On the Birkhoff Conjecture for Convex Billiards

(An analyst, a geometer and a probabilist walk into a bar... And play billiards!)

Alfonso Sorrentino

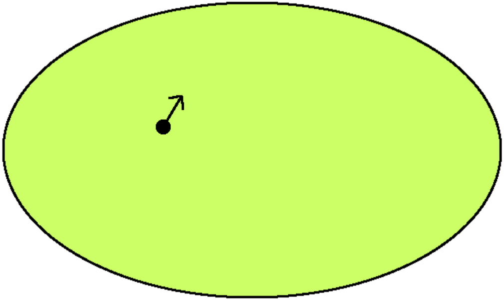
Cardiff (UK), 26^{th} June 2018

UNIVERSITA' DEGLI STUDI DI ROMA Tor Vergata

Matematica Dipartimento Università degli Studi di Roma Tor Vergata

Mathematical Billiards

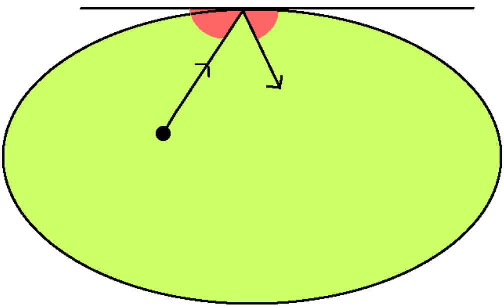
A mathematical billiard consists of a closed region in the plane (the *billiard table*) and a point-mass in its interior (the *ball*), which moves along straight lines with constant velocity.



Mathematical Billiards

A mathematical billiard consists of a closed region in the plane (the *billiard table*) and a point-mass in its interior (the *ball*), which moves along straight lines with constant velocity. When the ball hits the boundary, it reflects *elastically*, namely:

angle of incidence = angle of reflection.

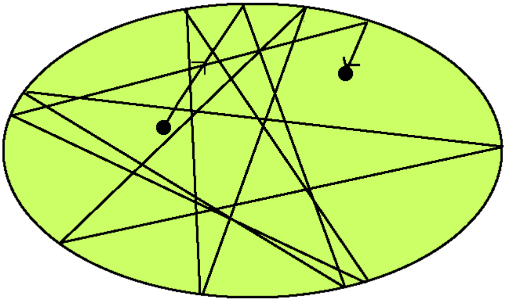


Mathematical Billiards

A mathematical billiard consists of a closed region in the plane (the *billiard table*) and a point-mass in its interior (the *ball*), which moves along straight lines with constant velocity. When the ball hits the boundary, it reflects *elastically*, namely:

angle of incidence = angle of reflection.

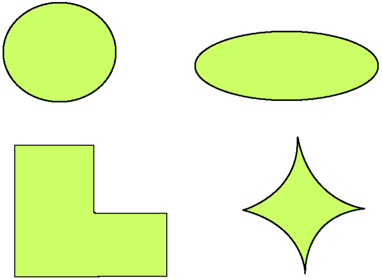
And it keeps on moving... Can we describe the evolution of its dynamics?



Geometry vs. Dynamics

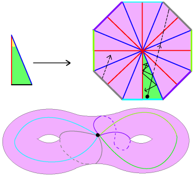
The dynamics inside a billiard is completely determined by its geometry (*i.e.*, its shape)!

One could choose billiard tables with different shapes:

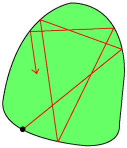


One could also assume that the domain lies inside a Riemannian manifold rather than the Euclidean plane: elastic reflection can be still defined and the ball will move along geodesics instead of straight lines.

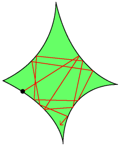
The study of the dynamics of billiards is a very active area of research. Dynamical behaviours and properties are strongly related to the shape of the table.



Polygonal billiards:
- Related to the study of the geodesic flow on a translation surface (with singular points);
- Teichmüller theory.

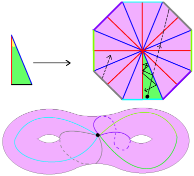


(Strictly) Convex Billiards:
- Birkhoff billiards (G. Birkhoff, 1927: paradigmatic example of Hamiltonian systems).
- The billiard map is a twist map.
- Coexistence of regular (KAM, Aubry-Mather) and chaotic dynamics.

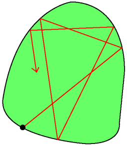


Concave Billiards (or dispersive):
- Nearby Orbits tend to move apart (exponentially).
- Hyperbolicity and chaotic behaviour (Y. Sinai, 1970).
- Study of statistical properties of orbits.

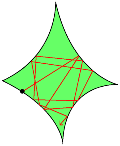
The study of the dynamics of billiards is a very active area of research. Dynamical behaviours and properties are strongly related to the shape of the table.



Polygonal billiards:
- Related to the study of the geodesic flow on a translation surface (with singular points);
- Teichmüller theory.



(Strictly) Convex Billiards:
- Birkhoff billiards (G. Birkhoff, 1927: paradigmatic example of Hamiltonian systems).
- The billiard map is a twist map.
- Coexistence of regular (KAM, Aubry-Mather) and chaotic dynamics.

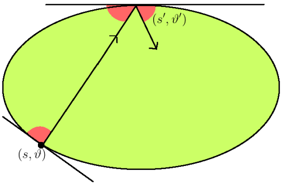


Concave Billiards (or dispersive):
- Nearby Orbits tend to move apart (exponentially).
- Hyperbolicity and chaotic behaviour (Y. Sinai, 1970).
- Study of statistical properties of orbits.

Birkhoff Billiards

Let Ω be a strictly convex domain in $\mathbb{R}^2$ with C^r boundary $\partial\Omega$, with $r \geq 3$. Let $\partial\Omega$ be parametrized by arc-length s (fix an orientation and denote by ℓ its length) and ϑ "shooting" angle (w.r.t. the positive tangent to $\partial\Omega$). The Billiard map is:

$$\begin{aligned} B : \mathbb{R}/\ell\mathbb{Z} \times (0,\pi) &\longrightarrow \mathbb{R}/\ell\mathbb{Z} \times (0,\pi) \\ (s,\vartheta) &\longmapsto (s',\vartheta'). \end{aligned}$$



This simple model has been first proposed by G.D. Birkhoff (1927) as a mathematical playground where "*the formal side, usually so formidable in dynamics, almost completely disappears and only the interesting qualitative questions need to be considered*".

Properties of Birkhoff billiard maps

- B is $C^{r-1}(\mathbb{R}/\ell\mathbb{Z} \times (0, \pi))$;

Properties of Birkhoff billiard maps

- B is $C^{r-1}(\mathbb{R}/\ell\mathbb{Z} \times (0, \pi))$;
- B can be extended continuously up to the boundary:
  $B(\cdot, 0) = B(\cdot, \pi) = Id$;

Properties of Birkhoff billiard maps

- B is $C^{r-1}(\mathbb{R}/\ell\mathbb{Z} \times (0, \pi))$;
- B can be extended continuously up to the boundary: $B(\cdot, 0) = B(\cdot, \pi) = Id$;
- B preserves the area form $\omega = \sin\vartheta \, d\vartheta \wedge ds$ (symplectic form);

Properties of Birkhoff billiard maps

- B is $C^{r-1}(\mathbb{R}/\ell\mathbb{Z} \times (0, \pi))$;
- B can be extended continuously up to the boundary: $B(\cdot, 0) = B(\cdot, \pi) = Id$;
- B preserves the area form $\omega = \sin\vartheta \, d\vartheta \wedge ds$ (symplectic form);
- B is a twist map $\leftarrow$ (Aubry-Mather theory, KAM theory, etc.);

Properties of Birkhoff billiard maps

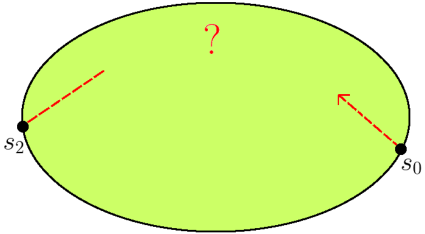
- B is $C^{r-1}(\mathbb{R}/\ell\mathbb{Z} \times (0, \pi))$;
- B can be extended continuously up to the boundary: $B(\cdot, 0) = B(\cdot, \pi) = Id$;
- B preserves the area form $\omega = \sin\vartheta\, d\vartheta \wedge ds$ (symplectic form);
- B is a twist map $\leftarrow$ (Aubry-Mather theory, KAM theory, etc.);
- B has a generating function:

$$h(s, s') := \|\gamma(s) - \gamma(s')\|,$$

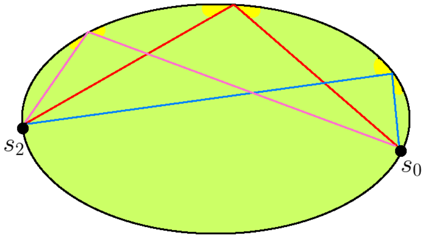
i.e., the Euclidean distance between two points on $\partial\Omega$. In particular if $B(s, \vartheta) = (s', \vartheta')$, then:

$$\begin{cases} \partial_1 h(s, s') = -\cos\vartheta \\ \partial_2 h(s, s') = \cos\vartheta' . \end{cases}$$

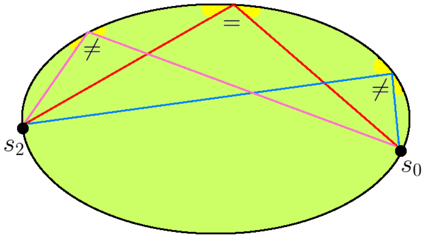
Dynamics and Length

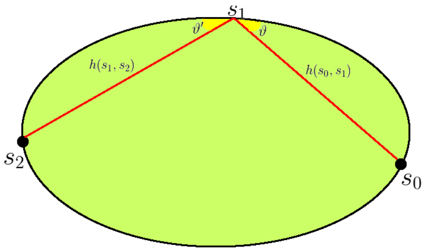


Dynamics and Length



Dynamics and Length





Let us consider the length functional:

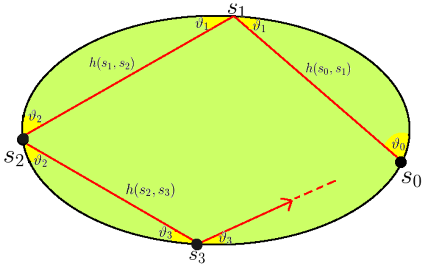
$$\mathcal{L}(s_1) := h(s_0, s_1) + h(s_1, s_2) \quad s_1 \in (s_0, s_2).$$

Then:

$$\frac{d}{ds}\mathcal{L}(s_1) \;=\; \partial_2 h(s_0, s_1) + \partial_1 h(s_1, s_2) = \cos\vartheta - \cos\vartheta'.$$

The real orbit (i.e., $\vartheta = \vartheta'$) correspond to $s_1 \in (s_0, s_2)$ such that $\frac{d}{ds}\mathcal{L}(s_1) = 0$ (i.e., s_1 is a critical point).

Dynamics and Length



$\{(s_n, \vartheta_n)\}_{n\in\mathbb{Z}}$ is an orbit $\iff$ $\{s_n\}_{n\in\mathbb{Z}}$ is a "critical configuration"

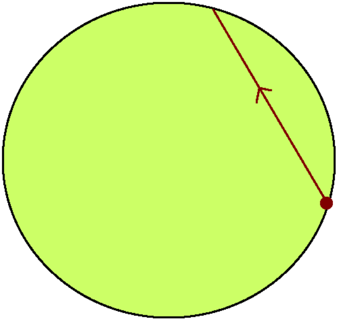
of the Length functional:

$$\mathcal{L}(\{s_n\}_n) := \sum_{n\in\mathbb{Z}} h(s_n, s_{n+1}).$$

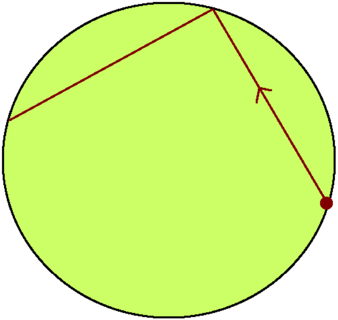
Relation between the Dynamics and the length of trajectories (Geometry).

Example I: Circular billiard

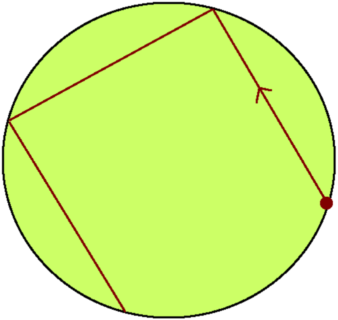
Example I: Circular billiard



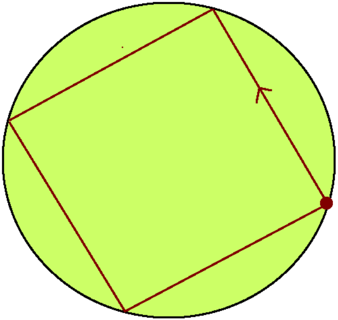
Example I: Circular billiard



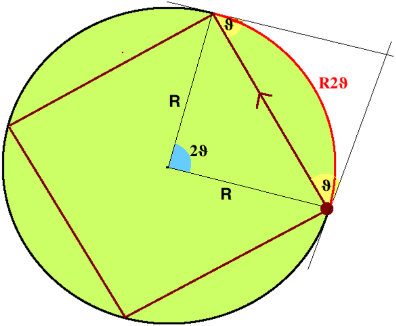
Example I: Circular billiard



Example I: Circular billiard



Example I: Circular billiard



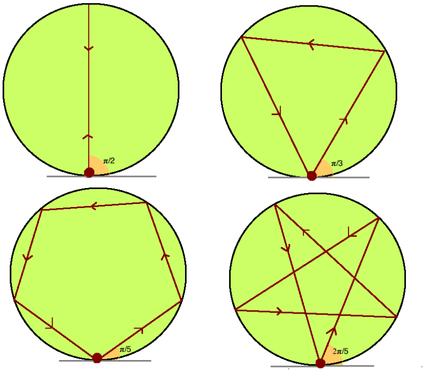
The corresponding Billiard map is:

$$
\begin{aligned}
B : \mathbb{R}/2\pi R\mathbb{Z} \times (0,\pi) &\longrightarrow \mathbb{R}/2\pi R\mathbb{Z} \times (0,\pi) \\
(s,\vartheta) &\longmapsto (s + 2R\vartheta, \vartheta).
\end{aligned}
$$

The angle remains constant at each bounce: it is an Integral of motion.

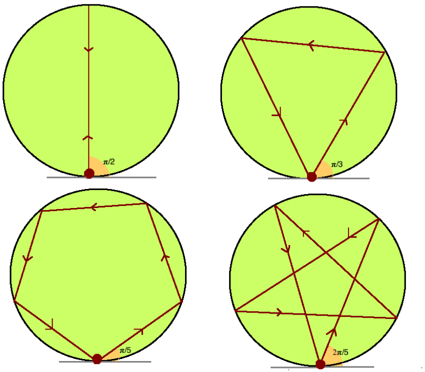
Example I: Circular billiard

If ϑ is a rational multiple of π, then the resulting orbit is periodic:



Example I: Circular billiard

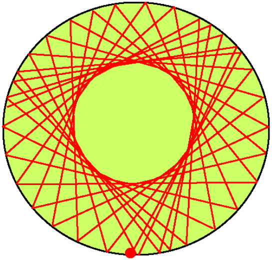
If ϑ is a rational multiple of π, then the resulting orbit is periodic:



For every rational $\frac{p}{q} \in (0, \frac{1}{2}]$ there exist infinitely many periodic orbits with q bounces (period) and which turn p times around before closing (winding number). $\frac{p}{q}$ is called rotation number.

Example I: Circular billiard

If ϑ is NOT a rational multiple of π, then the orbit hits the boundary on a dense set of points:



The trajectory does not fill in the table: there is a region (a disc) which is never crossed by the ball!

Observe that the trajectory is always tangent to a circle (this is an example of caustic).

What is true for general Birkhoff billiards?

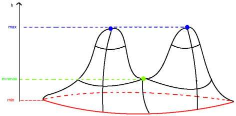
- Do there always exist periodic orbits? How many?

- Are there other integrable billiards?
- How often does the existence of caustics occur?

What is true for general Birkhoff billiards?

- Do there always exist periodic orbits? How many?
  YES! For every rotation number $\frac{p}{q} \in (0, \frac{1}{2}]$ there exist at least two distinct periodic orbits with that rotation number (Birkhoff, 1922).

A variation proof exploits the relation between orbits and lengths: one of the two orbits maximizes the length among all configurations with that rotation number, while the other is obtained via a min-max procedure.



(Mountain pass lemma)

- Are there other integrable billiards?
- How often does the existence of caustics occur?

Geometry $\longleftrightarrow$ Dynamics

The previous questions are all instances of a deep intertwine between geometry and dynamics: while the dependence of the dynamics on the geometry of the domain is well perceptible, an intriguing challenge is:

To what extent dynamical information can be used
to reconstruct the shape of the domain.

Geometry $\longleftrightarrow$ Dynamics

The previous questions are all instances of a deep intertwine between geometry and dynamics: while the dependence of the dynamics on the geometry of the domain is well perceptible, an intriguing challenge is:

To what extent dynamical information can be used
to reconstruct the shape of the domain.

- Which information on the geometry of the billiard domain, does the set of lengths periodic orbits (i.e., the Length spectrum) encode? What dynamical information can one infer from it?

Geometry $\longleftrightarrow$ Dynamics

The previous questions are all instances of a deep intertwine between geometry and dynamics: while the dependence of the dynamics on the geometry of the domain is well perceptible, an intriguing challenge is:

To what extent dynamical information can be used
to reconstruct the shape of the domain.

- Which information on the geometry of the billiard domain, does the set of lengths periodic orbits (i.e., the Length spectrum) encode? What dynamical information can one infer from it?
- Does integrability imply a certain shape? The famous Birkhoff conjecture.

Integrability of billiards

There are several ways to define integrability for Hamiltonian systems:

- Liouville-Arnol'd integrability (existence of integrals of motion);
- C^0 integrability (existence of a foliation by invariant Lagrangian submflds);

Integrability of billiards

There are several ways to define integrability for Hamiltonian systems:

- Liouville-Arnol'd integrability (existence of integrals of motion);
- C^0 integrability (existence of a foliation by invariant Lagrangian submflds);

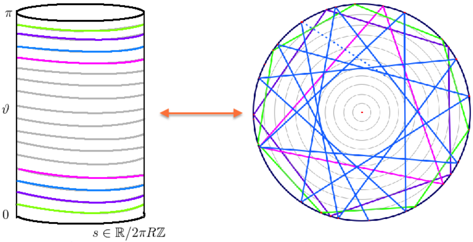
Is it possible to express the integrability of a billiard map in terms of property of the billiard table?

Integrability of billiards

There are several ways to define integrability for Hamiltonian systems:

- Liouville-Arnol'd integrability (existence of integrals of motion);
- C^0 integrability (existence of a foliation by invariant Lagrangian submflds);

Is it possible to express the integrability of a billiard map in terms of property of the billiard table?



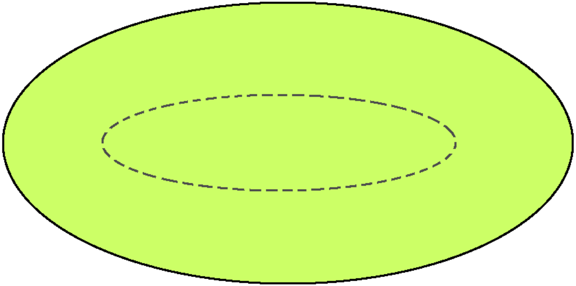
Phase space of the circular billiard map

Foliation of the circular billiard table by caustics

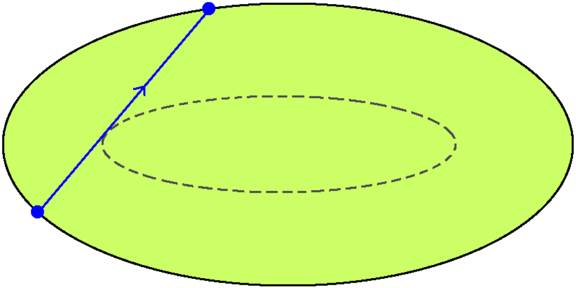
Integrability $\longleftrightarrow$ (Part of) the billiard table is foliated by caustics

Caustics

A convex caustic is a closed C^1 curve in the interior of Ω, bounding itself a strictly convex domain, with the property that each trajectory that is tangent to it stays tangent after each reflection.

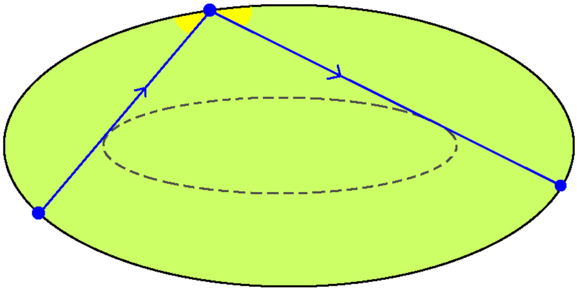


A convex caustic is a closed C^1 curve in the interior of Ω, bounding itself a strictly convex domain, with the property that each trajectory that is tangent to it stays tangent after each reflection.



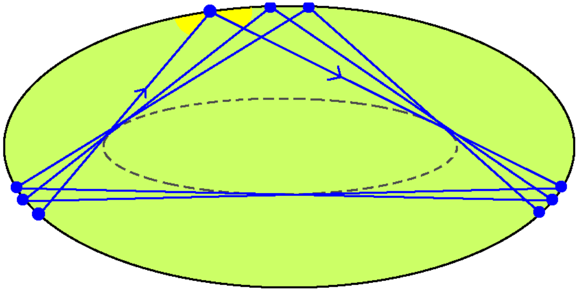
Caustics

A convex caustic is a closed C^1 curve in the interior of Ω, bounding itself a strictly convex domain, with the property that each trajectory that is tangent to it stays tangent after each reflection.



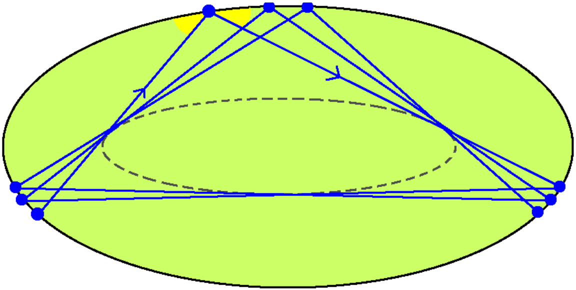
Caustics

A convex caustic is a closed C^1 curve in the interior of Ω, bounding itself a strictly convex domain, with the property that each trajectory that is tangent to it stays tangent after each reflection.



Caustics

A convex caustic is a closed C^1 curve in the interior of Ω, bounding itself a strictly convex domain, with the property that each trajectory that is tangent to it stays tangent after each reflection.

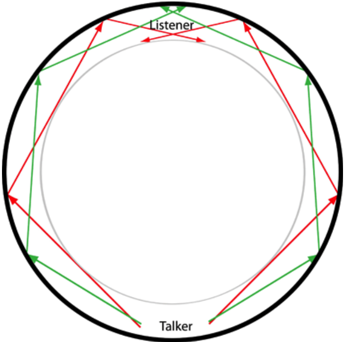


To a convex caustic in Ω corresponds an invariant circle for the billiard map.

(The converse is not entirely true: invariant curves give rise to caustics, but they might not be convex, nor differentiable).

Caustics and Whispering Galleries

Whispering Gallery in St. Paul Cathedral in London (Lord Rayleigh, 1878 ca.)

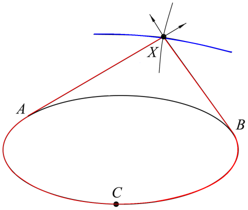


Existence of Caustics

- Do there exist other examples of billiards with at least one caustic?

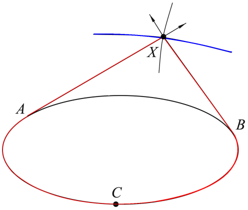
Existence of Caustics

- Do there exist other examples of billiards with at least one caustic?
  Easy to construct by means of the string construction:



Existence of Caustics

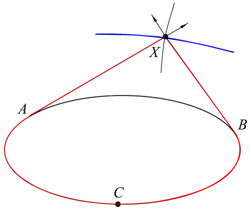
- Do there exist other examples of billiards with at least one caustic?
  Easy to construct by means of the string construction:



- Do there exist other examples of billiards with infinitely many caustic?

Existence of Caustics

- Do there exist other examples of billiards with at least one caustic?
  Easy to construct by means of the string construction:

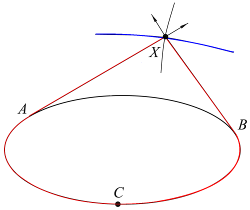


- Do there exist other examples of billiards with infinitely many caustic?

  YES! Lazutkin (1973) proved that by a suitable change of coordinates every Birkhoff billiard map becomes nearly integrable!
  Hence, if the domain is sufficiently smooth, he proved by means of KAM technique that there exists (at least) a Cantor set of invariant circles near the boundary (i.e., infinitely many caustics accumulating to the boundary of the table).

Existence of Caustics

- Do there exist other examples of billiards with at least one caustic?
  Easy to construct by means of the string construction:



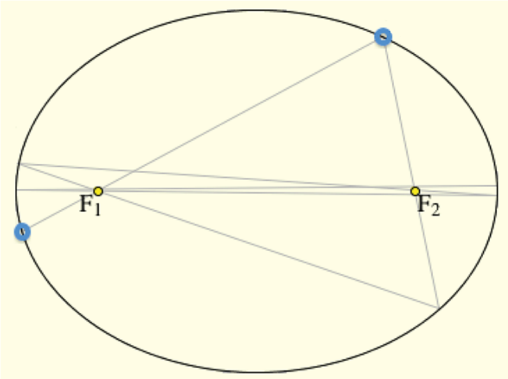
- Do there exist other examples of billiards with infinitely many caustic?

  YES! Lazutkin (1973) proved that by a suitable change of coordinates every Birkhoff billiard map becomes nearly integrable!
  Hence, if the domain is sufficiently smooth, he proved by means of KAM technique that there exists (at least) a Cantor set of invariant circles near the boundary (i.e., infinitely many caustics accumulating to the boundary of the table).

- Do there exist other examples of billiards admitting a foliation by caustics?

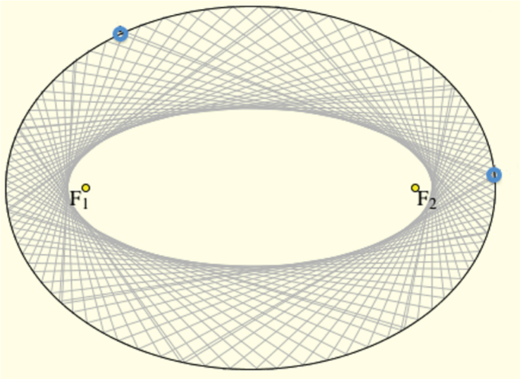
Example II: Elliptic billiard

Example II: Elliptic billiard



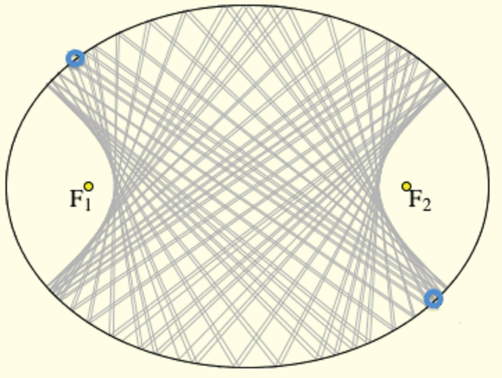
If the trajectory passes through one of the foci, then it always passes through them, alternatively.

Example II: Elliptic billiard



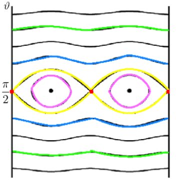
If the trajectory does not intersect the segment between the foci, then it never does and it is tangent to a confocal ellipse (a convex caustic).

Example II: Elliptic billiard

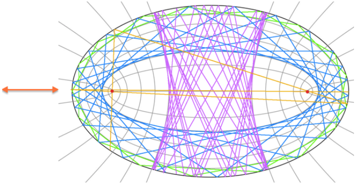


If the trajectory intersects the segment between the foci, then it always does and it is tangent to a confocal hyperbola (a non-convex caustic).

Example II: Elliptic billiard



Phase space of an elliptic billiard map


Dynamics inside an elliptic billiard and caustics

Some Properties of Elliptic billiards:

- For every rational $\frac{p}{q} \in (0, \frac{1}{2})$ there exist infinitely many periodic orbits rotation number $\frac{p}{q}$.
- There exist only two periodic orbits of period 2 (i.e., rotation number $\frac{1}{2}$): the two semi-axes.
- There exist infinitely many convex caustics (and also non-convex ones).

The ellipse, with the exception of the closed segment between the foci, is foliated by convex caustics. It is an Integrable billiard.

Birkhoff conjecture

Conjecture (Birkhoff-Poritsky)

The only integrable billiard maps correspond to billiards inside ellipses.

Although some vague indications of this question can be found in Birkhoff's works (1920's-30's), its first appearance was in a paper by Poritsky (1950), who was a National Research Fellow in Mathematics at Harvard University, presumably under the supervision of Birkhoff.

It quickly became one of the most famous - and hard to tackle - questions in dynamical systems.

Birkhoff conjecture

Conjecture (Birkhoff-Poritsky)

The only integrable billiard maps correspond to billiards inside ellipses.

Although some vague indications of this question can be found in Birkhoff's works (1920's-30's), its first appearance was in a paper by Poritsky (1950), who was a National Research Fellow in Mathematics at Harvard University, presumably under the supervision of Birkhoff.

It quickly became one of the most famous - and hard to tackle - questions in dynamical systems.

It is important to consider strictly convex domains!
Mather (1982) proved the non-existence of caustics (hence, some sort of non-integrability) if the curvature of the boundary vanishes at (at least) one point. See also Gutkin-Katok (1995).

Previous contributions

Despite its long history and the amount of attention that this conjecture has captured, it remains still open. Important contributions are the following:

- Bialy (1993): If the phase space of the billiard map is completely foliated by continuous invariant curves which are not null-homotopic, then it is a circular billiard.

Previous contributions

Despite its long history and the amount of attention that this conjecture has captured, it remains still open. Important contributions are the following:

- Bialy (1993): If the phase space of the billiard map is completely foliated by continuous invariant curves which are not null-homotopic, then it is a circular billiard.
- An integral-geometric approach to prove Bialy's result was proposed by Wojtkowski (1994), by means of the so-called mirror formula.

Previous contributions

Despite its long history and the amount of attention that this conjecture has captured, it remains still open. Important contributions are the following:

- Bialy (1993): If the phase space of the billiard map is completely foliated by continuous invariant curves which are not null-homotopic, then it is a circular billiard.
- An integral-geometric approach to prove Bialy's result was proposed by Wojtkowski (1994), by means of the so-called mirror formula.
- Innami (2002) showed that the existence of caustics with rotation numbers accumulating to $1/2$ implies that the billiard is an ellipse; the proof is based on Aubry-Mather theory (a simpler proof by Arnold-Bialy (2018).

Previous contributions

Despite its long history and the amount of attention that this conjecture has captured, it remains still open. Important contributions are the following:

- Bialy (1993): If the phase space of the billiard map is completely foliated by continuous invariant curves which are not null-homotopic, then it is a circular billiard.
- An integral-geometric approach to prove Bialy's result was proposed by Wojtkowski (1994), by means of the so-called mirror formula.
- Innami (2002) showed that the existence of caustics with rotation numbers accumulating to $1/2$ implies that the billiard is an ellipse; the proof is based on Aubry-Mather theory (a simpler proof by Arnold-Bialy (2018).
- In a different setting, when there exists an integral of motion that is polynomial in the velocity (Algebraic Birkhoff conjecture), the fact that the billiard is an ellipse has been recently proved by Glutsyuk (2018), based on previous results by Bialy-Mironov (2017).

Perturbative Birkhoff conjecture

One could restrict the analysis to what happens for domains that are sufficiently close to ellipses.

Perturbative Birkhoff conjecture

One could restrict the analysis to what happens for domains that are sufficiently close to ellipses.

Birkhoff Conjecture (Perturbative version)

A smooth strictly convex domain that is sufficiently close (w.r.t. some topology) to an ellipse and whose corresponding billiard map is integrable, is necessarily an ellipse.

- First results in this direction were obtained by:
  - Levallois (1993): Non-integrability of algebraic perturbations of elliptic billiards.
  - Delshams and Ramírez-Ros (1996): Non-integrability of entire symmetric perturbations of ellipses (these perturbations break integrability near the homoclinic solutions).

Perturbative Birkhoff conjecture

One could restrict the analysis to what happens for domains that are sufficiently close to ellipses.

Birkhoff Conjecture (Perturbative version)

A smooth strictly convex domain that is sufficiently close (w.r.t. some topology) to an ellipse and whose corresponding billiard map is integrable, is necessarily an ellipse.

- First results in this direction were obtained by:
  - Levallois (1993): Non-integrability of algebraic perturbations of elliptic billiards.
  - Delshams and Ramírez-Ros (1996): Non-integrability of entire symmetric perturbations of ellipses (these perturbations break integrability near the homoclinic solutions).
- Avila, De Simoi and Kaloshin (2016) proved that perturbative version of Birkhoff conjecture holds true for domains that are nearly circular.

Rational integrability

We consider a weaker notion of integrability: we focus on what happens when caustics with rational rotation numbers exist (very fragile objects!).

Rational integrability

Let Ω be a strictly convex domain.
(i) We say that Γ is an integrable rational caustic for the billiard map in Ω, if the corresponding (non-contractible) invariant curve Γ consists of periodic points; in particular, the corresponding rotation number is rational.

(ii) Let $q_0 \geq 2$ be a positive integer. If the billiard map inside Ω admits integrable rational caustics for all rotation numbers $0 < \frac{p}{q} < \frac{1}{q_0}$, we say that Ω is q_0-rationally integrable.

Main Result: the Perturbative Birkhoff Conjecture

Our main result is that the Perturbative Birkhoff conjecture holds true for any ellipse.

Main Result: the Perturbative Birkhoff Conjecture

Our main result is that the Perturbative Birkhoff conjecture holds true for any ellipse. More specifically:

Theorem [Kaloshin - S. (Ann. of Math, 2018)]

Let $\mathcal{E}_0$ be an ellipse of eccentricity $0 \leq e_0 < 1$ and semi-focal distance c; let $k \geq 39$. For every $K > 0$, there exists $\varepsilon = \varepsilon(e_0, c, K)$ such that the following holds.

Let Ω be a C^k domain such that:

- Ω admits integrable rational caustics of rotation number $1/q$ for $q \geq 3$ ($\Longleftarrow$ 2-rational integrability);
- $\partial\Omega$ is K-close to $\mathcal{E}_0$, with respect to the C^k-norm,
- $\partial\Omega$ is ε-close to $\mathcal{E}_0$, with respect to the C^1-norm,

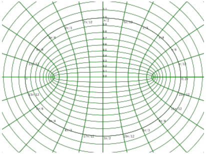
then Ω must be an ellipse.

Sketch of the proof of Theorem [Kaloshin-S.] 1/5

- Consider elliptic coordinates (μ, φ):

$$\begin{cases} x = c \cosh \mu \, \cos \varphi \\ y = c \sinh \mu \, \sin \varphi \end{cases}$$

describing confocal ellipses ($\mu = \mu_0$) and hyperbolae ($\varphi = \varphi_0$); $c > 0$ represents the semifocal distance.

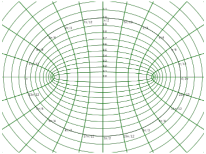


Sketch of the proof of Theorem [Kaloshin-S.] 1/5

- Consider elliptic coordinates (μ, φ):

$$\begin{cases} x = c \cosh \mu \, \cos \varphi \\ y = c \sinh \mu \, \sin \varphi \end{cases}$$

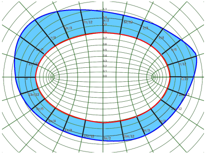
describing confocal ellipses ($\mu = \mu_0$) and hyperbolae ($\varphi = \varphi_0$); $c > 0$ represents the semifocal distance.



- We express a perturbation of a given ellipse $\{\mu = \mu_0\}$ as:

$$\mu_\varepsilon(\varphi) = \mu_0 + \varepsilon \mu_1(\varphi) + O(\varepsilon^2).$$

(Observe that the coordinate frame depends on the unperturbed ellipse)



Sketch of the proof of Theorem [Kaloshin-S.] 2/5

Let us start by considering a rationally integrable deformation Ω_ε of $\Omega_0 = \mathcal{E}_0$.

Action-angle coordinates for the billiard map in the ellipse $\mathcal{E}_0$. For $q \geq 3$, let $\varphi_q(\theta)$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, denote the parametrization of the boundary induced by the dynamics on the caustic of rotation number $1/q$:

$$B_{\mathcal{E}_0}(\mu_0, \varphi_q(\theta)) = (\mu_0, \varphi_q(\theta + 2\pi/q)).$$

Sketch of the proof of Theorem [Kaloshin-S.] 2/5

Let us start by considering a rationally integrable deformation Ω_ε of $\Omega_0 = \mathcal{E}_0$.

Action-angle coordinates for the billiard map in the ellipse $\mathcal{E}_0$. For $q \geq 3$, let $\varphi_q(\theta)$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, denote the parametrization of the boundary induced by the dynamics on the caustic of rotation number $1/q$:

$$B_{\mathcal{E}_0}(\mu_0, \varphi_q(\theta)) = (\mu_0, \varphi_q(\theta + 2\pi/q)).$$

Lemma [Pinto-de-Carvalho, Ramírez-Ros (2013)]

Let Ω_ε admit a rationally integrable caustic of rotation number $1/q$ for all ε. We denote by $\{\varphi_q^k\}_{k=0}^q$ the periodic orbit of the billiard map in $\mathcal{E}_0$ with rotation number $1/q$ and starting at φ; then $L_1(\varphi) = \sum_{k=1}^q \mu_1(\varphi_q^k) \equiv c_q$, where c_q is a constant independent of φ.

$L_1(\varphi)$ represents the subharmonic Melnikov potential of the elliptic caustic of rotation number $1/q$ under the deformation.

Sketch of the proof of Theorem [Kaloshin-S.] 3/5

Therefore, with respect to the action-angle variables we have that for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$:

$$\sum_{k=1}^{q} \mu_1(\varphi_q(\theta + 2\pi k/q)) \equiv \mathrm{c_q}.$$

Sketch of the proof of Theorem [Kaloshin-S.] 3/5

Therefore, with respect to the action-angle variables we have that for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$:

$$\sum_{k=1}^{q} \mu_1(\varphi_q(\theta + 2\pi k/q)) \equiv \mathrm{c_q}.$$

If $u(x)$ denotes either $\cos x$ and $\sin x$, then

$$\int_0^{2\pi} \mu_1(\varphi_q(\theta))\, u(q\,\theta)\, d\theta = 0,$$

which, using the expression for φ_q and by some change of variables, implies:

$$\int_0^{2\pi} \mu_1(\varphi)\, \frac{u\left(\frac{2\pi\, q}{4K(k_q)} F(\varphi; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2 \varphi}}\, d\varphi = 0.$$

- k_q is the eccentricity of the elliptic caustic of rotation number $1/q$
- $F(\varphi, k)$ the incomplete elliptic integral of the first kind;
- $K(k)$ the complete elliptic integral of the first kind, i.e. $K(k) = F(\pi/2, k)$.

Sketch of the proof of Theorem [Kaloshin-S.] 4/5

We define a family of dynamical modes $\{c_q, s_q\}_{q\geq 3}$ given by

$$c_q(\varphi) := \frac{\cos\left(\frac{2\pi\, q}{4K(k_q)} F(\varphi; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2 \varphi}} \qquad s_q(\varphi) := \frac{\sin\left(\frac{2\pi\, q}{4K(k_q)} F(\varphi; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2 \varphi}}.$$

These functions only depend on μ_0 and q.

Sketch of the proof of Theorem [Kaloshin-S.] 4/5

We define a family of dynamical modes $\{c_q, s_q\}_{q\geq 3}$ given by

$$c_q(\varphi) := \frac{\cos\left(\frac{2\pi\, q}{4K(k_q)} F(\varphi; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2 \varphi}} \qquad s_q(\varphi) := \frac{\sin\left(\frac{2\pi\, q}{4K(k_q)} F(\varphi; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2 \varphi}}.$$

These functions only depend on μ_0 and q.

Summarizing: if $\mu_\varepsilon(\varphi) = \mu_0 + \varepsilon\mu_1(\varphi) + O(\varepsilon^2)$ is a deformation of the ellipse $\mathcal{E}_0 = \{\mu = \mu_0\}$ which preserves the integrable caustic of rotation number $1/q$, then

$$< \mu_1, c_q >_{L^2} = < \mu_1, s_q >_{L^2} = 0$$

Sketch of the proof of Theorem [Kaloshin-S.] 4/5

We define a family of dynamical modes $\{c_q, s_q\}_{q\geq 3}$ given by

$$c_q(\varphi) := \frac{\cos\left(\frac{2\pi\, q}{4K(k_q)} F(\varphi; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2\varphi}} \qquad s_q(\varphi) := \frac{\sin\left(\frac{2\pi\, q}{4K(k_q)} F(\varphi; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2\varphi}}.$$

These functions only depend on μ_0 and q.

Summarizing: if $\mu_\varepsilon(\varphi) = \mu_0 + \varepsilon\mu_1(\varphi) + O(\varepsilon^2)$ is a deformation of the ellipse $\mathcal{E}_0 = \{\mu = \mu_0\}$ which preserves the integrable caustic of rotation number $1/q$, then

$$< \mu_1, c_q >_{L^2} = < \mu_1, s_q >_{L^2} = 0$$

Consider also five extra functions related to elliptic motions: $e_1, \ldots, e_5$: they correspond to infinitesimal generators of motions that transform ellipses into ellipses (translations, rotations, homotheties, hyperbolic rotations).

Sketch of the proof of Theorem [Kaloshin-S.] 5/5

Key result: Basis property

$\{e_j\}_{j=1}^5 \cup \{c_q, s_q\}_{q\geq 3}$ form a basis of $L^2(\mathbb{T})$.

Sketch of the proof of Theorem [Kaloshin-S.] 5/5

Key result: Basis property

$\{e_j\}_{j=1}^5 \cup \{c_q, s_q\}_{q\geq 3}$ form a basis of $L^2(\mathbb{T})$.

Idea: Make them (more) complex!

Sketch of the proof of Theorem [Kaloshin-S.] 5/5

Key result: Basis property

$\{e_j\}_{j=1}^5 \cup \{c_q, s_q\}_{q\geq 3}$ form a basis of $L^2(\mathbb{T})$.

Idea: Make them (more) complex!

- Consider complex analytic extensions of these functions.
- A detailed study of their complex singularities and the size of their maximal strips of analiticity, allow us to deduce their linear independence (both for finite and infinite combinations).
- By a codimension argument, show that they form a set of generators.

Sketch of the proof of Theorem [Kaloshin-S.] 5/5

Key result: Basis property

$\{e_j\}_{j=1}^5 \cup \{c_q, s_q\}_{q\geq 3}$ form a basis of $L^2(\mathbb{T})$.

Idea: Make them (more) complex!

- Consider complex analytic extensions of these functions.
- A detailed study of their complex singularities and the size of their maximal strips of analiticity, allow us to deduce their linear independence (both for finite and infinite combinations).
- By a codimension argument, show that they form a set of generators.

From Deformative to Perturbative Setting:

- Annihilation conditions are replaced by smallness condition;
- Approximate $\partial\Omega$ with its "best" approximating ellipse:

$$\partial\Omega = \{(\mu_0^* + \mu_{\rm pert}(\varphi), \varphi) : \ \varphi \in [0, 2\pi)\};$$

- Using smallness conditions and Basis property, deduce that $\|\mu_{\rm pert}\|_{L^2}$ must be zero.

Local integrability and Birkhoff conjecture

One could consider weaker notions of integrability.

For example: what can be said for locally integrable Birkhoff billiards? Namely, consider integrability in a neighborhood of the boundary of the billiard table, i.e., for sufficiently small rotation numbers.

Local integrability and Birkhoff conjecture

One could consider weaker notions of integrability.

For example: what can be said for locally integrable Birkhoff billiards? Namely, consider integrability in a neighborhood of the boundary of the billiard table, i.e., for sufficiently small rotation numbers.

The analogous conjecture would be:

Local Birkhoff Conjecture (LBC)

If Ω is a Birkhoff billiard admitting a foliation by caustics with rotation numbers in $(0, \delta)$, for some $0 < \delta \leq 1/2$, then Ω must be an ellipse.

Local integrability and Birkhoff conjecture

One could consider weaker notions of integrability.

For example: what can be said for locally integrable Birkhoff billiards? Namely, consider integrability in a neighborhood of the boundary of the billiard table, i.e., for sufficiently small rotation numbers.

The analogous conjecture would be:

Local Birkhoff Conjecture (LBC)

If Ω is a Birkhoff billiard admitting a foliation by caustics with rotation numbers in $(0, \delta)$, for some $0 < \delta \leq 1/2$, then Ω must be an ellipse.

For $\delta = 1/2$ it follows from a result by Innami (2002).

Local Perturbative Birkhoff conjecture (LPBC)

Let us consider a perturbative version of this conjecture.

Local Perturbative Birkhoff Conjecture (LBC)

For any integer $q_0 \geq 2$, there exist $e_0 = e_0(q_0) \in (0, 1)$, $m_0 = m_0(q_0)$, $n_0 = n_0(q_0) \in \mathbb{N}$ such that the following holds.

For each $0 < e \leq e_0$ and $c \geq 0$, there exists $\varepsilon = \varepsilon(e, c, q_0) > 0$ such that if

- $\mathcal{E}_0$ is an ellipse of eccentricity e and semi-focal distance c,
- Ω is q_0-rationally integrable,
- $\partial\Omega$ is C^{m_0} domain,
- $\partial\Omega$ is ε-close (in the C^{n_0} topology) to $\mathcal{E}_0$,

$\Longrightarrow$ Ω itself is an ellipse.

For $q_0 = 2$ it follows from our previous result [KS 2018] ($e_0 = 1$, $n_0 = 1$, $m_0 = 39$).

Local Perturbative Birkhoff conjecture (LPBC)

Theorem [Huang, Kaloshin, S. (GAFA, 2018)]

- LBC holds true for $q_0 = 2, 3, 4, 5$, with $m_0 = 40q_0$ and $n_0 = 3q_0$.
- LBC holds true for $q_0 > 5$ with $m_0 = 40q_0$ and $n_0 = 3q_0$, subject to checking that $q_0 - 2$ matrices (which are explicitely described) are invertible.

Local Perturbative Birkhoff conjecture (LPBC)

Theorem [Huang, Kaloshin, S. (GAFA, 2018)]

- LBC holds true for $q_0 = 2, 3, 4, 5$, with $m_0 = 40q_0$ and $n_0 = 3q_0$.
- LBC holds true for $q_0 > 5$ with $m_0 = 40q_0$ and $n_0 = 3q_0$, subject to checking that $q_0 - 2$ matrices (which are explicitely described) are invertible.

DIFFICULTY: We cannot use the preservation of integrable rational caustics for all rotation number $1/q$, with $q \geq 3$; hence, we need to recover the missing "annihilation" conditions.

Local Perturbative Birkhoff conjecture (LPBC)

Theorem [Huang, Kaloshin, S. (GAFA, 2018)]

- LBC holds true for $q_0 = 2, 3, 4, 5$, with $m_0 = 40q_0$ and $n_0 = 3q_0$.
- LBC holds true for $q_0 > 5$ with $m_0 = 40q_0$ and $n_0 = 3q_0$, subject to checking that $q_0 - 2$ matrices (which are explicitely described) are invertible.

DIFFICULTY: We cannot use the preservation of integrable rational caustics for all rotation number $1/q$, with $q \geq 3$; hence, we need to recover the missing "annihilation" conditions.

IDEA:

- Study the Taylor expansion, with respect to the eccentricity e, of the corresponding action-angle coordinates.
- Derive the necessary condition for the preservation of integrable rational caustics, in terms of the Fourier coefficients of the perturbation, up to the precision of order e^{2N}, for some positive integer $N = N(q_0)$.
- Combine several of these conditions (involving also the missing coefficients) to get a linear system to be solved.

The length spectrum

We define the Length spectrum of Ω:

$$\mathcal{L}(\Omega) := \mathbb{N}^+ \cdot \{\text{lengths of billiard periodic orbits in } \Omega\} \cup \ell \cdot \mathbb{N}^+.$$

The length spectrum

We define the Length spectrum of Ω:

$$\mathcal{L}(\Omega) := \mathbb{N}^+ \cdot \{\text{lengths of billiard periodic orbits in } \Omega\} \cup \ell \cdot \mathbb{N}^+.$$

There is a deep relation between this set and the spectrum of the Laplacian on Ω (e.g., with Dirichlet boundary conditions).

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The length spectrum

We define the Length spectrum of Ω:

$$\mathcal{L}(\Omega) := \mathbb{N}^+ \cdot \{\text{lengths of billiard periodic orbits in } \Omega\} \cup \ell \cdot \mathbb{N}^+.$$

There is a deep relation between this set and the spectrum of the Laplacian on Ω (e.g., with Dirichlet boundary conditions).

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem (Andersson and Melrose, 1977)

The wave trace $w(t) := \sum_{\lambda_i \in spec\Delta} \cos(t\sqrt{-\lambda_i})$ is well-defined as a distribution and it is smooth away from the length spectrum:

$$\text{sing. supp.}\big(w(t)\big) \subseteq \pm\mathcal{L}(\Omega) \cup \{0\}.$$

Generically, equality holds.

Hence, at least for generic domains, one can recover the length spectrum from the Laplace one.

The Marked Length Spectrum

One could also refine $\mathcal{L}(\Omega)$. Consider pairs (length, rotation number) and define the Marked Length spectrum $\mathcal{ML}(\Omega)$.

(This is also related to Mather's β-function for billiards)

The Marked Length Spectrum

One could also refine $\mathcal{L}(\Omega)$. Consider pairs (length, rotation number) and define the Marked Length spectrum $\mathcal{ML}(\Omega)$.

(This is also related to Mather's β-function for billiards)

Question: If $\mathcal{ML}(\Omega_1) \equiv \mathcal{ML}(\Omega_2)$, are Ω_1 and Ω_2 isometric?

The Marked Length Spectrum

One could also refine $\mathcal{L}(\Omega)$. Consider pairs (length, rotation number) and define the Marked Length spectrum $\mathcal{ML}(\Omega)$.

(This is also related to Mather's β-function for billiards)

Question: If $\mathcal{ML}(\Omega_1) \equiv \mathcal{ML}(\Omega_2)$, are Ω_1 and Ω_2 isometric?

Affirmative answer if one of the two is a disc (easy).

The Marked Length Spectrum

One could also refine $\mathcal{L}(\Omega)$. Consider pairs (length, rotation number) and define the Marked Length spectrum $\mathcal{ML}(\Omega)$.

(This is also related to Mather's β-function for billiards)

Question: If $\mathcal{ML}(\Omega_1) \equiv \mathcal{ML}(\Omega_2)$, are Ω_1 and Ω_2 isometric?

Affirmative answer if one of the two is a disc (easy).

- What about ellipses?

The Marked Length Spectrum

One could also refine $\mathcal{L}(\Omega)$. Consider pairs (length, rotation number) and define the Marked Length spectrum $\mathcal{ML}(\Omega)$.

(This is also related to Mather's β-function for billiards)

Question: If $\mathcal{ML}(\Omega_1) \equiv \mathcal{ML}(\Omega_2)$, are Ω_1 and Ω_2 isometric?

Affirmative answer if one of the two is a disc (easy).

- What about ellipses?

Corollary [KS 2018]

If a domain is "close" to an ellipse and has the same Marked Length spectrum of an ellipse, then it must be an ellipse.

From the spectrum to the dynamics

What dynamical information does $\mathcal{ML}(\Omega)$ encode?

From the spectrum to the dynamics

What dynamical information does $\mathcal{ML}(\Omega)$ encode?

Guillemin and Melrose (1979) asked whether the length spectrum and the eigenvalues of the linearizations of the (iterated) billiard map at periodic orbits constitute a complete set of symplectic invariants for the system.

From the spectrum to the dynamics

What dynamical information does $\mathcal{ML}(\Omega)$ encode?

Guillemin and Melrose (1979) asked whether the length spectrum and the eigenvalues of the linearizations of the (iterated) billiard map at periodic orbits constitute a complete set of symplectic invariants for the system.

Theorem [Huang, Kaloshin, S. (Duke Math. Journal, 2018)]

For generic billiard domain, it is possible to recover from the (maximal) marked length spectrum, the Lyapunov exponents of its Aubry-Mather (A-M) orbits), i.e., the periodic orbits with maximal length in their rotation number class.

From the spectrum to the dynamics

What dynamical information does $\mathcal{ML}(\Omega)$ encode?

Guillemin and Melrose (1979) asked whether the length spectrum and the eigenvalues of the linearizations of the (iterated) billiard map at periodic orbits constitute a complete set of symplectic invariants for the system.

Theorem [Huang, Kaloshin, S. (Duke Math. Journal, 2018)]

For generic billiard domain, it is possible to recover from the (maximal) marked length spectrum, the Lyapunov exponents of its Aubry-Mather (A-M) orbits), i.e., the periodic orbits with maximal length in their rotation number class.

(Vague) IDEA: Approximate an A-M orbit by a suitable sequence of other A-M orbits, do an asymptotic analysis of their minimal averaged action and show that this allows to recover its Lyapunov exponents....

Possible generalisations: from local to global

What about a global version of these results?

There are not even solid indications that these theorems should be true!

Possible generalisations: from local to global

What about a global version of these results?

There are not even solid indications that these theorems should be true!

Possible approach (Speculations...):
Find a geometric flow that:
- preserves (strict) convexity,
- preserves integrability,
- asymptotically transforms any convex domain into an ellipse (up to some normalization).

Possible generalisations: from local to global

What about a global version of these results?

There are not even solid indications that these theorems should be true!

Possible approach (Speculations...):
Find a geometric flow that:
- preserves (strict) convexity,
- preserves integrability,
- asymptotically transforms any convex domain into an ellipse (up to some normalization).

Candidates: curvature flow (NO!, it does not preserve integrability, Damasceno, Dias Carneir, Ramírez-Ros (2017)), affine curvature flow (maybe?), ... Any other suggestion?

Integrable geodesic flows on the Torus (Work in Progress)

Birkhoff conjecture can be thought as the analogue, in the case of billiards, of the following question: classify integrable (Riemannian) geodesic flows on $\mathbb{T}^2$.

The complexity of this question depends on the notion of integrability.

Integrable geodesic flows on the Torus (Work in Progress)

Birkhoff conjecture can be thought as the analogue, in the case of billiards, of the following question: classify integrable (Riemannian) geodesic flows on $\mathbb{T}^2$.

The complexity of this question depends on the notion of integrability.

- If one assumes that the whole space space is foliated by invariant Lagrangian graphs (C^0-integrability), then it follows from Hopf conjecture that the associated metric must be flat. (Similar to Bialy's result for billiards.)

Integrable geodesic flows on the Torus (Work in Progress)

Birkhoff conjecture can be thought as the analogue, in the case of billiards, of the following question: classify integrable (Riemannian) geodesic flows on $\mathbb{T}^2$.

The complexity of this question depends on the notion of integrability.

- If one assumes that the whole space space is foliated by invariant Lagrangian graphs (C^0-integrability), then it follows from Hopf conjecture that the associated metric must be flat. (Similar to Bialy's result for billiards.)
- This question is still open if one considers integrability only on an open and dense set (global integrability), or assumes the existence of an open set foliated by invariant Lagrangian graphs (local integrability).

Integrable geodesic flows on the Torus (Work in Progress)

Birkhoff conjecture can be thought as the analogue, in the case of billiards, of the following question: classify integrable (Riemannian) geodesic flows on $\mathbb{T}^2$.

The complexity of this question depends on the notion of integrability.

- If one assumes that the whole space space is foliated by invariant Lagrangian graphs (C^0-integrability), then it follows from Hopf conjecture that the associated metric must be flat. (Similar to Bialy's result for billiards.)
- This question is still open if one considers integrability only on an open and dense set (global integrability), or assumes the existence of an open set foliated by invariant Lagrangian graphs (local integrability).

Example of globally integrable (non-flat) geodesic flows on $\mathbb{T}^2$ are those associated to Liouville-type metrics:

$$ds^2 = (f_1(x_1) + f_2(x_2))\,(dx_1^2 + dx_2^2).$$

Folklore conjecture: these metrics are the only globally (resp. locally) integrable metrics on $\mathbb{T}^2$.

Work in progress (with Kaloshin and J. Zhang): apply similar ideas to prove a perturbative version of this conjecture.

Thank you for your attention