The Principle of Least Action in Hamiltonian dynamics, Analysis and Symplectic geometry

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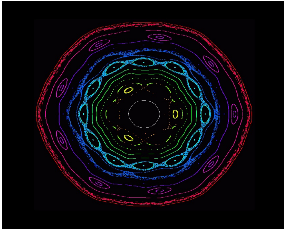
Introduction

Study of the dynamics of Hamiltonian systems

Order (stability) *versus* Chaos (instability)

- Methods from classical mechanics
- Perturbative methods (KAM theory,...)



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- Geometric methods (Symplectic geometry, Floer Homology,...)
- PDE methods (Hamilton-Jacobi, weak KAM theory, ...)

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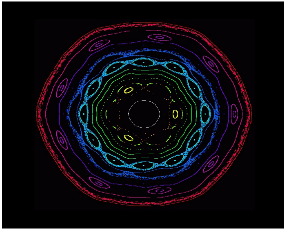

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Aubry - Mather theory

Variational methods based on the *Principle of Least Lagrangian Action* (*"Nature is thrifty in all its actions"*, Pierre Louis Moreau de Maupertuis, 1744).

- Serge Aubry & John Mather '80s: twist maps of the annulus;
- John Mather '90s: Hamiltonian flows of *Tonelli* type.

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Tonelli Hamiltonian

Let M be finite dimensional closed Riemannian manifold.
$H \in C^2(T^*M, \mathbb{R})$ is said to be *Tonelli* if:

- H is strictly convex in each fibre: $\partial^2_{pp} H(x,p) > 0$;
- H is super-linear in each fibre:

$$\lim_{\|p\| \to +\infty} \frac{H(x,p)}{\|p\|} = +\infty \quad \text{uniformly in } x.$$

Examples of Tonelli Hamiltonians

- Geodesic Flow
  Let g be a Riemannian metric on M. The Hamiltonian (or Kinetic energy)
  $$H(x,p) = \frac{1}{2}\|p\|_x^2 := \frac{1}{2} g_x(p,p)$$
  corresponds to the geodesic flow on M.
- Hamiltonians from classical mechanics (Kinetic Energy + Potential Energy):
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The dynamics associated to any vector field X on M can be *embedded* into the flow of a Tonelli Hamiltonian $H : T^*M \longrightarrow \mathbb{R}$:

$$H(x,p) = \frac{1}{2}\|p\|_x^2 + p \cdot X(x)\,.$$

Lagrangian formalism

Let $H : T^*M \longrightarrow \mathbb{R}$ a Tonelli Hamiltonian. We can associate to it the so-called Lagrangian function $L : TM \longrightarrow \mathbb{R}$, where

$$\boxed{L(x,v) := \sup_{p \in T_x^*M} \left(p \cdot v - H(x,p)\right)}$$

Euler-Lagrange equations: $\frac{d}{dt}\frac{\partial L}{\partial v} = \frac{\partial L}{\partial x} \quad \longrightarrow$ Euler-Lagrange flow.

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The Hamiltonian flow and the Euler-Lagrange flow are equivalent from a dynamical system point of view:

$$\begin{array}{ccc} TM & \xrightarrow{\Phi_t^L} & TM \\ \mathcal{L}\downarrow & & \downarrow\mathcal{L} \\ T^*M & \xrightarrow[\Phi_t^H]{} & T^*M \end{array}$$

Lagrangian formalism

The Euler-Lagrange flow has an interesting variational characterization in terms of the Lagrangian Action Functional.
If $\gamma : [a, b] \longrightarrow M$ is an abs. cont. curve, we define its action as:

$$A_L(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt.$$

γ is a solution of the Euler-Lagrange flow if and only if it is an extremal for the fixed-end variational problem.

Lagrangian formalism

These extremals are not necessarily minimisers, although they are *local* minimisers, *i.e.* for very short times.

Example: In the geodesic flow case not all geodesics are length minimising! But (global) minimising geodesics do exist.

Questions:

- Do global minimisers exist?
- What are their dynamical/geometric properties?
- Does this minimising property translate into some rigid structure?

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Study orbits and invariant probability measures that minimise the Lagrangian action of $L - \eta_c(x) \cdot v$, where η_c is any smooth closed 1-form on M with cohomology class c.

Observation:

- η_c closed $\Longrightarrow$ L and $L - \eta_c$ have the same Euler-Lagrange flow.
- The corresponding Hamiltonian is $H(x, \eta_c(x) + p)$.

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Tonelli Hamiltonian systems

↓ (Variational methods)

Aubry - Mather theory ⇜ Minim. Lagrangian action

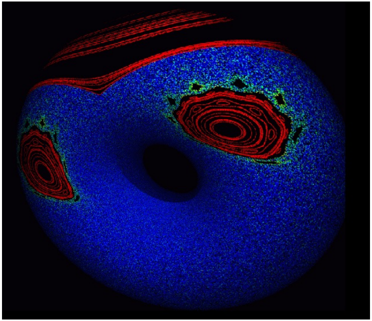
↓ (Invariant sets)

$\{\widetilde{\mathcal{M}}_c\}_{c \in H^1(M;\mathbb{R})}$ & $\{\widetilde{\mathcal{A}}_c\}_{c \in H^1(M;\mathbb{R})}$ ⇜ Aubry-Mather sets

Aubry-Mather sets

The Aubry-Mather sets are:

- non-empty and compact;
- invariant under the Hamiltonian flow;
- *supported* on Lipschitz graphs (Mather's graph theorem);



(Credits to Dr. Oliver Knill, Harvard)

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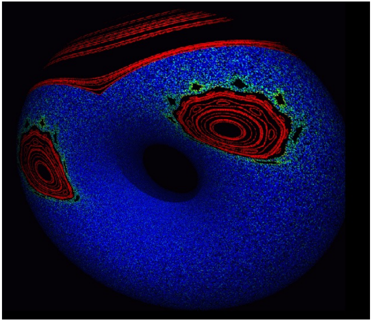

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In particular:

- Symplectic invariant $\longleftrightarrow$ Invariant under the action of simplectomorphisms.
- Lagrangian Structure $\longleftrightarrow$ The are supported on Lipschitz Lagrangian, graphs of cohomology class c.

Link to PDEs & Symplectic geometry

Proposition

If Λ is an invariant Lagrangian graph in $(T^*M, \omega_{\text{stand.}})$ with cohomology class c, then all orbits on Λ (resp. invariant prob. measures supported on Λ), minimises the action of $L - \eta_c(x) \cdot v$, where η_c is any smooth closed 1-form on M, with cohomology class c.

Therefore:

Aubry-Mather sets $\longleftrightarrow$ Invariant Lagrangian graphs (when they exist)

Invariant Lagrangian graphs are very rare. What if they do not exist?

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Proposition

If Λ is an invariant Lagrangian graph in $(T^*M, \omega_{\text{stand.}})$ with cohomology class c, then $\Lambda = \mathrm{Graph}(c + du)$ and

$$H(x, c + du(x)) = \alpha(c).$$

Therefore, one can study viscosity solutions and subsolutions of Hamilton-Jacobi equations:

Aubry-Mather sets $\longleftrightarrow$ supported on the "graphs" of the differentials of these weak solutions (Uniqueness set)

Weak KAM theory (Albert Fathi '90s) $\longleftrightarrow$ Homogenization of Hamilton-Jacobi equation (à la Lions-Papanicolaou-Varadhan and Evans)

A more geometric characterization

For a fixed c, the sets $\mathcal{M}_c$ and $\mathcal{A}_c$ lie in an energy level $\{H(x,p) = \alpha(c)\}$. The function

$$\alpha : H^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$$

is what we call Minimal average action or Effective Hamiltonian.

- It corresponds to the *Homogenized Hamiltonian*.

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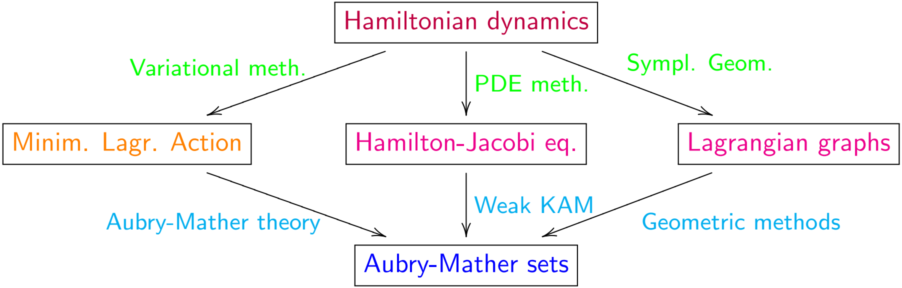
This function is also related to what is called Symplectic shape:

- $\alpha(c) = \inf\{k : \text{the sublevel } \{H(x,p) \leq k\} \text{ contains Lagrangian graphs of cohomology class } c\}$.
- α is related to Hofer geometry on the group of Hamiltonian diffeomorphisms (Sorrentino-Viterbo, *Geom&Top* 2010)


Hamiltonian dynamics
Variational meth.
PDE meth.
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Minim. Lagr. Action
Hamilton-Jacobi eq.
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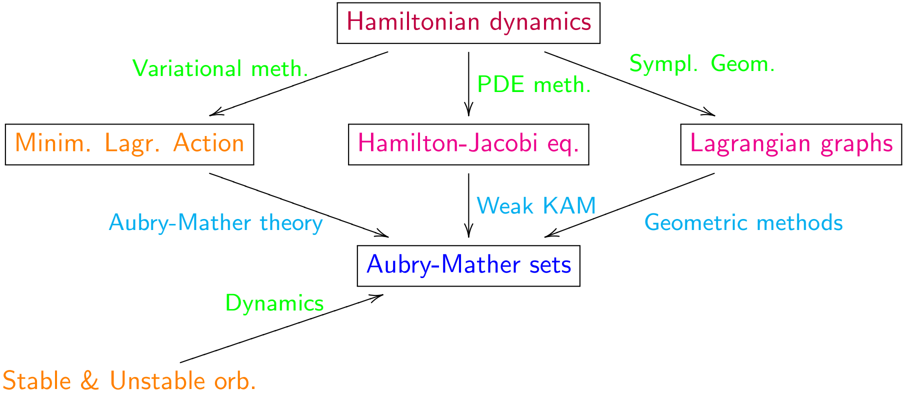
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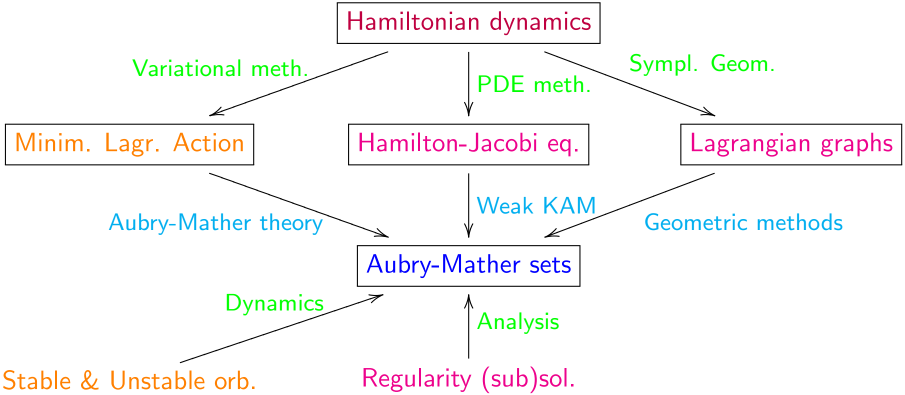
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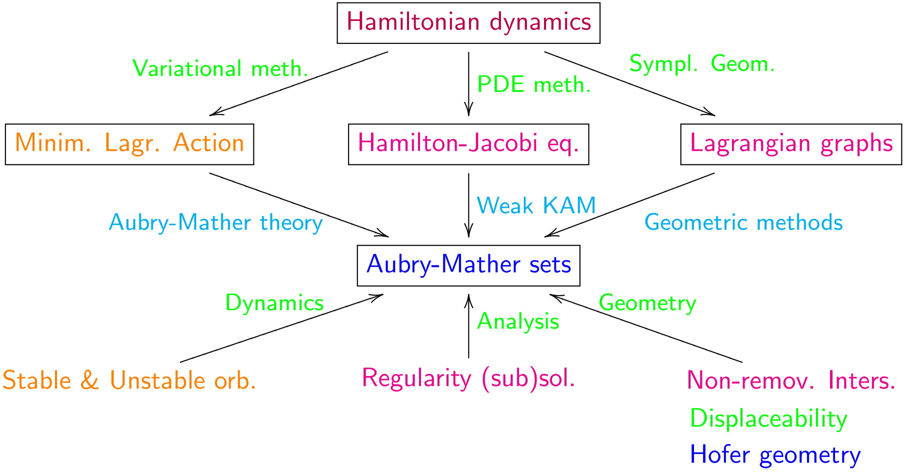
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Non-remov. Inters.
Displaceability
Hofer geometry

Research Interests

Structure of these Action-minimizing sets and H-J equation:

- Generic topological properties. Symplectic and contact properties.
- Implications to dynamics and Symplectic geometry;
- Implications to the regularity of viscosity solutions and subsolutions of Hamilton-Jacobi equation;
- Generalised forms of Homogenization of Hamilton-Jacobi equation.

Properties of the minimal average action:

- Symplectic properties and relation to Hofer geometry.
- Regularity, lack of regularity and geometric/dynamical implications.

Birkhoff Billiards (proposal for a SIR project 2014):

- Rigidity phenomena; (Length) Spectral properties.
- Integrability and Birkhoff conjecture.

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Some results on I

- A. S., *On the total disconnectedness of the quotient Aubry set*, Ergodic Theory Dynam. Systems 28 (2008), Vol. 1.
- A. Fathi, A. Giuliani and A.S., *Uniqueness of invariant Lagrangian graphs in a homology and cohomology class*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8 (2009), Vol. 4.
- A. S., *On the Integrability of Tonelli Hamiltonians*, Trans. Amer. Math. Soc. 363 (2011), Vol. 10.
- Leo Butler and A. S., *A weak Liouville-Arnol'd theorem*, Comm. Math. Phys. 315 (2012), Vol. 1.
- A. S., *A variational approach to the study of the existence of invariant Lagrangian graphs*, Boll. Unione Mat. Ital. Serie IX, Vol. VI (2013).
- G. Paternain and A.S., *Symplectic and contact properties of Mañé's energy level on the universal cover*, To appear on NoDEA (2014).

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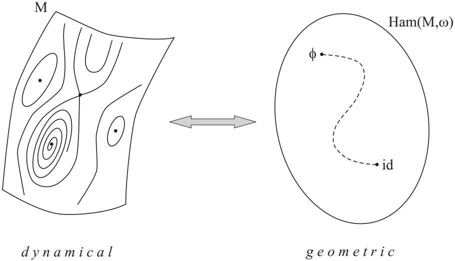
Some results on II

- A. S. and Claude Viterbo *Action minimizing properties and distances on the group of Hamiltonian diffeomorphisms*,
  Geom. & Topol. 14 (2010), no. 4.
- Daniel Massart and A. S., *Differentiability of Mather's average action and integrability on closed surfaces*,
  Nonlinearity 24 (2011), no. 6.
- A. S., *Computing Mathers beta-function for Birkhoff billiards*,
  To appear on Discrete and Contin. Dynamical Syst. - Series A (2014).

Relation to Hofer geometry

Consider the group of (compactly supported) Hamiltonian diffeomorphism $Ham(M, \omega)$:



Fundamental (unsolved) question

What is the relation between the geometry of this curve and the dynamics of the system?

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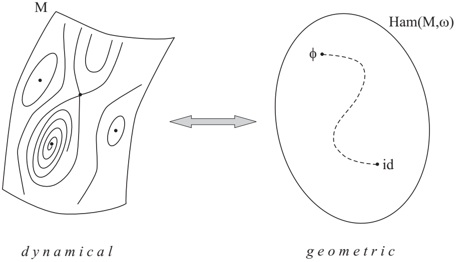

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Minimal average action $\longleftrightarrow$ Asymptotic distance from Identity

Research Projects

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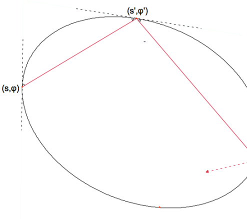
Playing with Billiards

What is a (mathematical) billiard?



The billiard ball moves on a rectilinear path: when it hits the boundary it reflects elastically according to the standard reflection law:

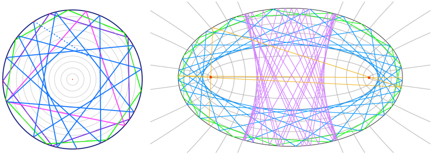
$$\text{angle of reflection} = \text{angle of incidence.}$$

This is a conceptually simple model, yet mathematically very complicated,

Playing with Billiards

In collaborations with Vadim Kaloshin (University of Maryland, USA) we have an ongoing project aimed at studying two important (and related) questions:

- Is it possible to hear the shape of a billiard? Can a planar convex domain be characterized in terms of the lengths of its periodic orbits, i.e., its Length spectrum (or Marked length spectrum), as conjectured by Guillemin and Melrose?
- Birkhoff conjecture on the integrability of convex billiards. Namely: the only integrable billiards are billiards in circles and ellipses.



Thank you
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