The Hamilton-Jacobi equation on networks:
From Aubry-Mather theory to homogenization

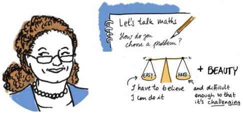
Alfonso Sorrentino

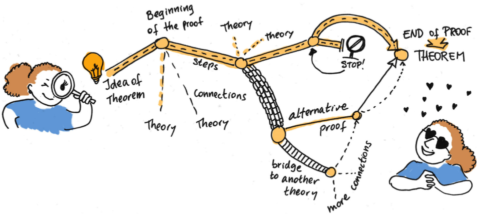
TOR VERGATA
UNIVERSITÀ DEGLI STUDI DI ROMA

Highlights in nonlinear analysis

In honour of Susanna Terracini's 60th birthday

Cetraro, 16th June 2023



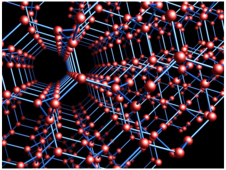


Introduction

Over the last years there has been an increasing interest in the study of the Hamilton-Jacobi (HJ) equation on networks and related problems.

These problems:

- involve a number of subtle theoretical issues;
- have a great impact in the applications in various fields, e.g., to data transmission, traffic management problems, *etc*...



While locally, *i.e.*, on each branch (arc) of the network, the study reduces to the analysis of 1-dimensional problems, the main difficulties arise in:

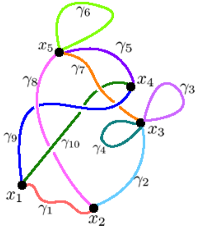
- matching together information converging at the juncture of two or more arcs;
- relating the local analysis at a juncture with the global structure (topology) of the network.

The network

An embedded network is a compact subset Γ in $(\mathbb{R}^N, d_{\mathrm{eucl}})$, or in any Riemannian manifold (M, g), of the form

$$\Gamma = \bigcup_{\gamma \in \mathcal{E}} \gamma \subset \mathbb{R}^N,$$

where $\mathcal{E}$ is a finite collection of arcs, *i.e.*, simple C^1 regular (oriented) curves $\gamma : [0,1] \to \mathbb{R}^N$, that are disjoint, except at the end-points (called vertices). We denote the set of vertices by $\mathcal{V}$.



Observe that Γ inherits:

- a metric d_Γ from the ambient space, hence a topology; we assume that Γ is path-connected.
- the structure of piecewise regular 1-dimensional manifold (vertices are special points).

The network

We introduce the following maps:

- A fixed-point-free involution $^{-} : \mathcal{E} \longrightarrow \mathcal{E}$ that to each arc $\gamma \in \mathcal{E}$ associates the arc $\overline{\gamma} \in \mathcal{E}$, *i.e.*, the same arc with opposite orientation(reversed arc).
- The map $\mathrm{o} : \mathcal{E} \longrightarrow \mathcal{V}$ which associates to each oriented arc $\gamma \in \mathcal{E}$ its initial vertex $\mathrm{o}(\gamma) \in \mathcal{V}$ (origin).
- The map $\mathrm{t} : \mathcal{E} \longrightarrow \mathcal{V}$ which associates to each oriented arc $\gamma \in \mathcal{E}$ its final vertex $\mathrm{t}(\gamma) \in \mathcal{V}$ (end).

In particular, for each $\gamma \in \mathcal{E}$:

$$\mathrm{t}(\gamma) = \mathrm{o}(\overline{\gamma}) \quad \text{and} \quad \mathrm{t}(\overline{\gamma}) = \mathrm{o}(\gamma).$$

It follows from the connectedness assumption on Γ, that the maps o and t are surjective.

Hamiltonians on the network

A Hamiltonian on a network Γ is a function $H : T^*\Gamma \longrightarrow \mathbb{R}$.
Equivalently, $H = \{H_\gamma\}_{\gamma\in\mathcal{E}}$ is a family of Hamiltonians, where H_γ denotes the restriction on the Hamiltonian on $T^*\gamma$ (vertices included).

We ask each H_γ to satisfy the following conditions (we use local coordinates $(x, p) \in T^*\gamma$):

- H_γ is continuous in x and differentiable in p;
- H_γ is strictly convex in p;
- H_γ is superlinear in p.

We also ask for the following compatibity condition:

$$H_\gamma(x, p) = H_{\overline{\gamma}}(x, -p) \qquad \forall\gamma \in \mathcal{E}.$$

Except for the above compatibility condition, Hamiltonians corresponding to geometrically different arcs are totally unrelated, even for arcs with some vertex in common.
No continuity or any mixed conditions at common vertices!

The Hamilton-Jacobi equation(s) on networks

- The stationary HJ equation:

$$H(x, du) = a \quad \text{on } \Gamma. \qquad (\mathcal{H}J)$$

This notation synthetically indicates the family (for γ varying in $\mathcal{E}$) of Hamilton–Jacobi equations $H_\gamma(x, d_x u) = a$ on $\gamma \setminus \{\mathrm{o}(\gamma), \mathrm{t}(\gamma)\}$.

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- The evolutive HJ equation on networks:

$$\partial_t u + H(x, du) = a \qquad \text{on } \Gamma \times (0, +\infty). \tag{$\mathcal{HJ}_t$}$$

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- Homogenization of the HJ equation:

Convergence result for solutions to ε–oscillating time dependent Hamilton–Jacobi equations on a network, as $\varepsilon \to 0^+$.

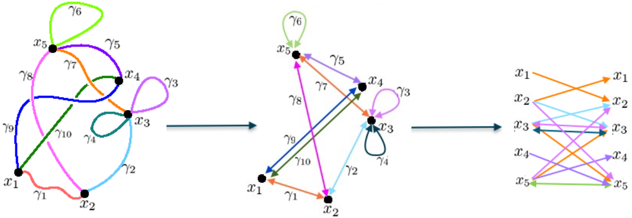
Local *versus* Global

1) The local problem on the arcs:

- (classical) 1-dimensional viscosity or variational techniques.

2) The global analysis on the network:

- we associate to the network an abstract graph, encoding all of the information on the complexity of the network;
- we relate the problems to discrete problems on the graph, to be studied by means of techniques inspired by weak KAM and Aubry-Mather theories.



The stationary HJ equation on networks

Let us consider the equation

$$H(x, du) = a \quad \text{on } \Gamma. \tag{$\mathcal{H}J$}$$

This notation synthetically indicates the family (for γ varying in $\mathcal{E}$) of Hamilton–Jacobi equations $H_\gamma(x, d_x u) = a$ on $\gamma \setminus \{\mathrm{o}(\gamma), \mathrm{t}(\gamma)\}$.

On a single arc, these equations possess infinitely many (viscosity) solutions, depending on the boundary data at $\mathrm{o}(\gamma)$ and $\mathrm{t}(\gamma)$.
We need to introduce suitable conditions on the vertices so to:

- select a unique solution on any arc;
- match these (local) solutions in a continuous way at vertices, so to obtain global solutions.

Notion of solution for $(\mathcal{H}J)$

Definition of solution

We say that $u : \Gamma \longrightarrow \mathbb{R}$ is solution to $(\mathcal{H}J)$ if
i) it is continuous on Γ;
ii) it is a (viscosity) solution on each $\gamma \setminus \{\mathrm{o}(\gamma), \mathrm{t}(\gamma)\}$, for any $\gamma \in \mathcal{E}$;
iii) For every vertex x there is at least one arc γ with $\mathrm{t}(\gamma) = x$ such that for any C^1 function φ satisfying

- $u = \varphi$ at $\mathrm{t}(\gamma)$,
- $u \geq \varphi$ in a sufficiently small open neighborhood of $\mathrm{t}(\gamma)$,

we have that $H_\gamma(x, d\varphi(x)) \geq a$.

Remark: Condition iii) is also called state constraint boundary condition: at $\mathrm{t}(\gamma)$ it allows to select the maximal solution taking a given value at $\mathrm{o}(\gamma)$.

Any function φ as in iii) is called constrained subtangent to u at γ (*cfr.* Soner, 1986)

Weak KAM theory on networks

Theorem (A. Siconolfi, A.S., 2018)

(i) (Existence) There exists a unique value $c = c(H)$, called Mañé critical value, for which the equation $H(x, du) = c$ admits global solutions. In particular, these solutions are Lipschitz continuous on Γ.

(ii) (Uniqueness) There exists a *uniqueness set* $\mathcal{A}_\Gamma := \mathcal{A}_\Gamma(H) \subseteq \mathcal{V}$ called the (projected) Aubry set of H, namely if u and v are two solutions to $H(x, du) = c$ and coincide on $\mathcal{A}_\Gamma$, then they coincide everywhere. (It is an intrisic boundary for the Cauchy problem).

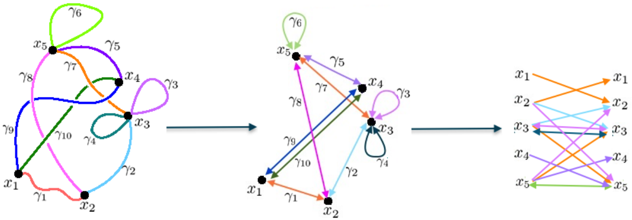
(iii) (Hopf–Lax type representation formulae) Explicit representation formulae are provided both for global solutions and for solutions on subsets of Γ.

Moreover we also discuss: subsolutions, critical subsolutions and their regularity, etc ...

Among several previous (partial) results, let us recall: Schieborn-Camilli (2013), Imbert-Monneau (2013, 2016).

From the network to the abstract graph

The main novelty of our method is to put in relation the HJ equation on the network to a discrete functional equation on the underlying abstract graph $\Gamma = (\mathcal{E}, \mathcal{V})$, where $\mathcal{E}$ is the (abstract) set of arcs and $\mathcal{V}$ the (abstract) set of vertices.



When referring to the abstract graph, we think of elements of $\mathcal{E}$ as immaterial edges (we use the same notation).

From the network to the abstract graph

The subsequent step is to transfer the Hamilton-Jacobi equation from Γ to the abstract graph, where it will take the form of a discrete functional equation.

For any $\gamma \in \mathcal{E}$ and $a \geq a_\gamma := \max_{x\in\gamma} \min_{T_x^*\gamma} H_\gamma$, the relevant information to transfer is

$$\sigma_a(\gamma) := \int_\gamma \sigma^+_{a,\gamma}(x)\, dx$$

where $\sigma^+_{a,\gamma}(x) = \max\{p : \ H_\gamma(x,p) = a\}$.

- $\sigma_a(\gamma)$ is the value at $\mathrm{t}(\gamma)$ of the maximal subsolution to $H_\gamma(x, du) = a$ on γ, vanishing at $\mathrm{o}(\gamma)$.
- We say that $\xi = (\gamma_1, \ldots, \gamma_M)$ is a path linking two vertices $x, y \in \mathcal{V}$ if
  - $\gamma_i \in \mathcal{E}$ for every $i = 1, \ldots, M$,
  - $\mathrm{o}(\gamma_1) = x$ and $\mathrm{o}(\gamma_M) = y$,
  - $\mathrm{t}(\gamma_i) = \mathrm{o}(\gamma_{i+1})$ for every $i = 1, \ldots, M-1$.

  Then: $\sigma_a(\xi) := \sum_{i=1}^M \sigma_a(\gamma_i)$.

The discrete functional equation

We introduce the following discrete functional equation:

$$u(x) = \min_{\gamma \in \mathcal{E}, \mathrm{o}(\gamma)=x} \left(u(\mathrm{t}(\gamma)) + \sigma_a(\overline{\gamma}) \right) \qquad \forall\, x \in \mathcal{V}. \tag{$\mathcal{D}FE$}$$

Remark: Equality is required only at (at least) one arc for each vertex. Moreover, the formulation of the discrete problem takes somehow into account the backward character of viscosity solutions.

Theorem

Any solution to ($\mathcal{D}FE$) can be (uniquely) extended to a solution of ($\mathcal{H}J$). Conversely, the trace on $\mathcal{V}$ of any solution to ($\mathcal{H}J$) is solution to ($\mathcal{D}FE$).

Therefore, the study of ($\mathcal{H}J$) reduces to the study of ($\mathcal{D}FE$).

Question: For which value(s) of a (if any) do ($\mathcal{D}FE$) admit solutions?

The critical value

Theorem

There exists a unique $c = c(\mathcal{H})$ such that $\mathcal{D}FE$ admits solutions.

- $c = c(\mathcal{H})$ is called critical value (or Mañé critical value).
- c can be characterized in terms of the existence of vanishing cycles (*i.e.*, closed path), i.e., there exists a closed path ξ such that $\sigma_a(\xi) = 0$ if and only if $a = c$.
- We define the Aubry set as

$$\mathcal{A}^*_\Gamma(H) := \{\gamma \in \mathcal{E} : \text{ belonging to some cycle with } \sigma_c(\xi) = 0\}$$

and the projected Aubry set as the set of vertices in $\mathcal{A}^*_\Gamma(H)$.

The evolutive HJ equation on networks

We consider the time-dependent equation

$$\partial_t u + H(x, du) = a \qquad \text{on } \Gamma \times (0, +\infty). \tag{$\mathcal{H}J_t$}$$

A solution is a continuous function $u : \Gamma \times (0, +\infty)$ such that

- $u(x, t)$ is solution to $(\mathcal{H}J_t)$ on $x \in \gamma \setminus \{\mathrm{o}(\gamma), \mathrm{t}(\gamma)\} \times (0, +\infty)\ \forall\, \gamma \in \mathcal{E}$;
- $u(x, t)$ satisfies suitable additional conditions at the discontinuity interfaces

$$\{(x, t) :\ x \in \mathcal{V},\ t \in (0, +\infty)\}.$$

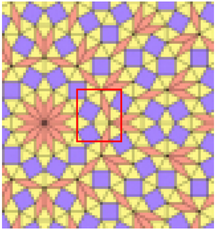
Existence and Uniqueness of the solution with initial continuous datum is prescribed at $t = 0$ and flux limiter at any vertex fixed. The flux limiter plays an essential role in the conditions on the discontinuity interfaces; Imbert-Monneau (2015), Siconolfi (2022).

Representation formula via Lax-Oleinik: Imbert-Monneau-Zidani (2012), Pozza-Siconolfi (2023).

Homogenization

Naively speaking, the goal is to describe the macroscopic structure and the global properties of a problem, by "neglecting" its microscopic oscillations and its local features.

Pictorially, we want to describe what remains visible to a (mathematical) observer, as she/he moves her/his (mathematical) point of view further and further.



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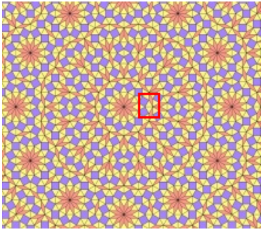

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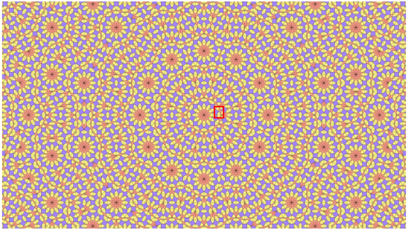

Periodic Homogenization of Hamilton-Jacobi in $\mathbb{R}^n$

Recall the classical result by Lions, Papanicolaou and Varadhan (LPV) in 1987.

Let $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Tonelli Hamiltonian[1] (*i.e.*, C^2, strictly convex and superlinear in the momentum variable p) $+$ $\mathbb{Z}^n$-periodic in the space variable x.

H can be also seen as the lift of a Tonelli Hamiltonian on $T^*\mathbb{T}^n$ (with $\mathbb{T}^n = \frac{\mathbb{R}^n}{\mathbb{Z}^n}$) to its universal cover.

Problem: Consider faster and faster oscillations of the x-variable and study the associated HJ equations:

$$(\mathrm{HJ}_\varepsilon) : \quad \boxed{\begin{cases} \partial_t u_\varepsilon(x,t) + H(\frac{x}{\varepsilon}, \partial_x u_\varepsilon(x,t)) = 0 & x \in \mathbb{R}^n, t > 0 \\ u_\varepsilon(x,0) = f_\varepsilon(x) \end{cases}}$$

where $\varepsilon > 0$ and $f_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$ is some initial datum.

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[1] Remark: Actually in LPV H is only asked to be continuous in (x,p) and coercive in p; no convexity (they use purely PDE techniques).

Periodic Homogenization of Hamilton-Jacobi in $\mathbb{R}^n$

Theorem (Lions, Papanicolaou & Varadhan, 1987)

Let $f_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$ be Lipschitz and assume that $f_\varepsilon \overset{\varepsilon \to 0^+}{\longrightarrow} \bar{f}$ locally uniformly. Then, as $\varepsilon \to 0^+$, the unique viscosity solution u_ε of $(\mathrm{HJ}_\varepsilon)$ converges locally uniformly to a function $\bar{u} : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$, which solves

$$(\overline{\mathrm{HJ}}) : \quad \boxed{\begin{cases} \partial_t \bar{u}(x,t) + \overline{H}(\partial_x \bar{u}(x,t)) = 0 & x \in \mathbb{R}^n, t > 0 \\ \bar{u}(x,0) = \bar{f}(x), & \end{cases}}$$

where $\overline{H} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called the effective Hamiltonian.

Remarks:

- $\overline{H}$ depends only on H and is independent of x (due to the limit process).
- $\overline{H}$ is in general not differentiable.
- $\overline{H}$ is convex, but not necessarily strictly convex.
- Representation formula for $\bar{u}$: $\bar{u}(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ \bar{f}(y) + t\overline{L}\left(\frac{x-y}{t}\right) \right\}$ for $x \in \mathbb{R}^n, t > 0$, where $\overline{L}(v) := \sup_{p \in \mathbb{R}^n} \left(p \cdot v - \overline{H}(p) \right)$ is the effective Lagrangian.

How to Generalize to a Non-Euclidean Setting?

- Rescaling (HJ): for $\varepsilon > 0$ consider the transformation $x \longmapsto \frac{x}{\varepsilon}$.
  The new Hamiltonian $H_\varepsilon(x, p) = H(\frac{x}{\varepsilon}, p)$ is still of Tonelli type, but it becomes $\varepsilon\mathbb{Z}^n$-periodic (its oscillations in the space variable become faster).

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  Rescale the metric, not the space!

  $v^\varepsilon(x,t) := u^\varepsilon(\varepsilon x, t)$ is a solution of HJ equation with $x \in (\mathbb{R}^n, \varepsilon d_{\mathrm{euc}})$:

$$(\widetilde{\mathrm{HJ}}_\varepsilon): \quad \begin{cases} \partial_t v^\varepsilon(x,t) + H(x, \frac{1}{\varepsilon}\partial_x v^\varepsilon(x,t)) = 0 & x \in \mathbb{R}^n, t > 0 \\ v^\varepsilon(x,0) = f_\varepsilon(\varepsilon x) =: \tilde{f}_\varepsilon(x). \end{cases}$$

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- Determine the limit problem:
  - The limit space is related to the periodicity group $(\mathbb{Z}^n, +)$.
    $\longrightarrow$ Asymptotic cone: $\lim_{\varepsilon \to 0^+}(\varepsilon\mathbb{Z}^n, \varepsilon d_{\mathrm{euc}}) = (\mathbb{R}^n, \|\cdot\|_{\mathrm{stable}})$.
  - The effective Hamiltonian can be interpreted in the context of Aubry-Mather theory: $\longrightarrow$ Mather's α-function (related to action-minimizing measures).

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  - The effective Hamiltonian can be interpreted in the context of Aubry-Mather theory: $\longrightarrow$ Mather's α-function (related to action-minimizing measures).
- Prove convergence of solutions to $(\mathrm{HJ}_\varepsilon)$ to solutions to $(\overline{\mathrm{HJ}})$ and obtain representation formulae in terms of the effective Lagrangian.

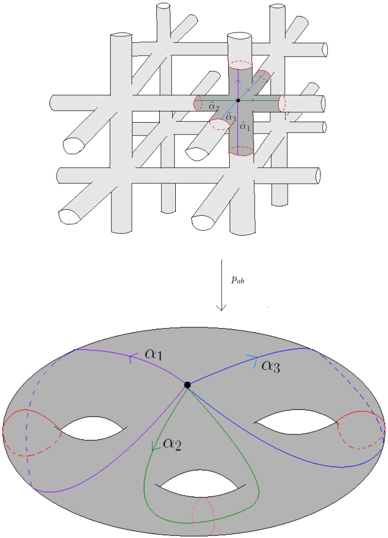
See [G. Contreras, R. Iturriaga, A. Siconolfi, 2015] and [A.S., 2015].

Remark

The dimension of the asymptotic cone is in general very different from the one of the ambient metric space!

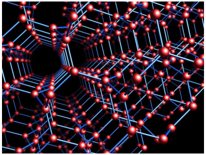
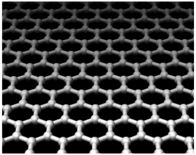
Let us consider a surface Σ_3 of genus 3 and consider a cover space whose group of Deck transformations is isomorphic to $\mathbb{Z}^3$.

(One could find a free abelian cover whose group of Deck transformations is isomorphic to $\mathbb{Z}^6$... I cannot draw it!)



Periodic networks

Given a finite network $\Gamma = (\mathcal{E}, \mathcal{V})$, we would like to "embed" Γ as the fundamental domain of a periodic network $\widetilde{\Gamma} = (\widetilde{\mathcal{E}}, \widetilde{\mathcal{V}})$.
Roughly speaking: there exists a group of symmetries G acting on $\widetilde{\Gamma}$ such that $\widetilde{\Gamma}/G = \Gamma$.

If $G \simeq \mathbb{Z}^N$, $\widetilde{\Gamma} = (\widetilde{\mathcal{E}}, \widetilde{\mathcal{V}})$ is called a N-dimensional topological crystal over $\Gamma = (\mathcal{E}, \mathcal{V})$.

Given a finite connected network Γ, there always exist N-dimensional topological crystals. They are related to abelian coverings of Γ. In particular, $N \leq b_1(\Gamma)$ where:

$$b_1(\Gamma) := \frac{1}{2}|\mathcal{E}| - |\mathcal{V}| + 1.$$

the first Betti number of Γ (*i.e.*, the rank of the first homology group of Γ, $H_1(\Gamma, \mathbb{Z})$).

Remark: These notions naturally extends to the associated abstract graphs.

Homogenization on periodic networks

We consider a maximal tological crystal $\widetilde{\Gamma} = (\widetilde{\mathcal{E}}, \widetilde{\mathcal{V}})$ over $\Gamma = (\mathcal{E}, \mathcal{V})$, *i.e.*, the acting abelian group has rank $\mathsf{N} = b_1(\Gamma)$.

The Hamiltonian H on $T^*\Gamma$ can be extended by periodicity to a Hamiltonian on $T^*\widetilde{\Gamma}$.

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The Hamiltonian H on $T^*\Gamma$ can be extended by periodicity to a Hamiltonian on $T^*\widetilde{\Gamma}$.

Theorem (M. Pozza, A. Siconolfi, A.S., 2023)

Let $\varepsilon > 0$ and let $u_\varepsilon : \widetilde{\Gamma} \times [0, +\infty) \longrightarrow \mathbb{R}$ be a solution to

$$\begin{cases} \partial_t u_\varepsilon + H(x, \partial_x u_\varepsilon) = 0 & x \in (\widetilde{\Gamma}, \varepsilon d_{\widetilde{\Gamma}}),\ t > 0 \\ u_\varepsilon(x, 0) = f_\varepsilon(x) \end{cases}$$

where $f_\varepsilon : (\widetilde{\Gamma}, \varepsilon d_{\widetilde{\Gamma}}) \longrightarrow \mathbb{R}$ are Lipschitz functions such that f_ε locally uniformly converge to $\bar{f} : \mathbb{R}^{b_1(\Gamma)} \to \mathbb{R}$, as $\varepsilon \to 0^+$. Then, u_ε locally uniformly converge to a function $\bar{u} : \mathbb{R}^{b_1(\Gamma)} \times [0, +\infty) \to \mathbb{R}$, which solves

$$\begin{cases} \partial_t \bar{u}(z,t) + \overline{H}(\partial_z \bar{u}(z,t)) = 0 & z \in \mathbb{R}^{b_1(\Gamma)}, t > 0 \\ \bar{u}(z, 0) = \bar{f}(z), \end{cases}$$

where $\overline{H} : \mathbb{R}^{b_1(\Gamma)} \longrightarrow \mathbb{R}$ is convex and superlinear. Moreover:

$$\bar{u}(z,t) = \inf_{y \in \mathbb{R}^{b_1(\Gamma)}} \left\{ \bar{f}(y) + t\overline{L}\left(\frac{z-y}{t}\right) \right\} \qquad z \in \mathbb{R}^{b_1(\Gamma)}, t > 0,$$

where $\overline{L}(v) := \sup_{p \in \mathbb{R}^{b_1(\Gamma)}} \left(p \cdot v - \overline{H}(p) \right)$.

A few related contributions in the literature

- Imbert, Monneau (2014)
  They consider the periodic network $\mathbb{Z}^N \subset \mathbb{R}^N$ and prove a homogenization result in their setting with PDE techniques.

- Camilli (2023)
  He also considers the periodic network $\mathbb{Z}^N \subset \mathbb{R}^N$ and provides estimates on the convergence rate.

- Other related results with applications to traffic models: Galise, Imbert, Monneau (2015), Forcadel and several coauthors, etc...

Convergence of spaces: $\lim_{\varepsilon \to 0^+} (\widetilde{\Gamma}, \varepsilon d_{\tilde{\Gamma}}) = (\mathbb{R}^{b_1(\Gamma)}, \bar{d})$

Gromov-Hausdorff (GH) "distance": Let $\widetilde{X}_1 := (X_1, d_1)$ and $\widetilde{X}_2 := (X_2, d_2)$ be metric spaces. We say that $d_{\mathrm{GH}}(\widetilde{X}_1, \widetilde{X}_2) < r$ if there exist a metric space (Z, d) and two subspaces $Z_1, Z_2 \subset Z$ isometric (respectively) to $\widetilde{X}_1$ and $\widetilde{X}_2$, s.t. their Hausdorff distance in (Z, d) is $d_H(Z_1, Z_2) < r$.

[Recall that $d_H(A, B) = \inf\{r > 0 : \mathcal{N}_r(A) \supset B \text{ and } \mathcal{N}_r(B) \supset A\}$, where $\mathcal{N}_r(\cdot)$ denotes the open neighborhood of size r]

Intuitively: $(X_n, d_n, x_n) \to (X, d, x_0)$ if balls of radius $r > 0$ and centers at x_n (in X_n) converge (in the GH distance) to the ball of radius r and center at x_0 (in X).

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Main ideas:

- Metric spaces at finite GH distance have the same limit when rescaled;
- $\widetilde{\Gamma}$ and its set of vertices $\widetilde{\mathcal{V}}$ are at finite GH distance;
- $\widetilde{\mathcal{V}}$ can be identified with $\mathcal{V} \times \mathbb{Z}^{b_1(\Gamma)}$;
- The map $F : \mathcal{V} \times \mathbb{Z}^{b_1(\Gamma)} \to \mathbb{R}^{b_1(\Gamma)}$ defined as $F(x, h) := h$ is a quasi-isometry.

A map $F : (X, d_X) \to (Y, d_Y)$ is called a quasi–isometry if $\exists\ k \geq 1,\ A \geq 0$:

$$k^{-1} d_X(x_1, x_2) - A \leq d_Y(F(x_1), F(x_2)) \leq k d_X(x_1, x_2) + A \qquad \forall x_1, x_2 \in X.$$

Convergence of functions

For $\varepsilon > 0$ let us define the rescaling maps:

$$\begin{array}{rcl} F_\varepsilon : \mathcal{V} \times \mathbb{Z}^{b_1(\Gamma)} & \longrightarrow & \mathbb{R}^{b_1(\Gamma)} \\ (x, h) & \longmapsto & \varepsilon\, h. \end{array}$$

We say that a sequence $v_\varepsilon : \mathcal{V} \times \mathbb{Z}^{b_1(\Gamma)} \to \mathbb{R}$ locally uniformly converges to $v : \mathbb{R}^{b_1(\Gamma)} \to \mathbb{R}$ if for any subsequence $\{(x_{\varepsilon_n}, h_{\varepsilon_n})\}_n$, where $\varepsilon_n \to 0^+$ and $F_{\varepsilon_n}(h_{\varepsilon_n}) = \varepsilon_n h_{\varepsilon_n} \to \bar{h}$, we have

$$\lim_{n \to +\infty} v_{\varepsilon_n}(x_{\varepsilon_n}, h_{\varepsilon_n}) = v(\bar{h}).$$

Remark: This convergence extends to functions defined on $\widetilde{\Gamma}$ and not just on vertices, using the fact that they are a finite GH distance.

The limit Hamiltonian: an action-minimizing approach

Let us consider the Lagrangian on a network Γ, namely $L = \{L_\gamma\}_{\gamma\in\mathcal{E}}$ where each L_γ is the Fenchel-Legendre trasnform of H_γ. Extend it by periodicity to $\widetilde{\Gamma}$.

- Solutions u_ε have a Lax-Oleinik representation formula (Pozza-Siconolfi, 2022):

$$u_\varepsilon(x,t) = \inf\left\{ f_\varepsilon(\xi(0)) + \varepsilon \int_0^{\frac{t}{\varepsilon}} L(\xi(s),\dot{\xi}(s))\, ds \right\}$$

(The infimum taken over the absolutely continuous curves ξ from $[0, \frac{t}{\varepsilon}]$ to $\widetilde{\Gamma}$ satisfying $\xi(\frac{t}{\varepsilon}) = x$)

- Consider the minimal action

$$\Phi(x,y,t) := \inf\left\{ \int_0^t L(\xi(s),\dot{\xi}(s))\, ds : \ \xi(0) = x,\ \xi(t) = y \right\}$$

and investigate the asymptotic problem

$$\lim_{\varepsilon\to 0^+} \varepsilon\, \Phi(x_\varepsilon, y_\varepsilon, t/\varepsilon)$$

for suitable sequences $\{x_\varepsilon\}_\varepsilon$, $\{y_\varepsilon\}_\varepsilon$ in $\widetilde{\Gamma}$.

This is related to the so-called Aubry-Mather theory, i.e., the study of curves or probability measures that minimize the Lagrangian action.

Aubry-Mather theory on graphs (A. Siconolfi, A.S, 2022)

Let us first define the discrete Hamiltonian/Lagrangian on the abstract graph.

Recall that for any $\gamma \in \mathcal{E}$ and $a \geq a_\gamma := \max_{x\in\gamma} \min_{T_x^*\gamma} H_\gamma$, we have defined

$$\sigma(\gamma, a) := \sigma_a(\gamma) = \int_\gamma \sigma^+_{a,\gamma}(x)\, dx$$

where $\sigma^+_{a,\gamma}(x) = \max\{p : \; H_\gamma(x,p) = a\}$.

The function $a \longmapsto \sigma(\gamma, a)$ from $[a_\gamma, \mathbb{R})$ is continuous and strictly increasing.

We define the discrete Hamiltonian $\mathcal{H} : \mathcal{E} \times \mathbb{R} \longrightarrow \mathbb{R}$ as

$$\mathcal{H}(\gamma, p) := \begin{cases} \sigma^{-1}(\gamma, p) & \text{if } p \geq \sigma(\gamma, a_\gamma) \\ \sigma^{-1}(\overline{\gamma}, p) & \text{if } p \leq \sigma(\gamma, a_\gamma). \end{cases}$$

For every $\gamma \in \mathcal{E}$, $\mathcal{H}(\gamma, \cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ is convex, differentiable and superlinear.
In particular:

- $\mathcal{H}(\gamma, p) = \mathcal{H}(\overline{\gamma}, -p)$ for every $\gamma \in \mathcal{E}$ and $p \in \mathbb{R}$;
- $a_\gamma = \mathcal{H}(\gamma, p_\gamma)$ where $p_\gamma := \sigma(\gamma, a_\gamma)$.

Discrete Lagrangian on the abstract graph

We define the discrete Lagrangian on the graph to be the function $\mathcal{L} : \mathcal{E} \times [0, +\infty) \longrightarrow \mathbb{R}$ obtained by convex duality:

$$\mathcal{L}(\gamma, q) \quad := \quad \max_{p \in \mathbb{R}} (q\, p - \mathcal{H}(\gamma, p)) \;=\; \max_{a \geq a_\gamma} (q\, \sigma(\gamma, a) - a)$$

- $\mathcal{L}(\gamma, \cdot)$ is strictly convex and superlinear;
- $\mathcal{L}(\gamma, 0) = \mathcal{L}(\overline{\gamma}, 0) = -a_\gamma = -a_{\overline{\gamma}}$ (because of the compatibility conditions).

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$\mathcal{L}(\gamma, q)$ has an interpretation on the network in terms of "optimal cost".

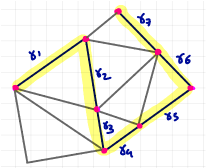
For every $\gamma \in \mathcal{E}$, let $L_\gamma : T\gamma \longrightarrow \mathbb{R}$ be the Lagrangian on the arc γ associated to the Hamiltonian H_γ (via the Legendre transform). Then, for $q > 0$

$$\mathcal{L}(\gamma, q) = q \cdot \min \left\{ \int_0^{\frac{1}{q}} L_\gamma(\xi(t), \dot{\xi}(t))dt : \ \xi \text{ abs. cont. param. of } \gamma \text{ on } [0, 1/q] \right\}.$$

Parametrized path on the abstract graph

A parametrized path on $\Gamma = (\mathcal{E}, \mathcal{V})$ is a sequence $\xi = \{(\gamma_i, q_i, T_i)\}_{i=1}^{M}$ such that:

- $\gamma_i \in \mathcal{E}$ for $i = 1, \ldots, M$;
- If $q_i > 0$, then $T_i = 1/q_i$; otherwise, if $q_i = 0$, then T_i can be any positive number.
  q_i must be meant as an average velocity.
- Concatenation condition:
  - if $q_i > 0$, then $\mathrm{o}(\gamma_{i+1}) = \mathrm{t}(\gamma_i)$;
  - if $q_i = 0$, then $\mathrm{o}(\gamma_{i+1}) = \mathrm{o}(\gamma_i)$.



We call $T_\xi := \sum_i T_i$ the total time of the parametrization of ξ.

The (discrete) action of ξ is defined as

$$A_{\mathcal{L}}(\xi) := \sum_{i=1}^{M} T_i \, \mathcal{L}(\gamma_i, q_i).$$

Discrete measures on the abstract graph

We introduce the set $\mathbb{M} = \mathbb{M}(\Gamma)$ of discrete probability measures on Γ, consisting of probability measures on $\mathcal{E} \times [0, +\infty)$ with finite first momentum:

$$\mu = \sum_{\gamma \in \mathcal{E}} \lambda_\gamma \, \mu_\gamma$$

with $\lambda_\gamma > 0$, $\sum_\gamma \lambda_{\gamma \in \mathcal{E}} = 1$, μ_γ prob. measures on $[0, +\infty)$ with $\int_0^{+\infty} q \, d\mu_\gamma < +\infty$.

Examples:

- $\delta(\gamma, T)$ the Dirac delta measure on the copy of $[0, +\infty)$ indexed by γ, concentrated at $T \geq 0$. It follows from compatibility condition that $\delta(\gamma, 0) = \delta(\overline{\gamma}, 0)$ for every $\gamma \in \mathcal{E}$.
- Given a parametrized closed path $\xi = \{(\gamma_i, q_i, T_i)\}_{i=1}^{M}$ we define the occupation measure supported on ξ:

$$\mu_\xi := \frac{1}{T_\xi} \sum_{i=1}^{M} T_i \, \delta(\gamma_i, q_i).$$

Homology of a graph

- 0-chain group $\mathfrak{C}_0(\Gamma,\mathbb{R})$: the free abelian group on $\mathcal{V}$ with coefficients in $\mathbb{R}$.
- 1-chain group $\mathfrak{C}_1(\Gamma,\mathbb{R})$: the free abelian group on $\mathcal{E}$ with coefficients in $\mathbb{R}$ and with the relation $\overline{\gamma} = -\gamma$.
- Boundary operator $\partial : \mathfrak{C}_1(\Gamma,\mathbb{R}) \to \mathfrak{C}_0(\Gamma,\mathbb{R})$ by setting for any arc $\partial\gamma = \mathrm{t}(\gamma) - \mathrm{o}(\gamma)$.

Example: Let $\xi = \{\gamma_i\}_{i=1}^{M}$ be a path, *i.e.*, $\mathrm{o}(\gamma_{i+1}) = \mathrm{t}(\gamma_i)$ for every $i = 0, \dots, M-1$. Then: $[\xi] := \sum_{i=1}^{M} \gamma_i \in \mathfrak{C}_1(\Gamma,\mathbb{R})$. Observe that ξ is closed if and only if $\partial[\xi] = 0$.

The First Homology group of Γ with coefficients $\mathbb{R}$ is defined as

$$H_1(\Gamma,\mathbb{R}) := \mathrm{Ker}\,\partial \simeq \mathbb{R}^{b_1(\Gamma)}.$$

An element of $H_1(\Gamma,\mathbb{R})$ is called a 1-cycle.

Remark: A 1-chain $\sum_{\gamma\in\mathcal{E}} a_\gamma \gamma$ is a 1-cycle if and only if for every $x \in \mathcal{V}$:

$$\sum_{\gamma\in\mathcal{E},\ \mathrm{t}(\gamma)=x} a_\gamma = \sum_{\gamma\in\mathcal{E},\ \mathrm{o}(\gamma)=x} a_\gamma .$$

Closed measures

Let $\mu = \sum_{\gamma \in \mathcal{E}} \lambda_\gamma \, \mu_\gamma$. We can associate to an element of $\mathfrak{C}_1(\Gamma, \mathbb{R})$:

$$[\mu] := \sum_{\gamma \in \mathcal{E}} \left(\lambda_\gamma \int_0^{+\infty} q \, d\mu_\gamma \right) \gamma.$$

We say that $\mu \in \mathbb{M}$ is a closed measure if $\partial[\mu] = 0$. In particular, $[\mu] \in H_1(\Gamma, \mathbb{R})$ is called homology class of μ (or rotation vector).

We denote the space of closed measures on Γ by $\mathbb{M}^0 = \mathbb{M}^0(\Gamma)$.

Example: If μ_ξ is the occupation measure supported on a parametrized closed path $\xi = \{(\gamma_i, q_i, T_i)\}_{i=1}^{M}$, then μ_ξ is a closed measure and $[\mu_\xi] = \frac{[\xi]}{T_\xi}$, where $[\xi] = \sum_{i:\ q_i \neq 0} \gamma_i$.

Occupation measures are dense in $\mathbb{M}^0$ w.r.t. the Wasserstein topology.
$\longrightarrow$ For every $h \in H_1(\Gamma, \mathbb{R})$, there exists $\mu \in \mathbb{M}^0$ with $[\mu] = h$.

Action-Minimizing measures (or Mather measures)

We define the Action functional $A_{\mathcal{L}}$

$$\begin{array}{rcl} \mathbb{M}^0 & \longrightarrow & \mathbb{R} \\ \mu := \displaystyle\sum_{\gamma \in \mathcal{E}} \lambda_\gamma \, \mu_\gamma & \longmapsto & \displaystyle\int \mathcal{L} \, d\mu := \sum_{\gamma \in \mathcal{E}} a_\gamma \int_0^{+\infty} \mathcal{L}(\gamma, q) \, d\mu_\gamma(q). \end{array}$$

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- We define the minimal average action with homology $h \in H_1(\Gamma, \mathbb{R})$ as

$$\beta(h) := \inf_{[\nu]=h} \int \mathcal{L}\, d\nu.$$

  It is a minimum and minimizers are called Mather measures with homology h.

- We call the function $\beta : H_1(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$ Mather's β function (or effective Lagrangian). It is convex and superlinear.

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- We call the function $\beta : H_1(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$ Mather's β function (or effective Lagrangian). It is convex and superlinear.

- We call $\alpha : H^1(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$ its Fenchel-Legendre transform

$$\alpha(c) = \max_{h\in H_1(\Gamma,\mathbb{R})} \left(\langle c, h\rangle - \beta(h)\right).$$

  α is called Mather's α function (or effective Hamiltonian). It is also convex and superlinear.

  Remark: $-\alpha$ can be also obtained as the minimal values of the action of suitable modified Lagrangians.

A glimpse on how to prove the homogenization result

Key convergence result

Let $t_\varepsilon > 0$ be a sequence of times converging to $\tau > 0$ and let

$$\tilde{x}_\varepsilon := (x_\varepsilon, h_\varepsilon), \quad \tilde{y}_\varepsilon := (y_\varepsilon, \ell_\varepsilon) \in \widetilde{\mathcal{V}} \simeq \mathcal{V} \times \mathbb{Z}^{b_1(\Gamma)} \qquad \forall\, \varepsilon > 0$$

such that $\lim_{\varepsilon \to 0^+} \varepsilon\, h_\varepsilon = h \in \mathbb{R}^{b_1(\Gamma)}$ and $\lim_{\varepsilon \to 0^+} \varepsilon\, \ell_\varepsilon = \ell \in \mathbb{R}^{b_1(\Gamma)}$.
Identifying $H_1(\Gamma, \mathbb{R}) \simeq \mathbb{R}^{b_1(\Gamma)}$, we conclude:

$$\lim_{\varepsilon \to 0^+} \varepsilon\, \Phi\big(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon, \frac{t_\varepsilon}{\varepsilon}\big) = \tau\, \beta \left(\frac{\ell - h}{\tau} \right).$$

One can then prove the convergence of solutions, via the Lax-Oleinik representation formula:

$$u_\varepsilon(x, t) = \inf_{y \in \widetilde{\Gamma}} \left\{ f_\varepsilon(y) + \varepsilon \Phi(y, x, \frac{t}{\varepsilon}) \right\} \quad x \in \widetilde{\Gamma},\ t > 0$$

and showing that for the limit problem:

- Effective Lagrangian $\overline{L}$ $\longleftrightarrow$ $\beta : H_1(\Gamma, \mathbb{R}) \simeq \mathbb{R}^{b_1(\Gamma)} \longrightarrow \mathbb{R}$
- Effective Hamiltonian $\overline{H}$ $\longleftrightarrow$ $\alpha : H^1(\Gamma, \mathbb{R}) \simeq \mathbb{R}^{b_1(\Gamma)} \longrightarrow \mathbb{R}$

with limit solution $\bar{u}(z, t) = \inf_{p \in \mathbb{R}^{b_1(\Gamma)}} \left\{ \bar{f}(p) + t\beta(\frac{z-p}{t}) \right\}$.

Thank you for your attention.
Any question?