\xi) : \; \xi \text{ is a path in } \gamma \text{ linking } x \text{ to } y$$

where if $\xi = (\gamma_1, \ldots, \gamma_M)$ then $\sigma_a(\xi) := \sum_{i=1}^M \sigma_a(\gamma_i)$.

The discrete functional equation

Let $a \geq a_0$. Let us start by observing the following admissibility condition:

- there exists a subsolution on γ attaining the values α and β at, respectively, $\mathrm{o}(\gamma)$ and $\mathrm{t}(\gamma)$, if and only if $-\sigma_a(\overline{\gamma}) \leq \beta - \alpha \leq \sigma_a(\gamma)$.

If $u : \Gamma \to \mathbb{R}$ is a subsolution to $(\mathcal{H}Ja)$, then

$$u(x) \leq \min_{\gamma \in \mathcal{E}, \mathrm{o}(\gamma) = x} \big(u(\mathrm{t}(\gamma)) + \sigma_a(\overline{\gamma}) \big) \qquad \forall\, x \in \mathcal{V}.$$

If $u : \Gamma \to \mathbb{R}$ is a solution to $(\mathcal{H}Ja)$, then equality holds at each $x \in \mathcal{V}$.

We introduce the following discrete functional equation:

$$\boxed{u(x) = \min_{\gamma \in \mathcal{E}, \mathrm{o}(\gamma) = x} \big(u(\mathrm{t}(\gamma)) + \sigma_a(\overline{\gamma}) \big)} \qquad \forall\, x \in \mathcal{V}. \qquad (\mathcal{D}FEa)$$

Note: Equality is required only at (at least) one arc for each vertex!
Moreover, the formulation of the discrete problem takes somehow into account the backward character of viscosity solutions.

Relation between $(\mathcal{H}Ja)$ and $(\mathcal{D}FEa)$

Proposition

Let $a \geq a_0$. Then:

- Any solution to $(\mathcal{D}FEa)$ can be (uniquely) extended to a solution of $(\mathcal{H}Ja)$ in Γ. Conversely, the trace on $\mathcal{V}$ of any solution to $(\mathcal{H}Ja)$ in Γ is solution to $(\mathcal{D}FEa)$.
- Any subsolution to $(\mathcal{D}FEa)$ can be (uniquely) extended to a subsolution of $(\mathcal{H}Ja)$ in Γ. Conversely, the trace on $\mathcal{V}$ of any subsolution of $(\mathcal{H}Ja)$ in Γ is subsolution to $(\mathcal{D}FEa)$.
- Similar results can be stated for subsets of Γ and, consequently, of $\mathcal{V}$.

Therefore, the study of $(\mathcal{H}Ja)$ reduces to the study of $(\mathcal{D}FEa)$.

Question: For which value(s) of a (if any) do $(\mathcal{D}FEa)$ admit solutions?

Critical Value

Theorem

There exists a unique $c = c(\mathcal{H})$ such that $\mathcal{D}FEc$ admits solutions.

- $c \geq a_0$ is called critical value (or Mañé critical value).
- c can be characterized in terms of the finiteness of the intrinsic semidistance $S_a(\cdot,\cdot)$ or in terms of the the existence of vanishing cycles (*i.e.*, closed path). More specifically:
  - $S_a(\cdot,\cdot) \not\equiv -\infty$ if and only if $a \geq c$.
  - There exists a closed path ξ such that $\sigma_a(\xi) = 0$ if and only if $a = c$ (which is equivalent to say that $S_a(x,x) = 0$ for some $x \in \mathcal{V}$).
- We define the Aubry set as

$$\mathcal{A}^*_\Gamma(\mathcal{H}) := \{\gamma \in \mathcal{E} : \text{ belonging to some cycle with } \sigma_c(\xi) = 0\}$$

and the projected Aubry set as

$$\mathcal{A}_\Gamma(\mathcal{H}) := \{x \in \mathcal{V} : \; S_c(x,x) = 0\}.$$

Main results I

I. Global Solutions

(i) (Existence) There exists a unique value $c = c(\mathcal{H}) \geq a_0$ – called Mañé critical value – for which the equation $\mathcal{H}(x, du) = c$ admits global solutions. In particular, these solutions are Lipschitz continuous on Γ.

(ii) (Uniqueness) There exists a *uniqueness set* $\mathcal{A}_\Gamma := \mathcal{A}_\Gamma(\mathcal{H}) \subseteq \mathcal{V}$ called the (projected) Aubry set of $\mathcal{H}$, such that the following holds. Given any *admissible trace* g on $\mathcal{A}_\Gamma$, i.e., a function $g : \mathcal{A}_\Gamma \longrightarrow \mathbb{R}$ such that for every $x, y \in \mathcal{A}_\Gamma$

$$g(x) - g(y) \leq S_c(y, x),$$

there exists a unique global solution $u \in C(\Gamma, \mathbb{R})$ to $\mathcal{H}(x, du) = c$ agreeing with g on $\mathcal{A}_\Gamma$:

$$u(x) = \min\{g(y) + \sigma_c(\xi) : \ y \in \mathcal{A}_\Gamma, \ \xi \text{ path linking } x \text{ to } y\}$$

Conversely, for any solution u to $\mathcal{H}(x, du) = c$, the function $g = u_{|\mathcal{A}_\Gamma}$ gives rise to an admissible trace on $\mathcal{A}_\Gamma$.

(iii) (Hopf–Lax type representation formulae) Explicit representation formulae are provided both for global solutions and for solutions on subsets of Γ.

Main results II

II. SUBSOLUTIONS

(i) (Maximal subsolutions) For $a \geq c$, $y \in \Gamma$, the maximal subsolution to $(\mathcal{H}Ja)$ taking an assigned value at y is solution in $\Gamma \setminus \{y\}$.

(ii) (PDE characterization of the Aubry set) Let $\mathcal{A}^*_\Gamma = \mathcal{A}^*_\Gamma(\mathcal{H}) \subset \Gamma$ be the Aubry set (on the network). The maximal subsolution to $(\mathcal{H}Jc)$ taking a given value at a point $y \in \Gamma$ is a solution on the whole network if and only if $y \in \mathcal{A}^*_\Gamma$.

(iii) (Regularity of critical subsolutions) Any subsolution $v : \Gamma \to \mathbb{R}$ to $\mathcal{H}(x, du) = c$ is of class $C^1(\mathcal{A}^*_\Gamma \setminus \mathcal{V})$ and they all possess the same differential on $\mathcal{A}^*_\Gamma \setminus \mathcal{V}$.

(iv) (Existence of C^1 critical subsolutions) Given an admissible trace $g : \mathcal{V} \longrightarrow \mathbb{R}$ there exists a critical subsolution v on Γ, with $v = g$ on $\mathcal{V}$, which is of class C^1 on $\Gamma \setminus \mathcal{V}$.
In addition, there exists a critical subsolution v of class $C^1(\Gamma \setminus \mathcal{V})$ which is strict outside $\mathcal{A}^*_\Gamma$.

(v) (Hopf–Lax type representation formulae) Explicit representation formulae are provided both for critical and supercritical subsolutions.

Advantages and novelties

- Global analysis that goes beyond what happens at a single juncture.
- The Network is only assumed to be finite and connected
  $\longrightarrow$ multiple arcs between two vertices and loops are allowed.
- Hamiltonians are assumed continuous, quasi-convex and coercive.
  $\longrightarrow$ No compatibility conditions at the vertices are required.
- We prove uniqueness and comparison principles in a simple way
  $\longrightarrow$ completely bypassing the difficulties involved in the Crandall-Lions doubling variable method, in favor of a more direct analysis of a discrete equation.
- We identify an intrinsic boundary (Aubry set) on which admissible traces can be assigned to get unique global solutions
  $\longrightarrow$ Formulating boundary problems on the network and determining "natural" subsets on which to assign boundary data is a subtle issue, yet not well settled in the literature.

Comparison with previous literature

Large amount of literature related to differential equations on networks, or others non-regular geometric structures (ramified/stratified spaces), in various contexts: hyperbolic problems, traffic flows, evolutionary equations, (regional) control problems, Hamilton-Jacobi equations, etc...

Some references closer related to our work:

- Schieborn-Camilli (2013):
  - PDE approach;
  - Eikonal equation in the supercritical case;
  - restrictions on the topology of the network;
  - they require a-priori existence of a regular strict subsolution;
  - continuity of Hamiltonians at vertices (and, accordingly, mixed conditions on the test functions at vertices).

  Other related contributions by: Achdou, Cutrí, Marchi, Oudet and Tchou.

Comparison with previous literature

- Imbert-Monneau (2013, 2016):
  - rather different point of view and techniques from ours;
  - local analysis at a juncture;
  - they use the doubling variable method by introducing an extra parameter (flux limiter), a companion equation (junction condition) and by using special vertex test functions.

  Other related contributions in collaborations with Galise and Zidani.

- Lions-Souganidis (2016):
  - one dimensional junction-type problems for non convex discounted HJ equations;
  - we adopt the same notion of solution.

- Discrete weak KAM and Aubry Mather theories:
  - Bernard-Buffoni (2006-2007): optimal transport maps.
  - Zavidovique (2010-2012): more systematic development.

  Other related contributions by: Gomes (2005), Iturriaga-Sánchez Morgado (2017), Su-Thieullen (2016).

From weak KAM to Aubry-Mather theory

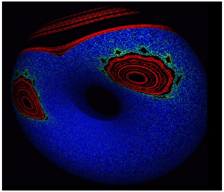
From a dynamical systems point of view, if $H : T^*M \longrightarrow \mathbb{R}$ is a Hamiltonian on a closed Riemannian manifold (M, g) then:

- Regular solutions to $H(x, du) = k \longleftrightarrow$ invariant (exact) Lagrangian graph in $(T^*M, \omega_{\mathrm{stand.}})$:

$$\Lambda := \{(x, d(x)) \in T^*M : \ x \in M\}.$$

- Weak solutions of H-J $\longleftrightarrow$ Aubry sets

$$\mathcal{A}^* \subseteq \{(x, d(x)) \in T^*M : \ x \in M \ u \text{ is differentiable at } x\}.$$



(Credits to Dr. Oliver Knill, Harvard)

A variational approach: Aubry-Mather theory

Aubry - Mather theory

Variational methods based on the *Principle of Least Lagrangian Action* (*"Nature is thrifty in all its actions"*, Pierre Louis Moreau de Maupertuis, 1744).

- Serge Aubry & John Mather '80s: twist maps of the annulus;
- John Mather '90s: Hamiltonian flows of *Tonelli* type.

Let us introduce:

- the Lagrangian function $L : TM \longrightarrow \mathbb{R}$:
  $L(x,v) := \sup_{p \in T_x^* M} (p \cdot v - H(x,p))$,
- the Euler-Lagrange equations: $\frac{d}{dt}\frac{\partial L}{\partial v} = \frac{\partial L}{\partial x}$.
- The Hamiltonian flow in T^*M and the Euler-Lagrange flow in TM are dynamically equivalent.

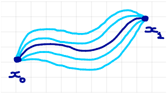
$$\begin{tikzcd} TM \arrow[r, "\Phi_t^L"] \arrow[d, "\mathcal{L}"'] & TM \arrow[d, "\mathcal{L}"] \\ T^*M \arrow[r, "\Phi_t^H"'] & T^*M \end{tikzcd}$$

A variational approach: Aubry-Mather theory

The Euler-Lagrange flow has an interesting variational characterization in terms of the Lagrangian Action Functional. If $\gamma : [a, b] \longrightarrow M$ is an abs. cont. curve, we define its action as:

$$A_L(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt.$$

γ is a solution of the Euler-Lagrange flow if and only if it is an extremal for the fixed-end variational problem.



These extremals are not necessarily minimizers, although they are local minimizers, *i.e.* for very short times.

Aubry-Mather theory is concerned with the study of global action-minimizing orbits or (invariant) probability measures.

$\longrightarrow$ Construct invariant sets: Mather and Aubry sets.

Remark: Orbits/Invariant probability measures on invariant Lagrangian graphs are global action-minimizers.

Discrete Lagrangian on the abstract graph

For every $\gamma \in \mathcal{E}$, let $L_\gamma : T\gamma \longrightarrow \mathbb{R}$ be the Lagrangian on the arc γ associated to the Hamiltonian H_γ:

$$L_\gamma(x, v) := \sup_{p \in T_x^* \gamma} \left(p \cdot v - H_\gamma(x, p) \right).$$

We want to define a discrete Lagrangian on the abstract graph $\Gamma = (\mathcal{E}, \mathcal{V})$, namely $\mathcal{L} : \mathcal{E} \times [0, +\infty) \longrightarrow \mathbb{R}$ defined as (for $q > 0$):

$$\mathcal{L}(\gamma, q) = q \cdot \min \left\{ \int_0^{\frac{1}{q}} L_\gamma(\xi(t), \dot{\xi}(t)) dt : \ \xi \text{ abs. cont. param. of } \gamma \text{ on } [0, 1/q] \right\}.$$

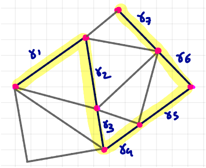
- $\mathcal{L}(\gamma, q) \to -a_\gamma$ as $q \to 0^+$;
- $\mathcal{L}(\gamma, 0) = \mathcal{L}(\overline{\gamma}, 0)$ (because of the compatibility conditions).
- $\mathcal{L}(\gamma, \cdot)$ is strictly convex and $\mathcal{L}(\gamma, q) \to +\infty$ as $q \to +\infty$.

One can consider the associated discrete Hamiltonian $\mathcal{H} : \mathcal{E} \times [p_\gamma, +\infty) \to \mathbb{R}$ (defined by convex duality), where $p_\gamma := \max \partial_q \mathcal{L}(\gamma, 0)$. In paticular, $p_\gamma = -p_{\overline{\gamma}}$.

Parametrized path on the abstract graph

A parametrized path on $\Gamma = (\mathcal{E}, \mathcal{V})$ is a sequence $\xi = \{(\gamma_i, q_i, T_i)\}_{i=1}^{M}$ such that:

- $\gamma_i \in \mathcal{E}$ for $i = 1, \dots, M$;
- $\mathrm{t}(\gamma_i) = \mathrm{o}(\gamma_{i+1})$ for $i = 1, \dots, M-1$ (concatenation);
- If $q_i > 0$, then $T_i = 1/q_i$; otherwise, if $q_i = 0$, then T_i can be any positive number.



We say that a parametrized path is singular if there exists γ_i such that $q_i = 0$, otherwise we say that it is non-singular.

We call $T_\xi := \sum_i T_i$ the total time of the parametrization of ξ.

The (discrete) action of ξ is defined as

$$A_{\mathcal{L}}(\xi) := \sum_{i=1}^{M} T_i \, \mathcal{L}(\gamma_i, q_i).$$

Discrete measures on the abstract graph

We introduce the set $\mathbb{M} = \mathbb{M}(\Gamma)$ of discrete probability measures on Γ, consisting of probability measures on $\mathcal{E} \times [0, +\infty)$ with finite first momentum:

$$\mu = \sum_{\gamma \in \mathcal{E}} \lambda_\gamma \, \mu_\gamma$$

with $\lambda_\gamma > 0$, $\sum_\gamma \lambda_{\gamma \in \mathcal{E}} = 1$, μ_γ prob. measures on $[0, +\infty)$ with $\int_0^{+\infty} q \, d\mu_\gamma < +\infty$.

Examples:

- $\delta(\gamma, T)$ the Dirac delta measure on the copy of $[0, +\infty)$ indexed by γ, concentrated at $T \geq 0$. It follows from compatibility condition that $\delta(\gamma, 0) = \delta(\overline{\gamma}, 0)$ for every $\gamma \in \mathcal{E}$.
- Given a parametrized closed path $\xi = \{(\gamma_i, q_i, T_i)\}_{i=1}^M$ we define the holonomic measure supported on ξ:

$$\mu_\xi := \frac{1}{T_\xi} \sum_{i=1}^{M} T_i \, \delta(\gamma_i, q_i).$$

We say that μ_ξ is singular if the corresponding parametrized path is singular.

Crash course on algebraic topology on a graph

- 0-chain group $\mathfrak{C}_0(\Gamma, \mathbb{R})$: the free abelian group on $\mathcal{V}$ with coefficients in $\mathbb{R}$.
- 1-chain group $\mathfrak{C}_1(\Gamma, \mathbb{R})$: the free abelian group on $\mathcal{E}$ with coefficients in $\mathbb{R}$ and with the relation $\overline{\gamma} = -\gamma$.
- boundary operator $\partial : \mathfrak{C}_1(\Gamma, \mathbb{R}) \to \mathfrak{C}_0(\Gamma, \mathbb{R})$ by setting for any arc $\partial\gamma = \mathrm{t}(\gamma) - \mathrm{o}(\gamma)$.
- First Homology group of Γ with coefficients $\mathbb{R}$: $H_1(\Gamma, \mathbb{R}) := \mathrm{Ker}\, \partial$.
  An element of $H_1(\Gamma, \mathbb{R})$ is called a 1-cycle. In particular, a 1-chain $\sum_{\gamma \in \mathcal{E}} a_\gamma \gamma$ is a 1-cycle if and only if for every $x \in \mathcal{V}$: $\sum_{\gamma \in \mathcal{E},\ \mathrm{t}(\gamma) = x} a_\gamma = \sum_{\gamma \in \mathcal{E},\ \mathrm{o}(\gamma) = x} a_\gamma$.

- 0-cochain group $\mathfrak{C}^0(\Gamma, \mathbb{R})$: the space of functions from $\mathcal{V}$ to $\mathbb{R}$.
- 1-cochain group $\mathfrak{C}^1(\Gamma, \mathbb{R})$: the space of functions $\omega : \mathcal{E} \to \mathbb{R}$ such that $\omega(\overline{\gamma}) = -\omega(\gamma)$.
- coboundary operator (differential) $d : \mathfrak{C}^0(\Gamma, \mathbb{R}) \to \mathfrak{C}_1(\Gamma, \mathbb{R})$ by setting for any $f \in \mathfrak{C}^0(\Gamma, \mathbb{R})$ $df(\gamma) = f(\mathrm{t}(\gamma)) - f(\mathrm{o}(\gamma))$.
- First Cohomology group of Γ with coefficients $\mathbb{R}$: $H^1(\Gamma, \mathbb{R}) := \mathfrak{C}^1(\Gamma, *)/\mathrm{Im}\, d$.

Pairing between $\mathfrak{C}^1(\Gamma, \mathbb{R})$ and $\mathfrak{C}_1(\Gamma, \mathbb{R})$: $\langle \omega, \sum_{\gamma \in \mathcal{E}} a_\gamma \gamma \rangle := \sum_{\gamma \in \mathcal{E}} a_\gamma \omega(\gamma)$.

Closed measures

Let $\mu = \sum_{\gamma\in\mathcal{E}} \lambda_\gamma\, \mu_\gamma$. Given $\omega \in \mathfrak{C}^1(\Gamma, \mathbb{R})$, we define:

$$\int \omega\, d\mu := \sum_{\gamma\in\mathcal{E}} \lambda_\gamma \int_0^{+\infty} \langle \omega, q\,\gamma\rangle\, d\mu_\gamma = \left\langle \omega, \underbrace{\sum_{\gamma\in\mathcal{E}} \left[\lambda_\gamma \int_0^{+\infty} q\, d\mu_\gamma\right] \gamma}_{[\mu]\in\mathfrak{C}_1(\Gamma,\mathbb{R})} \right\rangle.$$

We say that $\mu \in \mathbb{M}$ is a closed measure if $\partial[\mu] = 0$. In particular, $[\mu] \in H_1(\Gamma; \mathbb{R})$ is called homology class of μ (or rotation vector).

We denote the space of closed measures on Γ by $\mathbb{M}^0 = \mathbb{M}^0(\Gamma)$.

Example: If μ_ξ is the holonomic measure supported on a parametrized closed path $\xi = \{(\gamma_i, q_i, T_i)\}_{i=1}^M$, then μ_ξ is a closed measure and $[\mu_\xi] = \frac{[\xi]}{T_\xi}$, where $[\xi] = \sum_{i:\ q_i\neq 0} \gamma_i$.

Action-Minimizing measures (or Mather's measures)

We define the Action functional:

$$\begin{array}{rcl} A_{\mathcal{L}} : \mathbb{M}^0 & \longrightarrow & \mathbb{R} \\ \mu & \longmapsto & \int \mathcal{L}\, d\mu \end{array}$$

- $\mu \in \mathbb{M}^0$ is a Mather measure (or action-minimizing measure) with homology $h \in H_1(\Gamma, \mathbb{R})$ if

  $$A_{\mathcal{L}}(\mu) = \min_{[\nu]=h} \int \mathcal{L}\, d\nu =: \beta(h).$$

  We denote the subset of these measures by $\mathfrak{M}^h$.

- We define the Mather set of homology h as the set

  $$\widetilde{\mathcal{M}}^h := \bigcup_{\mu \in \mathfrak{M}^h} \bigcup_{\gamma \in \mathrm{supp}_{\mathcal{E}}\, \mu} \{\gamma\} \times \mathrm{supp}\, \mu_\gamma$$

  (for a given μ we denote by μ_γ its restriction on the edge γ).

- We call the function $\beta : H_1(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$ Mather's β function (or effective Lagrangian). It is convex and coercive.

Action-Minimizing measures (or Mather's measures)

- We say that a measure $\mu \in \mathbb{M}^0$ is a Mather measure (or action-minimizing measure) with cohomology $c \in H^1(\Gamma, \mathbb{R})$ if

$$A_{\mathcal{L}-\omega_c}(\mu) = \min_{\nu \in \mathbb{M}^0} \int (\mathcal{L} - \omega_c)\, d\nu =: -\alpha(c).$$

  Being μ closed, this notion does not depend on the choice of the representative ω_c, but only on its cohomology class (we mean that $\omega_c(\gamma, q) := \langle \omega_c, q\gamma \rangle$). We denote the subset of these measures by $\mathfrak{M}_c$.

- We define the Mather set of cohomology c as the set

$$\widetilde{\mathcal{M}}_c := \bigcup_{\mu \in \mathfrak{M}_c} \bigcup_{\gamma \in \mathrm{supp}_{\,\mathcal{E}}\, \mu} \{\gamma\} \times \mathrm{supp}\, \mu_\gamma$$

  (for a given μ we denote by μ_γ its restriction on the edge γ).

- We call the function $\alpha : H^1(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$ Mather's α function (or effective Hamiltonian). It is convex and coercive.

- α and β are convex conjugate to each other:

$$\alpha(c) = \max_{h \in H_1(\Gamma, \mathbb{R})} (\langle c, h \rangle - \beta(h)) \qquad \text{and} \qquad \beta(h) = \max_{c \in H^1(\Gamma, \mathbb{R})} (\langle c, h \rangle - \alpha(c)).$$

Properties of Mather sets

Structural properties of Mather's measures

- If $\mu \in \mathfrak{M}_c$, then $\mu_\gamma = \delta(\gamma, q_\gamma)$, $q_\gamma \geq 0$, for every $\gamma \in \operatorname{supp}_{\mathcal{E}}\mu$. In other words, the restriction of a Mather measure to any arc in its support is a Dirac delta measure.
- If $\mu, \nu \in \mathfrak{M}_c$ and $\gamma \in \operatorname{supp}_{\mathcal{E}}\mu \cap \operatorname{supp}_{\mathcal{E}}\nu$, then there exists a unique $\alpha \geq 0$ such that $\mu_\gamma = \nu_\gamma = \delta(\gamma, \alpha)$.

(As usual, μ_γ and ν_γ denote the restriction of μ and ν to the arc γ.)

Properties of Mather's sets

- $\widetilde{\mathcal{M}}^h \subseteq \widetilde{\mathcal{M}}_c$ if and only if $h \in \partial\alpha(c)$ (if and only if $c \in \partial\beta(h)$).
- In particular: $\widetilde{\mathcal{M}}_c = \bigcup_{h \in \partial\alpha(c)} \widetilde{\mathcal{M}}^h$.
- (Graph property) Let $\pi_{\mathcal{E}} : \mathcal{E} \times [0, +\infty) \longrightarrow \mathcal{E}$ denote the projection. Then:

$$\pi_{\mathcal{E}}|\widetilde{\mathcal{M}}_c : \widetilde{\mathcal{M}}_c \longrightarrow \mathcal{E} \qquad \text{and} \qquad \pi_{\mathcal{E}}|\widetilde{\mathcal{M}}^h : \widetilde{\mathcal{M}}^h \longrightarrow \mathcal{E}$$

  are injective maps for every $c \in H^1(\Gamma, \mathbb{R})$ and $h \in H_1(\Gamma, \mathbb{R})$.
  The same is true if we consider $\pi_{\mathcal{E}^+} : \mathcal{E} \times [0, +\infty) \longrightarrow \mathcal{E}^+$, where $\mathcal{E}^+$ is an orientation (i.e., we choose an element for each pair $\gamma, \overline{\gamma}$).

Relation between weak KAM and Aubry-Mather theories

- Mañé's critical value $c(\mathcal{H})$ coincides with $\alpha(0)$.
  Note: For any $c \in H^1(\Gamma, \mathbb{R})$, $\alpha(c)$ corresponds to the critical value for the modified Hamilton-Jacobi equation $\mathcal{H}(x, \eta_c + du) = k$, for some closed 1-form η_c on the network with cohomology class c.

- Aubry set and Mather set: $\pi_{\mathcal{E}}(\widetilde{\mathcal{M}}_0) = \mathcal{A}^*_\Gamma$.
  Note: Similarly, for any $c \in H^1(\Gamma, \mathbb{R})$, $\pi_{\mathcal{E}}(\widetilde{\mathcal{M}}_c)$ coincides with the Aubry set $\mathcal{A}^*_{\Gamma,c}$ corresponding to the modified Hamilton-Jacobi equation $\mathcal{H}(x, \eta_c + du) = k$, for some closed 1-form η_c on the network with cohomology class c.

- Graph property and (sub)solutions. Let $\pi_0^{-1} : \pi_{\mathcal{E}}(\widetilde{\mathcal{M}}_0) \longrightarrow \widetilde{\mathcal{M}}_0$. If $\gamma \in \pi_{\mathcal{E}}(\widetilde{\mathcal{M}}_0)$, the value $\pi_0^{-1}(\gamma)$ is univocally determined by the condition

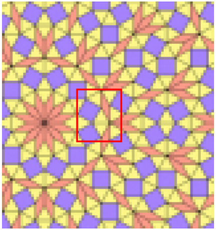
$$\langle du, \gamma \rangle \, \pi_0^{-1}(\gamma) = \mathcal{H}(\gamma, \langle du, \gamma \rangle) + \mathcal{L}(\gamma, \pi_0^{-1}(\gamma)),$$

  where u is any critical subsolution of the Hamilton–Jacobi equation on the network.
  Note: Similarly, as above, one can extend this result to any $c \in H^1(\Gamma, \mathbb{R})$ by considering the modified Hamiltonian and Lagrangian.

Per aspera ad... Homogenization

Naively speaking, the goal is to describe the macroscopic structure and the global properties of a problem, by "neglecting" its microscopic oscillations and its local features.

Pictorially, we want to describe what remains visible to a (mathematical) observer, as she/he moves her/his (mathematical) point of view further and further.



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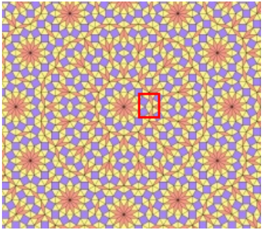

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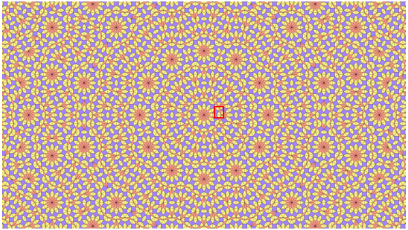


Periodic Homogenization of Hamilton-Jacobi in $\mathbb{R}^n$

Recall the classical result by Lions, Papanicolaou and Varadhan (LPV) in their famous preprint from 1987.

Let $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Tonelli Hamiltonian (i.e., C^2, strictly convex and superlinear in the momentum variable p) + $\mathbb{Z}^n$-periodic in the space variable x.

H can be also seen as the lift of a Tonelli Hamiltonian on $T^*\mathbb{T}^n$ (with $\mathbb{T}^n = \frac{\mathbb{R}^n}{\mathbb{Z}^n}$) to its universal cover.

Problem: Consider faster and faster oscillations of the x-variable and study the associated HJ equations:

$$(\mathrm{HJ}_\varepsilon): \quad \boxed{\begin{cases} \partial_t u^\varepsilon(x,t) + H(\frac{x}{\varepsilon}, \partial_x u^\varepsilon(x,t)) = 0 & x \in \mathbb{R}^n, t > 0 \\ u^\varepsilon(x,0) = f_\varepsilon(x) \end{cases}}$$

where $\varepsilon > 0$ and $f_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$ is some initial datum.

Periodic Homogenization of Hamilton-Jacobi in $\mathbb{R}^n$

Theorem (Lions, Papanicolaou & Varadhan, 1987)

Let $f_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$ be Lipschitz and assume that $f_\varepsilon \overset{\varepsilon\to 0^+}{\longrightarrow} \bar{f}$ uniformly.
Then, as $\varepsilon \to 0^+$, the unique viscosity solution u^ε of $(\mathrm{HJ}_\varepsilon)$ converges locally uniformly to a function $\bar{u} : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$, which solves

$$(\overline{\mathrm{HJ}}) : \quad \boxed{\begin{cases} \partial_t \bar{u}(x,t) + \overline{H}(\partial_x \bar{u}(x,t)) = 0 & x \in \mathbb{R}^n, t > 0 \\ \bar{u}(x,0) = \bar{f}(x), & \end{cases}}$$

where $\overline{H} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called the effective Hamiltonian.

Remarks:

- $\overline{H}$ depends only on H and is independent of x (due to the limit process).
- $\overline{H}$ is in general not differentiable.
- $\overline{H}$ is convex, but not necessarily strictly convex.
- Representation formula for $\bar{u}$: $\bar{u}(x,t) = \inf_{y\in\mathbb{R}^n}\left\{\bar{f}(y) + t\overline{L}\left(\frac{x-y}{t}\right)\right\}$ for $x \in \mathbb{R}^n, t > 0$, where $\overline{L}(v) := \sup_{p\in\mathbb{R}^n}\left(p\cdot v - \overline{H}(p)\right)$ is the effective Lagrangian.

How to Generalize to a Non-Euclidean Setting?

Main steps in LPV's Theorem:

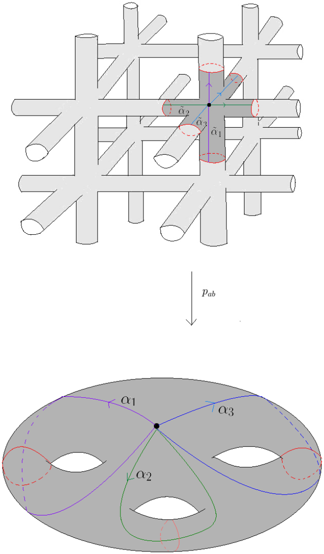
- Rescale (HJ): for $\varepsilon > 0$ consider the transformation $x \longmapsto \frac{x}{\varepsilon}$.
  The new Hamiltonian $H_\varepsilon(x,p) = H(\frac{x}{\varepsilon}, p)$ is still of Tonelli type, but it becomes $\varepsilon\mathbb{Z}^n$-periodic (its oscillations in the space variable become faster).
- Determine the limit problem, i.e., the effective Hamiltonian $\overline{H}$ and the limit space in which it is defined (in LPV's case, this is $\mathbb{R}^n$).
- Prove the convergence of solutions to $(\mathrm{HJ}_\varepsilon)$ to solutions to $(\overline{\mathrm{HJ}})$, as $\varepsilon \to 0^+$.
- Find a representation formula for the solution to $(\overline{\mathrm{HJ}})$ in terms of the effective Lagrangian $\overline{L}$.

A first generalization of [LPV] to non Euclidean setting has been proved in:

- G. Contreras, R. Iturriaga and A. Siconolfi, 2015 (in the "abelian case")).
- A.S., 2015 (Inspired by the work by Gromov, Pansu et al. on asymptotic cones).

A glimpse of our strategy

- Rescaling $\longleftrightarrow$ Rescale the distance, not the space! One has to consider a non-compact space: consider the maximal free abelian covering of Γ (i.e., the Deck transformation group is given by the free part of $H_1(\Gamma, \mathbb{Z})$). In otherwords, we introduce periodicity (or invariance) under a suitable action of a group.
- Limit space (in the Gromov-Hausdorff sense) $\longleftrightarrow$ $H_1(\Gamma, \mathbb{R})$ (first homology group). This corresponds to the asymptotic cone of $H_1(\Gamma, \mathbb{Z})$.
- Effective Hamiltonian $\longleftrightarrow$ $\alpha : H^1(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$
- Effective Lagrangian $\longleftrightarrow$ $\beta : H_1(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$
- ... Prove the convergence of solutions.



Thank you for your attention

... And keep safe!