Action-Minimizing Methods in Hamiltonian Dynamics and Invariant Lagrangian Graphs

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Seminar: "Geometry, Topology and their Applications"

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GOAL OF THIS TALK: DESCRIBE WHAT KIND OF INFORMATION THE PRINCIPLE OF LEAST ACTION CONVEYS INTO THE STUDY OF THE EXISTENCE OR THE NON-EXISTENCE OF INVARIANT LAGRANGIAN GRAPHS.

IN PARTICULAR:

- WEAK INTEGRABILITY $\longrightarrow$ A NON-COMMUTATIVE VERSION OF LIOUVILLE-ARNOLD THEOREM FOR CERTAIN CLASSES OF HAMILTONIANS.
- REGULARITY OF THE MINIMAL AVERAGED ACTION

1. THE PRINCIPLE OF LEAST ACTION IN HAMILTONIAN DYNAMICS

⟶ AUBRY–MATHER THEORY (1980-90's)

"NATURE IS THRIFTY IN ALL ITS ACTIONS" – PIERRE-LOUIS MOREAU DE MAUPERTUIS (1744)

HISTORICAL REMARK:

KÖNIG PUBLISHED A NOTE CLAIMING PRIORITY FOR LEIBNIZ IN THE BERLIN ACADEMY CORRESPONDENCES OVERSEEN BY MAUPERTUIS.

PRIORITY DISPUTE BROUGHT IN EULER, VOLTAIRE AND ULTIMATELY A COMMITTEE CONVENED BY THE KING OF PRUSSIA.

IN 1913, THE BERLIN ACADEMY REVERSED ITS PREVIOUS DECISION AND FOUND LEIBNIZ HAD PRIORITY

1.1 CARTOON EXAMPLE - A (VERY SPECIAL) INTEGRABLE HAMILTONIAN SYSTEM

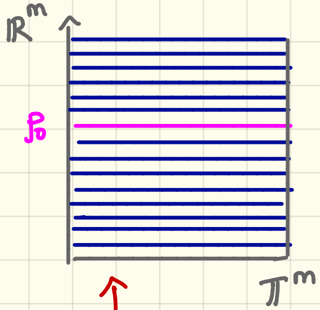
$$h : \mathbb{T}^m \times \mathbb{R}^m \longrightarrow \mathbb{R} \qquad \text{HAMILTONIAN}$$
$$(x,p) \longmapsto h(p)$$

ASSUMPTIONS

- h ONLY DEPENDS ON p
- h IS C^2
- h IS STRICTLY CONVEX IN p (I.E., POSITIVE DEFINITE HESSIAN)
- h SUPERLINEAR IN p (I.E., $\lim_{\|p\|\to\infty} \frac{h(p)}{\|p\|} = +\infty$)

$$\begin{cases} \dot{x} = \partial_p h(p) \\ \dot{p} = 0 \end{cases} \quad \leftarrow \text{ EQUATIONS OF MOTION}$$

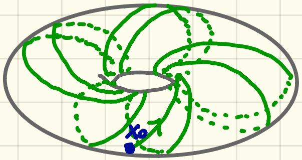
$$\phi_h^t(x_0,p_0) = \big(x_0 + \underbrace{\partial_p h(p_0)}_{\rho(p_0)}\, t\,,\ p_0\big) \quad \leftarrow \text{ FLOW}$$



$\leftarrow \quad \mathcal{T}^*_{p_0} = \mathbb{T}^m \times \{p_0\}$

INVARIANT TORUS

THE PHASE SPACE IS FOLIATED BY INVARIANT TORI

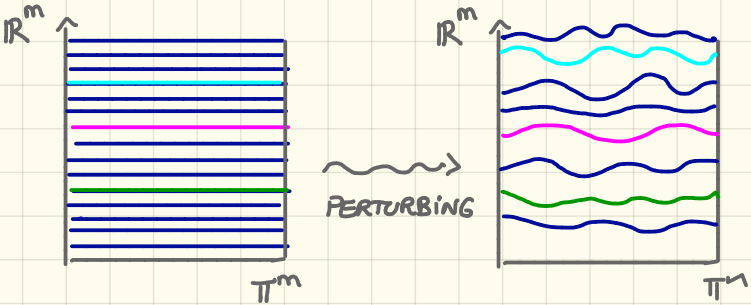


$\leftarrow$ ROTATION WITH ROTATION VECTOR $\rho(p_0)$

QUESTION: WHAT HAPPENS TO THIS FOLIATION IF ONE PERTURBS THE SYSTEM?

- TORI FOLIATED BY PERIODIC ORBITS WILL IN GENERAL DISAPPEAR
- IN 1954 KOLMOGOROV (AND LATER ARNOL'D AND MOSER IN DIFFERENT SETTINGS) PROVED THAT THE MAJORITY OF THESE TORI WILL SURVIVE ← KAM THEORY

THOSE CORRESPONDING TO DIOPHANTINE ROTATION VECTORS



- WHAT HAPPENS TO THE DESTROYED TORI?
- WHAT HAPPENS IN THOSE GAPS?

CHANGE OF POINT OF VIEW: LAGRANGIAN POINT OF VIEW

$H : \mathbb{T}^m \times \mathbb{R}^m \to \mathbb{R}$ HAMILTONIAN $\rightsquigarrow$ $L : \mathbb{T}^m \times \mathbb{R}^m \to \mathbb{R}$ LAGRANGIAN

$$L(x,v) = \sup_{p \in \mathbb{R}^m} \left(\langle p, v \rangle - H(x,p) \right)$$

FENCHEL-LEGENDRE INEQUALITY

$$\boxed{\begin{aligned} L(x,v) + H(x,p) &\geq \langle p, v \rangle \quad \forall\, v, p \in \mathbb{R}^m \\ = \quad &\iff \quad p = \partial_v L(x,v) \\ &\iff \quad v = \partial_p H(x,p) \end{aligned}}$$

$$\begin{tikzcd} \mathbb{T}^m \times \mathbb{R}^m \arrow[r, "\phi_H^t"] \arrow[d, "{(x,\partial_p H)}"'] & \mathbb{T}^m \times \mathbb{R}^m \arrow[d, "{(x,\partial_p H)}"] \\ \mathbb{T}^m \times \mathbb{R}^m \arrow[r, "\phi_L^t"'] & \mathbb{T}^m \times \mathbb{R}^m \end{tikzcd}$$

$$\frac{d}{dt} \frac{\partial L}{\partial v} = \frac{\partial L}{\partial x} \quad \leftarrow \quad \text{EULER-LAGRANGE EQUATION (E-L)}$$

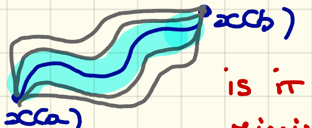
ACTION FUNCTIONAL

$(x(t), v(t))$ $t \in [a,b]$ (E-L)

$\Updownarrow$

$x(t)$ is a "CRITICAL POINT" FOR THE FUNCTIONAL

$$A_L(x(t)) = \int_a^b L(x(t), v(t))\, dt$$



IS IT A MINIMUM?

(BACK TO THE CARTOON EXAMPLE)

$$\ell(v) = \sup_{p \in \mathbb{R}^m} \left(\langle p, v \rangle - h(p) \right)$$

$$\mathcal{T}^*_{p_0} = \mathbb{T}^m_x \{p_0\} \quad \rightsquigarrow \quad \tilde{\mathcal{T}}_{p_0} = \mathbb{T}^m_x \{ \underset{\partial_p h(p_0)}{\underset{\shortparallel}{v_0}} \}$$

INVARIANT FOR ϕ^t_h INVARIANT FOR ϕ^t_ℓ

$$\boxed{\text{GOAL: CHARACTERIZE } \tilde{\mathcal{T}}_{p_0} \text{ IN TERMS OF ITS ACTION PROPERTIES}}$$

IDEA 1 (J. MATHER): LOOK AT INVARIANT PROBABILITY MEASURES + THEIR ACTION PROPERTIES

- μ_0 INVARIANT PROB. MEASURE SUPPORTED ON $\tilde{\mathcal{T}}_{p_0} = \mathbb{T}^m_x \{v_0\}$

$$A_\ell(\mu_0) = \int_{\mathbb{T}^m_x \mathbb{R}^m} \ell(v)\, d\mu_0 = \ell(v_0) = \langle p_0, v_0 \rangle - h(p_0)$$

- ν ANY INVARIANT PROB. MEASURE

$$A_\ell(\nu) = \int_{\mathbb{T}^m_x \mathbb{R}^m} \ell(v)\, d\nu \geq \int_{\mathbb{T}^m_x \mathbb{R}^m} \left(\langle p_0, v \rangle - h(p_0) \right) d\nu = \langle p_0, \int v\, d\nu \rangle - h(p_0)$$

HOW TO COMPARE THEM?

IDEA 2 (J. MATHER): MODIFY THE LAGRANGIAN : $\ell(v) - \langle p_0, v\rangle$ ← SAME E-L FLOW BUT DIFFERENT ACTION

$$A_{\ell - \langle p_0, v\rangle}(\mu_0) = \int_{\mathbb{T}^m \times \mathbb{R}^m} (\ell(v) - \langle p_0, v\rangle)\, d\mu_0 = -h(p_0)$$

$$A_{\ell - \langle p_0, v\rangle}(\nu) \geq \int_{\mathbb{T}^m \times \mathbb{R}^m} (\ell(v) - \langle p_0, v\rangle)\, d\nu = -h(p_0)$$

$$\Longrightarrow \boxed{A_{\ell - \langle p_0, v\rangle}(\mu_0) \leq A_{\ell - \langle p_0, v\rangle}(\nu)}$$

CONCLUSION 1:

- EVERY INV. PROB. MEASURE SUPPORTED ON $\tilde{\mathcal{T}}_{p_0}$ MINIMIZES $A_{\ell - \langle p_0, v\rangle}(\cdot)$
- $\boxed{\tilde{\mathcal{T}}_{p_0} = \bigcup \{ \operatorname{supp} \mu_0 : \mu_0 \text{ MINIMIZES } A_{\ell - \langle p_0, v\rangle}(\cdot) \}}$ ← VARIATIONAL CHARACTERIZ. OF THE TORUS
- $\boxed{\min_{\mu \text{ INV. PB MEAS}} A_{\ell - \langle p_0, v\rangle}(\mu) = -h(p_0)}$ ← VARIATIONAL CHARACT. OF THE HAMILTONIAN

MATHER SET

MINIMAL AVERAGE ACTION

QUESTION: CAN WE AVOID TO CHANGE THE LAGRANGIAN?

DEFINITION: μ INV. PROB. MEASURE $\Longrightarrow \rho(\mu) := \int_{\mathbb{T}^m \times \mathbb{R}^m} v \, d\mu \in \mathbb{R}^m$ ROTATION VECTOR OF μ

IDEA: COMPARE ONLY INV. PROB. MEASURES WITH THE SAME ROTATION VECTOR

- μ_0 INVARIANT PROB. MEASURE SUPPORTED ON $\tilde{\mathcal{G}}_{p_0} = \mathbb{T}^m \times \{v_0\}$ $\leftarrow$ ROTATION VECTOR $\rho(\mu_0) = v_0$

$$A_\ell(\mu_0) = \int_{\mathbb{T}^m \times \mathbb{R}^m} \ell(v) \, d\mu_0 = \ell(v_0) = \langle p_0, v_0 \rangle - h(p_0)$$

- ν ANY INVARIANT PROB. MEASURE WITH $\rho(\nu) = v_0$

$$A_\ell(\nu) = \int_{\mathbb{T}^m \times \mathbb{R}^m} \ell(v) \, d\nu \geq \int_{\mathbb{T}^m \times \mathbb{R}^m} (\langle p_0, v \rangle - h(p_0)) \, d\nu = \langle p_0, \underbrace{\int v \, d\nu}_{= v_0} \rangle - h(p_0)$$

$$\Longrightarrow \boxed{A_\ell(\mu_0) \leq A_\ell(\nu)}$$

CONCLUSION 2: • EVERY INV. PROB. MEASURE SUPPORTED ON $\tilde{\mathcal{T}}_{\rho_0}$ MINIMIZES $A_\ell(\cdot)$

AMONG ALL INV. PB MEASURES WITH THE SAME ROTATION VECTOR

MATHER SET

• $\boxed{\tilde{\mathcal{T}}_{\rho_0} = \bigcup \{\operatorname{supp} \mu_0 : \mu_0 \text{ MINIMIZES } A_\ell \text{ AMONG MEAS. WITH SAME ROT. VECT}\}}$

VARIATIONAL CHARACTERIZ. OF THE TORUS

MINIMAL AVERAGE ACTION

• $\boxed{\min_{\substack{\mu \text{ inv.} \\ \rho(\mu)=\nu_0}} A_\ell(\mu) = \ell(\nu_0)}$

VARIATIONAL CHARACT. OF THE LAGRANGIAN

SUMMARY OF THIS CARTOON EXAMPLE

$$\bigcup\left\{\mathrm{supp}\,\mu:\ \mu \text{ minimizes } A_{\ell-\langle p_0, v\rangle}\right\} \longleftarrow \tilde{\mathcal{T}}_{p_0} = \mathbb{T}^m \times \{v_0\} \longrightarrow \bigcup\left\{\mathrm{supp}\,\mu:\ \mu \text{ minimizes } A_\ell \text{ with constraint } \rho(\mu) = v_0\right\}$$

INVARIANT TORUS

$$\tilde{\mathcal{M}}_{p_0} = \tilde{\mathcal{M}}^{v_0}$$

$$p_0 = \partial_v \ell(v_0)$$

$$v_0 = \partial_p h(p_0)$$

$$h(p_0) = -\min_{\substack{\mu \text{ inv.} \\ \text{PB MEAS.}}} A_{\ell-\langle p_0, v\rangle}(\mu) \longleftrightarrow \ell(v_0) = \min_{\substack{\mu \text{ inv} \\ \text{PB meas} \\ \rho(\mu)=v_0}} A_\ell(\mu)$$

FENCHEL-LEGENDRE DUALITY

$$h(p) = \sup_v \left(\langle p, v\rangle - \ell(v)\right) \qquad \ell(v) = \sup_p \left(\langle p, v\rangle - h(p)\right)$$

1.2 Setting: Tonelli Lagrangian and Hamiltonian

- M compact connected manifold w/out boundary $\dim M = m$
- g Riemannian metric

$L: \underset{(x,v)}{TM} \to \mathbb{R}$ is a **Tonelli Lagrangian** if

- $L \in C^2(TM)$
- L fiberwise strictly convex: $\partial^2_{vv} L > 0$
- L superlinear in the fiber

$$\lim_{\|v\| \to \infty} \frac{L(x,v)}{\|v\|} = +\infty \quad \text{unif. in } x$$

$H: T^*M \to \mathbb{R}$ is a **Tonelli Hamiltonian** if

- $H \in C^2(T^*M)$
- H fiberwise strictly convex: $\partial^2_{pp} H > 0$
- H superlinear in the fiber

$$\lim_{\|p\| \to \infty} \frac{H(x,p)}{\|p\|} = +\infty \quad \text{unif. in } x$$

Fenchel Legendre duality

$$L(x,v) = \sup_{p \in T_x^*M} \left(\langle p, v \rangle_x - H(x,p) \right)$$

$$H(x,p) = \sup_{v \in T_xM} \left(\langle p, v \rangle_x - L(x,v) \right)$$

EXAMPLES:

- GEODESIC FLOW $\qquad L(x,v)=\frac{1}{2}\|v\|_x^2$

- MECHANICAL SYSTEM $\qquad L(x,v)=\frac{1}{2}\|v\|_x^2 - V(x) \qquad V\in C^2(M)$

- LET X BE A C^2 VECTOR FIELD ON M $\qquad$ LET φ_t^X BE THE FLOW OF X

  LET $L(x,v)=\frac{1}{2}\|v-X(x)\|_x^2$ $\qquad$ ← MAÑÉ LAGRANGIAN

GRAPH$(X)=\{(x,X(x)):\ x\in M\}$ is INVARIANT UNDER Φ_L^t

$$\begin{tikzcd}
\mathrm{GRAPH}(X) \arrow[r, "\Phi_L^t"] \arrow[d, "\pi"'] & \mathrm{GRAPH}(X) \arrow[d, "\pi"] \\
M \arrow[r, "\varphi_t^X"] & M
\end{tikzcd}$$

THE FLOW OF X IS EMBEDDED IN THE FLOW OF A TONELLI LAGRANGIAN

1.3 MATHER THEORY FOR TONELLI LAGRANGIAN

$L : TM \longrightarrow \mathbb{R}$ TONELLI LAGRANGIAN

$\Phi_L : TM \longrightarrow TM$ E-L FLOW

$\mathfrak{M}_L = \{$ INVARIANT PROBABILITY MEASURES OF $\Phi_L \} \neq \emptyset$

$A_L : \mathfrak{M}_L \longrightarrow \mathbb{R}$ ACTION FUNCTIONAL

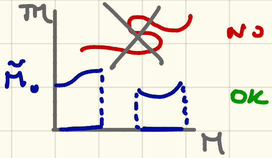
- $\exists\, \mu \in \mathfrak{M}_L$ MINIMIZING A_L
- $\alpha(0) := -\min\limits_{\mu \in \mathfrak{M}_L} A_L(\mu)$
- $\boxed{\tilde{M}_0 := \bigcup \{ \operatorname{supp} \mu : \; A_L(\mu) = -\alpha(0) \}}$
  - IT IS NON-EMPTY, INVARIANT, COMPACT
  - IT LIES IN THE ENERGY VALUE $= \alpha(0)$ (CARNEIRO, 1995)
  - IT IS SUPPORTED ON A LIPSCHITZ GRAPH (MATHER GRAPH THEOREM 1991)

MATHER GRAPH THEOREM

$\tilde{M}_0 \subset TM$

$\downarrow \pi$ BI-LIPSCHITZ HOMEOM

$\pi(\tilde{M}_0) \subset M$



How to modify the Lagrangian?

- If η is a closed one form on M $\Rightarrow$ $L_\eta(x,v) = L(x,v) - \langle \eta, v\rangle$ has the same E-L flow as L (EXERCISE: just write E-L equations)
- If $\eta = df$, $f \in C^1(M)$ $\Rightarrow$ $\int_{TM} \langle df, v\rangle \, d\mu = 0 \quad \forall \mu \in \mathfrak{M}_L$

STRATEGY:

- Let $c \in H^1(M,\mathbb{R})$ and η_c a representative
- $L_{\eta_c}(x,v) = L(x,v) - \langle \eta_c, v\rangle$ is a Tonelli Lagrangian
- Find minimizing measures for $A_{L_{\eta_c}}$
- Define $\alpha(c) := -\min_{\mathfrak{M}_L} A_{L_{\eta_c}}$
- Construct Mather sets $\boxed{\tilde{M}_c := \bigcup \{ \operatorname{supp} \mu : A_{L_{\eta_c}}(\mu) = -\alpha(c) \}}$

These definitions depend only on c

↳ compact, invariant, lies in energy level $\alpha(c)$, supported on a Lipschitz graph

ROTATION VECTOR OF AN INVARIANT MEASURE

(SCHWARTZMAN ASYMPTOTIC CYCLE)

$\rho : \mathfrak{M}_L \longrightarrow H_1(M, \mathbb{R})$

LET $\mu \in \mathfrak{M}_L$ DEFINE $I_\mu : H^1(M,\mathbb{R}) \longrightarrow \mathbb{R}$

$$c \longmapsto \int_{TM} \langle \eta_c, v \rangle \, d\mu$$

IT IS INDEPENDENT OF THE CHOICE OF THE REPRESENTATIVE η_c

I_μ LINEAR FUNCTIONAL $\xrightarrow{\text{RIESZ THEOREM}}$ $\exists!\ \rho(\mu) \in (H^1(M,\mathbb{R}))^* \simeq H_1(M,\mathbb{R})$

S.T. $I_\mu(c) = \langle \rho(\mu), c \rangle \quad \forall c \in H^1(M,\mathbb{R})$

STRATEGY:

- $\forall h \in H_1(M,\mathbb{R})$, $\mathfrak{M}_L^h = \{\mu \in \mathfrak{M}_L : \rho(\mu) = h\} \neq \emptyset$
- FIND MINIMIZING MEASURES FOR A_L ON $\mathfrak{M}_L^h$
- DEFINE $\beta(h) := \min_{\mathfrak{M}_L^h} A_L$
- CONSTRUCT MATHER SETS $\boxed{\tilde{M}^h := \bigcup \{\operatorname{supp} \mu : \rho(\mu) = h,\ A_L(\mu) = \beta(h)\}}$

$\hookrightarrow$ COMPACT, INVARIANT, SUPPORTED ON LIPSCHITZ GRAPHS

SUMMARIZING:

$$\{\tilde{\mathcal{M}}_c\}_{c\in H^1(M,\mathbb{R})} \quad \overset{?}{\longleftrightarrow} \quad \{\tilde{\mathcal{M}}^h\}_{h\in H_1(M,\mathbb{R})}$$

MATHER SET OF COHOMOLOGY CLASS c

MATHER SET OF HOMOLOGY CLASS h

$$\alpha: H^1(M,\mathbb{R}) \longrightarrow \mathbb{R}, \qquad c \longmapsto -\min_{m_L} A_{L-\langle \eta_c, v\rangle}$$

MATHER'S α-FUNCTION (EFFECTIVE HAMILTONIAN)

FENCHEL-LEGENDRE DUALITY

$$\beta: H_1(M,\mathbb{R}) \longrightarrow \mathbb{R}, \qquad h \longmapsto \min_{m_L^h} A_L$$

MATHER'S β-FUNCTION (EFFECTIVE LAGRANGIAN)

$$\alpha(c) = \sup_{h\in H_1(M,\mathbb{R})} \left(\langle c,h\rangle - \beta(h)\right)$$

$$\beta(h) = \sup_{c\in H^1(M,\mathbb{R})} \left(\langle c,h\rangle - \alpha(c)\right)$$

FENCHEL-LEGENDRE INEQUALITY:

$$\alpha(c)+\beta(h) \geq \langle c,h\rangle \qquad \forall c\in H^1(M,\mathbb{R}) \ \forall h\in H_1(M,\mathbb{R})$$

$$= \iff c = \partial\beta(h) \iff h\in \partial\alpha(c)$$

SUBDERIVATIVES

WARNING:
α, β ARE CONVEX BUT NOT NECESSARILY STRICTLY CONVEX OR DIFFERENTIABLE

$$\{\tilde{M}_c\}_{c\in H^1(M,\mathbb{R})} \overset{?}{\longleftarrow\text{-}\text{-}\text{-}\longrightarrow} \{\tilde{M}^h\}_{h\in H_1(M,\mathbb{R})}$$

RELATIONS:

- $$\bigcup_{c\in H^1(M,\mathbb{R})} \tilde{M}_c = \bigcup_{h\in H_1(M,\mathbb{R})} \tilde{M}^h$$
- $$\tilde{M}^h \subseteq \tilde{M}_c \iff h\in \partial\alpha(c) \iff c\in\partial\beta(h)$$
- IF $c,c'\in\partial\beta(h) \Rightarrow \tilde{M}^h \subseteq \tilde{M}_c \cap \tilde{M}_{c'}$ ($\tilde{M}_c$ AND $\tilde{M}_{c'}$ ARE NOT DISJOINT)
- IF $h,h'\in\partial\alpha(c) \Rightarrow \tilde{M}^h \cup \tilde{M}^{h'} \subseteq \tilde{M}_c$ ($\tilde{M}_c$ CONTAINS MOTIONS OF DIFFERENT ROT. VECTORS)

<u>QUESTIONS</u>

- IF β IS C^1, IS IT TRUE THAT THE SYSTEM IS INTEGRABLE?
- CAN ONE DETECT THE EXISTENCE OF AN INVARIANT LAGRANGIAN GRAPH FROM PROPERTIES OF β?

2. DIFFERENTIABILITY OF MATHER'S β-FUNCTION AND INTEGRABILITY

$\beta : H_1(M, \mathbb{R}) \longrightarrow \mathbb{R}$ IS A CONVEX FUNCTION

$\hookrightarrow$ NOT NECESSARILY STRICTLY CONVEX NOR DIFFERENTIABLE

- IF $\exists$ AN INVARIANT TORUS, WHOSE MOTION IS CONJUGATED TO A ROTATION OF ROTATION VECTOR h $\Rightarrow$ β IS DIFFERENTIABLE AT h

- $\boxed{\text{IF THE SYSTEM IS INTEGRABLE} \Rightarrow \beta \text{ IS } C^1 \text{ ON SOME OPEN SET}}$

$\overset{?}{\not\Leftarrow}$

- IF $\exists$ AN INVARIANT LAGRANGIAN GRAPHS Λ, S.T. ALL ITS INVARIANT MEASURES HAVE ROTATION VECTOR h AND THE UNION OF THEIR SUPPORTS COVERS Λ

$\Rightarrow$ β IS DIFFERENTIABLE AT h.

- IF $\dim H_1(M,\mathbb{R})=0 \Rightarrow \beta$ IS DEFINED ONLY AT A POINT

- IF $\dim H_1(M,\mathbb{R})=1 \Rightarrow \beta$ IS DIFFERENTIABLE EVERYWHERE (WITH THE POSSIBLE EXCEPTION OF THE ORIGIN)

(CARNEIRO '95: β ALWAYS DIFF. IN RADIAL DIRECTION)

$\boxed{\text{QUESTION: WITH THE EXCEPTION OF THESE TRIVIAL CASES, DOES THE DIFFERENTIABILITY OF } \beta \text{ IMPLY ANYTHING ON INTEGRABILITY / } \exists \text{ INV. LAGRANGIAN GRAPHS?}}$

DEFINITIONS:

- $h \in H_1(M,\mathbb{R})$ IS K-IRRATIONAL IF K IS THE DIMENSION OF THE SMALLEST SUBSPACE GENERATED BY ELEMENTS OF $H_1(M,\mathbb{Z})$, CONTAINING h.

$\longrightarrow$ 1-IRRATIONAL: "ON A LINE OF RATIONAL SLOPE"

$\longrightarrow$ COMPLETELY IRRATIONAL: $k = \dim H_1(M,\mathbb{R})$

RESULTS ON CLOSED SURFACES (with Daniel Massart, Nonlinearity 2011)

THEOREM: Let M be a closed oriented surface of genus g.

Let $h_0 \in H_1(M,\mathbb{R})$ 1-irrational and non-singular

(i) $\Longrightarrow \dim \partial\beta(h_0) \geq g$

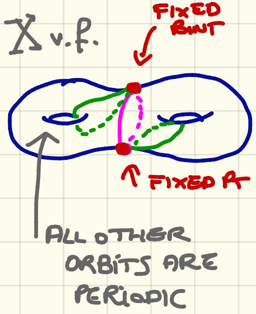
(ii) if $M=\mathbb{T}^2$ and β is differentiable at h_0

$\Longrightarrow \exists$ an invariant Lagrangian graph foliated by periodic orbits, whose homology is a multiple of h_0

NON-SINGULAR means that $\bigcup_{c\in\partial\beta(h_0)} \tilde{M}_c$ does not contain fixed points of the flow

REMARKS:

- The above results are false if h_0 is singular



- (ii) generalizes a result by Mather for twist maps of the annulus, and a result by Bangert for geodesic flows on $\mathbb{T}^2$

DEFINITION (M-C. ARNAUD) A TONELLI HAMILTONIAN $H: T^*M$ IS SAID TO BE C^0-INTEGRABLE IF

$\exists$ A FOLIATION OF T^*M BY INVARIANT LAGRANGIAN GRAPHS, ONE FOR EACH COHOMOLOGY CLASS

REMARKS: LIOUVILLE-ARNOLD INTEGRABILITY ON T^*M $\overset{\Longrightarrow}{\underset{?}{\not\Longleftarrow}}$ C^0 INTEGRABILITY

$$\boxed{\text{THEOREM 2} \qquad \beta \in C^1(H_1(M,\mathbb{R})) \iff \text{THE SYSTEM IS } C^0 \text{ INTEGRABLE}}$$

$$\Downarrow$$

$$(M \cong \mathbb{T}^2)$$

IN PARTICULAR:

- h 1-IRRATIONAL: $\exists$ INV. LAGRANGIAN GRAPH FOLIATED BY PER. ORBITS WITH HOMOLOGY h AND THE SAME MINIMAL PERIOD
- h COMPLETELY IRRATIONAL: $\exists$ INV. LAG. GRAPH WHOSE MOTION IS CONJUGATED TO AN IRRATIONAL ROTATION OR A DENJOY TYPE HOMEOMORPHISM.

  $\hookrightarrow$ $\exists$ G_δ SET OF (CO)HOMOLOGIES FOR WHICH THE MOTION IS CONJUGATED TO A ROTATION
- $h = 0$ $\exists$ C^1 INVARIANT TORUS CONSISTING OF FIXED POINTS

SOME COMMENTS

- THIS ANALYSIS CAN BE EXTENDED TO NON ORIENTABLE SURFACES

IF $M \neq \mathbb{RP}^2$, KLEIN BOTTLE $\Rightarrow$ $\exists\, h \in H_1(M,\mathbb{R})$ S.T. β IS NOT DIFFERENTIABLE AT h

- ONE OF THE KEY PROPERTY IN DIMENSION 2 IS THAT IF h IS 1-IRRATIONAL AND NON-SINGULAR $\Rightarrow$ $\tilde{M}^h$ CONSISTS OF PERIODIC ORBITS.

- IN HIGHER DIMENSION NOT MANY RESULTS ARE KNOWN:

$\longrightarrow$ FOR NEARLY INTEGRABLE SYSTEMS

USING KAM THEORY ONE CAN DEDUCE SOME REGULARITY, IN THE WHITNEY SENSE, ON SOME SET OF DIOPHANTINE HOMOLOGIES.

3. WEAK INTEGRABILITY

RECALL LET (V,ω) BE A SYMPLECTIC MANIFOLD $H: V \longrightarrow \mathbb{R}$ HAMILTONIAN

(ω 2-FORM CLOSED AND NON DEGENERATE)

- HAMILTONIAN VECTOR FIELD: $i_{X_H}\omega = dH$

POISSON BRACKETS $\{F,H\}$

- $F: V \longrightarrow \mathbb{R}$ IS AN INTEGRAL OF MOTION FOR X_H IF $\omega(X_F, X_H) = 0$ $\left(= \{F,H\} \right)$

$\rightarrow$ F REMAINS CONSTANT ALONG THE ORBITS OF X_H $\longleftrightarrow$ ORBITS LIE IN $\{F = c\}$

- IF $F_1, \dots, F_m$ ARE INTEGRALS OF MOTION $\Rightarrow$ $\{F_1 = c_1, \dots, F_m = c_m\}$ IS AN INVARIANT SET

THE MORE INTEGRALS OF MOTION EXIST $\Rightarrow$ THE MORE ORBITS ARE CONSTRAINED ON "SMALLER" SUBMANIFOLDS

THEY MUST HAVE SOME "TRANSVERSALITY"

$\Rightarrow$ FUNCTIONAL INDEPENDENCE

THEOREM (LIOUVILLE-ARNOL'D)

LET $m = \frac{1}{2} \dim V$, $F_1, \dots, F_m : V \to \mathbb{R}$ INTEGRALS OF MOTION.

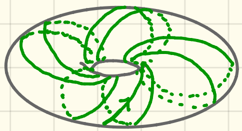
ASSUME:
- $F_1, \dots, F_m$ ARE FUNCTIONALLY INDEPENDENT ($dF_1, \dots, dF_m$ LINEAR INDEP.)
- $\{F_i, F_j\} = 0 \quad \forall i, j = 1 \dots m \longleftrightarrow$ PAIRWISE IN INVOLUTION (POISSON COMMUTE)

CONSIDER $M_c := \{F_1 = c_1, \dots, F_m = c_m\} \neq \emptyset \qquad c = (c_1, \dots c_m) \in \mathbb{R}^m$

THEN:
- M_c IS AN INVARIANT LAGRANGIAN SUBMANIFOLD

$\uparrow$ INTEGRALS OF MOTION $\quad \uparrow$ INVOLUTION $\quad \uparrow$ INDEPENDENCE

- IF M_c IS COMPACT (AND CONNECTED), THEN $M_c \cong \mathbb{T}^m$ ON WHICH THE MOTION IS CONJUGATED TO A ROTATION



- $\exists$ A NEIGHBOURHOOD OF M_c AND A CHANGE OF COORDINATES (ACTION-ANGLE) TRANSFORMING THE HAMILTONIAN: $\hat{H} : \mathcal{U} \subseteq T^*\mathbb{T}^m \to \mathbb{R}$, $(x, p) \mapsto \hat{H}(p)$ $\qquad \{p = 0\} = M_c$

CAN ONE WEAKEN THE HYPOTHESES IN LIOUVILLE-ARNOL'D THEOREM ?

REMARK: IN LIOUVILLE-ARNOL'D THEOREM, INVOLUTION IS FUNDAMENTAL TO CONCLUDE THAT THE INVARIANT MANIFOLD IS LAGRANGIAN AND ITS DYNAMICAL PROPERTIES.

DEFINITION (WEAK-INTEGRABILITY) ← [SORRENTINO, TRANS. AMS 2011]

(V,ω) SYMPLECTIC MANIFOLD, $\dim V = 2m$, $H: V \to \mathbb{R}$ HAMILTONIAN

H IS SAID WEAKLY-INTEGRABLE IF IT HAS m INDEPENDENT INTEGRALS OF MOTION.

LIOUVILLE-ARNOL'D INTEGRABILITY $\Longrightarrow$ WEAK INTEGRABILITY

$\overset{?}{\not\Longleftarrow}$

NO!

LET G A COMPACT SEMI-SIMPLE LIE GROUP OF RANK ≥ 2

IN ANY NBHD OF THE BI-INVARIANT METRIC THERE ARE LEFT INV. METRICS WITH POSITIVE TOP. ENTROPY (BUTLER-PATERNAIN)

THESE LEFT INV. METRICS ARE WEAKLY INTEGRABLE BUT NOT LIOUVILLE-ARNOLD

EXAMPLE OF WEAKLY-INTEGRABLE SYSTEMS (SOME IDEAS)

G COMPACT LIE GROUP, $\mathfrak{g}$ LIE ALGEBRA

CONSIDER A LEFT INVARIANT RIEMANNIAN METRIC ON G $\longrightarrow$ GEODESIC FLOW

↳ $A: \mathfrak{g} \to \mathfrak{g}^*$ EUCLIDEAN STRUCTURE ON G DEFINING THE METRIC (SYMMETRIC POSITIVE DEFINITE)

↳ $A_g : T_g G \longrightarrow T_g^* G$
$\quad \dot{g} \longmapsto L^*_{g^{-1}} A L_{g^{-1}*} \dot{g}$

IT CAN BE EXTENDED TO THE WHOLE TG

MAPS INDUCED BY LEFT TRANSLATION ON COTANGENT AND TANGENT SPACE

$\tilde{A}(g,\dot{g}) = (g, A_g(\dot{g}))$ MOMENT OF INERTIA OPERATOR

$H: T^*G \to \mathbb{R}$
$(g,p) \longmapsto \frac{1}{2} \langle p, A_g^{-1} p \rangle$

$p \rightsquigarrow$ $p_b = L_g^* p$ $\longrightarrow$ ANGULAR MOMENTUM RELATIVE TO THE BODY

$p_s = R_g^* p$ $\longrightarrow$ " " " TO THE SPACE

$$\boxed{\frac{d}{dt} p_s = 0}$$

INTEGRALS OF MOTION

ONE FOR EVERY $g \in G$

CO-ADJOINT REPR OF THE GROUP

$$\frac{d}{dt} p_b = ad^*_{A^{-1}p_b} p_b$$

WEAKLY-INTEGRABLE TONELLI HAMILTONIANS $\quad H: T^*M \longrightarrow \mathbb{R}$

IDEA: STUDY HOW THE PRESENCE OF INTEGRAL OF MOTIONS, IS REFLECTED BY THE ACTION-MINIMIZING PROPERTIES OF THE SYSTEM (i.E, ITS MATHER SETS $\{M_c^*\}_{c \in H^1(M,\mathbb{R})}$)

MATHER SETS IN T^*M VIA THE LEGENDRE TRANSFORM

KEY REMARKS:

- $\{M_c^*\}_c$ ARE SYMPLECTIC INVARIANT. LET $F: T^*M \to \mathbb{R}$ INTEGRAL OF MOTION AND LET $\phi_F^t : T^*M \longrightarrow T^*M$ BE ITS FLOW.

  THEN: $\phi_F^t(M_c^*(H)) = M_c^*(H \circ \phi_F^{-t}) = M_c^*(H) \Rightarrow M_c^*$ IS PRESERVED BY ϕ_F^t

- IF H HAS k INDEPENDENT INTEGRALS OF MOTION $\Rightarrow$ RANK $T_{(x,p)} M_c^* \geq k$

  M_c^* ARE NOT NEC. SUBMANIFOLDS

  $\forall (x,p) \in M_c^* \quad \forall c \in H^1(M,\mathbb{R})$

  THE RANK OF $T^*_{(x,p)} M_c^*$ PROVIDES A CONSTRAINT TO HOW MANY INDEP. INTEGRALS OF MOTION, CAN EXIST IN A NEIGHBORHOOD OF (x,p)

  $T_{(x,p)} M_c^* = \{$ SET OF ALL VECTORS TANGENT TO M_c^* AT $(x,p)\}$

- MATHER SETS FORCE INTEGRAL OF MOTIONS TO POISSON COMMUTE.

LET $F_1, F_2 : T^*M \to \mathbb{R}$ C^2 INTEGRALS OF MOTION. THEN $\forall c \in H^1(M, \mathbb{R})$

$$\{F_1, F_2\}\left(x, \pi_c^{-1}(x)\right) = 0 \qquad \forall x \in \overline{\mathrm{Int}\left(\pi_c(\tilde{M}_c^*)\right)}^{(*)}$$

WHERE $\pi : TM \to M$ AND $\pi_c := \pi|\tilde{M}_c^*$

THIS SET MIGHT BE EMPTY

THIS FOLLOWS FROM THE FACT THAT $\tilde{M}_c^*$ IN A LIPSCHITZ GRAPH, WHICH IS A (SUB) SOLUTION ON HAMILTON-JACOBI EQUATION $H(x, \eta_c + du) = \alpha(c)$ $[\eta_c] = c$

$\Rightarrow$ IT INHERITS A ISOTROPIC TANGENT STRUCTURE ALMOST EVERYWHERE.

THEOREM [BUTLER - SORRENTINO] (S. TRANS. AMS 2011, BUTLER-S. CMP 2012)

LET $H: T^*M^m \to \mathbb{R}$ WEAKLY-INTEGRABLE C^r TONELLI HAMILTONIAN.

SUPPOSE $\exists\, c \in H^1(M,\mathbb{R})$: $\mathcal{M}_c^*$ INTERSECTS A REGULAR LEVEL SET OF $F = (F_1 \ldots F_m)$

THEN:

i) • $\Lambda_c := \mathcal{M}_c^*$ is a C^r INVARIANT LAGRANGIAN GRAPH;

• Λ_c is STRICTLY SCHWARTZMAN ERGODIC (i.e., ALL INVARIANT PB MEASURES SUPPORTED ON IT, HAVE THE SAME ROTATION VECTOR, AND THE UNION OF THEIR SUPPORTS EQUALS Λ_c)

IN PARTICULAR, ALL ORBITS ARE CONJUGATE BY A SMOOTH DIFFEO ISOTOPIC TO THE IDENTITY

• Λ_c ADMITS THE STRUCTURE OF A SMOOTH $\mathbb{T}^d$-BUNDLE OVER A BASIS B^{m-d} THAT is PARALLELISABLE, WITH $H_1(B) = 0$ $(d>0)$

ii) • THE SAME is TRUE FOR c' IN AN OPEN SET $\mathcal{O} \subseteq H^1(M,\mathbb{R})$

• α-FUNCTION is DIFFERENTIABLE ON $\mathcal{O}$ & β-FUNCTION is DIFFERENTIABLE ON $\partial\alpha(\mathcal{O})$

iii) • IF $\exists\ \Lambda_{c_0} \subseteq \mathrm{Int}\left(\bigcup_{c' \in \mathcal{O}} \Lambda_{c'}\right) \implies \Lambda_{c_0}$ is DIFFEOMORPHIC TO $\mathbb{T}^m$ AND THE MOTION ON IT IS CONJUGATE TO A ROTATION.

(WE HAVE A FOLIATION)

• IN PARTICULAR, $M \cong \mathbb{T}^m$ AND THE SYSTEM IS LIOUVILLE-ARNOL'D INTEGRABLE IN A NEIGHBORHOOD OF Λ_{c_0}

iv) IF $\dim H^1(M, \mathbb{R}) \geq \dim M$, THEN (iii) IS SATISFIED:

H IS LIOUVILLE-ARNOL'D INTEGRABLE IN A NEIGHBORHOOD OF Λ_c AND $M \cong \mathbb{T}^m$.

(v) • IF $\dim M \leq 3$, THEN $M \cong \mathbb{T}^m$

• IF $\dim M = 4$, $M \cong \mathbb{T}^4$ OR $\mathbb{T}^1 \times E$ WHERE E = ORIENTABLE 3-MANIFOLD FINITELY COVERED BY S^3

THANK YOU

FOR YOUR

ATTENTION!

ALL PUBLICATIONS CAN BE FOUND ON MY WEB-PAGE:

WWW.MAT.UNIROMA2.IT/~SORRENTI/