

Markov numbers, Mather's β function and stable norm

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Abstract

Fock (1997 (arXiv:dg-ga/9702018v3); Fock et al 2007 Handbook of Teichmüller Theory (Zürich: European Mathematical Society)) introduced an interesting function $\psi(x)$, $x \in \mathbb{R}$ related to Markov numbers. We explain its relation to Federer–Gromov's stable norm and Mather's β -function, and use this to study its properties. We prove that ψ and its natural generalisations are differentiable at every irrational x and non-differentiable otherwise, by exploiting the relation with length of simple closed geodesics on the punctured or one-holed tori with the hyperbolic metric and the results by Bangert (1994 *Calculus Variations Partial Differ. Equ.* **2** 49–63) and McShane–Rivin (1995 *C. R. Acad. Sci. Paris I* **320**).

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1. Introduction

In 1880, Markov [15] discovered a remarkable relation between the theory of binary quadratic forms and the following Diophantine equation known as the *Markov equation*

$$x^2 + y^2 + z^2 = 3xyz. \quad (1)$$

Markov showed that all positive integer solutions (known as *Markov triples*) can be obtained from the obvious $(1, 1, 1)$ by applying the *Vieta symmetry*

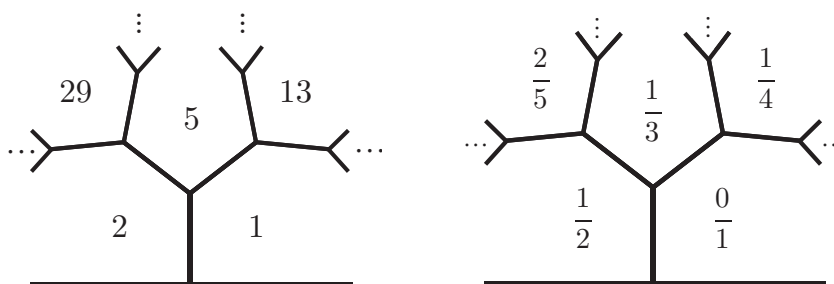


Figure 1. The Markov tree and the corresponding branch of the Farey tree.

$$(x, y, z) \mapsto (x, y, 3xy - z) \tag{2}$$

and permutations. The elements of Markov triples are the famous *Markov numbers*

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, \dots$$

which play a very important role in number theory [1], in the theory of Frobenius manifolds and related Painlevé-VI equation, Teichmüller spaces and algebraic geometry (see [30] and references therein).

The Markov numbers can be naturally labelled by rationals $x \in [0, 1/2]$ using the Farey tree (see figure 1). Recall that at each vertex of the Farey tree we have fractions $\frac{a}{b}, \frac{c}{d}$ and their *Farey mediant* $\frac{a+c}{b+d}$ (see e.g. [30]). On the Markov tree the triples at two neighbouring vertices are related by the involution (2).

Let $m\left(\frac{p}{q}\right)$ be the Markov number corresponding to $\frac{p}{q}$ on the Farey tree. The function $m\left(\frac{p}{q}\right)$ can be extended to all rationals $\frac{p}{q}$ using the symmetry

$$m(1 - x) = m(1/x) = m(x).$$

Following Fock [6], consider the following function $\psi(x), x \in \mathbb{R}$. At a rational $x = \frac{p}{q}$ this function is defined as

$$\psi\left(\frac{p}{q}\right) := \frac{1}{q} \operatorname{arcosh}\left(\frac{3}{2}m\left(\frac{p}{q}\right)\right). \tag{3}$$

Fock was motivated by Thurston’s approach to Teichmüller theory based on measured laminations [31] and by the link of Markov numbers with hyperbolic geometry discovered by Gorshkov and Cohn [4, 8].

Theorem 1 (Fock). *Function ψ can be extended to a continuous convex function on the whole \mathbb{R} .*

We will present the proof of this result below, which is essentially equivalent to Fock’s original proof.

The main aim of this note is to explain the relation of this result with Aubry–Mather theory, more specifically with the so-called Mather’s β -function [19, 20, 29], and Federer–Gromov’s stable norm. The latter concept appeared for the first time in Federer [5] and was named *stable norm* in [9].

Combining this theory with the interpretation of Markov numbers in terms of the lengths of simple closed geodesics on a punctured torus [4, 8] we prove the following

Theorem 2. *Fock’s function ψ is differentiable at all irrationals and not differentiable at any rational.*

We show also that these results can be naturally generalised to the solutions of the Diophantine equation [30]

$$X^2 + Y^2 + Z^2 = XYZ + 4 - 4a^6, \quad a \in \mathbb{N},$$

which are related to hyperbolic tori with a hole (see theorem 3 in the next section).

2. Markov equation and hyperbolic tori

We explain now in more detail the relation of Markov numbers with the simple closed geodesics on the punctured torus with hyperbolic metric, which was found by Gorshkov [8] in his thesis in 1953 and, independently, by Cohn [4] (see also [10, 27]).

The punctured torus \mathcal{T}^2 is homotopically equivalent to the bouquet of two circles, so its fundamental group $\pi_1(\mathcal{T}^2)$ is the free group F_2 . The hyperbolic structure corresponds to a realisation of $\pi_1(\mathcal{T}^2) = F_2$ as a discrete (Fuchsian) subgroup of $SL_2(\mathbb{R})$. Let $A, B \in SL_2(\mathbb{R})$ be the corresponding generators of the group.

We have the classical *Fricke identities*: for any $A, B \in SL_2(\mathbb{R}), C = AB$

$$\begin{aligned} \text{tr } AB + \text{tr } AB^{-1} &= \text{tr } A \text{tr } B, \\ (\text{tr } A)^2 + (\text{tr } B)^2 + (\text{tr } C)^2 &= \text{tr } A \text{tr } B \text{tr } C + \text{tr } (ABA^{-1}B^{-1}) + 2, \end{aligned} \tag{4}$$

where $\text{tr}(\cdot)$ denotes the trace of a matrix.

The puncture condition means that the commutator of the generators is a parabolic element with

$$\text{tr } (ABA^{-1}B^{-1}) = -2,$$

so (4) implies that the corresponding $X = \text{tr } A, Y = \text{tr } B, Z = \text{tr } C$ satisfy the *real* Markov equation

$$X^2 + Y^2 + Z^2 = XYZ, \quad X, Y, Z \in \mathbb{R}. \tag{5}$$

One can show that the positive component of this surface with $X, Y, Z > 0$ is the Teichmüller space of the punctured tori (see [13]).

The corresponding mapping class group $SL_2(\mathbb{Z})$ acts by permutations of X, Y, Z and by Vieta involution

$$(X, Y, Z) \mapsto (Y, X, XY - Z).$$

The orbit starting from the symmetric solution $(3, 3, 3)$ simply consists of Markov triples multiplied by 3:

$$X = 3x, Y = 3y, Z = 3z.$$

It corresponds to the punctured equianharmonic (i.e. rhombic with angle $\pi/3$) hyperbolic torus with three-fold symmetry, which implies that the Markov numbers are related to the lengths l of simple closed geodesics by

$$m = \frac{2}{3} \cosh \frac{l}{2}$$

(see [4, 8, 10, 27]). The matrices A, B in this particular case can be chosen as

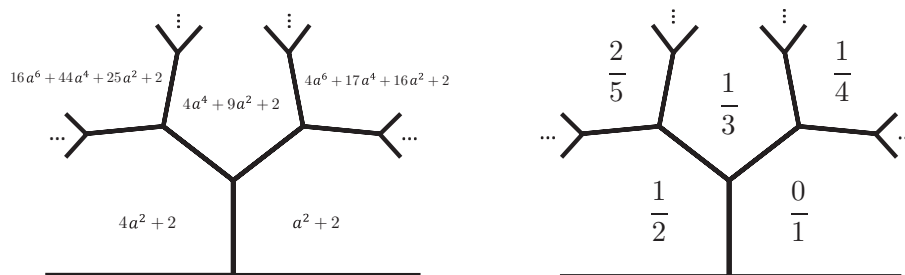


Figure 2. The a -generalisation of the Markov tree and corresponding Farey fractions.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \tag{6}$$

and generate the commutator subgroup of $SL_2(\mathbb{Z})$.

Consider the following generalisation of these matrices proposed in [30] in relation with the computation of the Lyapunov exponents of Markov and Euclid trees:

$$A_a = \begin{pmatrix} 1 - a + a^2 & a^2 \\ a & a + 1 \end{pmatrix}, \quad B_a = \begin{pmatrix} 1 - 2a + 4a^2 & 4a^2 \\ 2a & 2a + 1 \end{pmatrix}, \quad a \in \mathbb{N}. \tag{7}$$

The trace of the corresponding commutator is $2 - 4a^6$, so we have a hyperbolic torus with a hole of length $l = 2 \operatorname{arcosh}(2a^6 - 1)$. When $a = 1$ we have the punctured torus with $l = 0$.

The corresponding traces are solutions of the Diophantine equation

$$X^2 + Y^2 + Z^2 = XYZ + 4 - 4a^6, \tag{8}$$

which is a particular case of the one studied by Mordell [25]. It has no fully symmetric integer solutions, but has a solution with $X = Y$:

$$X = Y = a^2 + 2, \quad Z = 4a^2 + 2. \tag{9}$$

Applying the permutations and Vieta involution we have the following generalisation of the Markov tree [30].

Let $X(\frac{p}{q})$, where $\frac{p}{q} \in [0, \frac{1}{2}]$, be the a -Markov number corresponding to Farey fraction $\frac{p}{q}$ (see figure 2). Introduce the a -analogue of Fock function

$$\psi_a\left(\frac{p}{q}\right) := \frac{1}{q} \operatorname{arcosh}\left(\frac{1}{2}X\left(\frac{p}{q}\right)\right). \tag{10}$$

Theorem 3. *The function ψ_a can be extended to a continuous convex function of real $x \in [0, \frac{1}{2}]$, which is differentiable at all irrational x and not differentiable at rational x .*

We will explain now how to prove both our theorems (theorems 2 and 3) by establishing links with known results about the stable norm and Mather β -function.

3. Stable norm and Mather β -function

3.1. The stable norm on (\mathbb{T}^2, g)

In this section we introduce Federer–Gromov’s *stable norm* on $H_1(\mathbb{T}^2; \mathbb{R})$ and state some of its properties. We discuss only the case of the two-dimensional torus, which presents several simplifications; however, we refer the reader to [3, 9] for a more general presentation.

Let g be a Riemannian metric on $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ and let L_g be the associated length functional.

For any integral class $h \in H_1(\mathbb{T}^2; \mathbb{Z}) \subset H_1(\mathbb{T}^2; \mathbb{R})$ we define

$$\ell_g(h) := \min \{L_g(\gamma) : \gamma \text{ is a smooth closed curve representing } h\}.$$

Observe that $\ell_g(h) = \ell_g(-h)$, and $\ell_g(h) = 0$ if and only if $h = 0$. It follows from a result by Hedlund [11] (see also [3, theorem 8.5.10]) that $\ell_g : H_1(\mathbb{T}^2; \mathbb{Z}) \rightarrow [0, +\infty)$ is positively homogeneous:

$$\ell_g(nh) = n\ell_g(h) \quad \forall h \in H_1(\mathbb{T}^2; \mathbb{Z}) \text{ and } \forall n \in \mathbb{N};$$

observe that this result is a peculiarity of the 2-dimensional case.

The *stable norm* $\|\cdot\|_s$ corresponds to the unique norm on $H_1(\mathbb{T}^2; \mathbb{R})$, which coincides with ℓ_g on $H_1(\mathbb{T}^2; \mathbb{Z})$ (see [3, proposition 8.5.3]). This norm can be actually constructed quite easily from ℓ_g in the following way:

- for every $h \in H_1(\mathbb{T}^2; \mathbb{Z})$, we define $\|h\|_s := \ell_g(h)$;
- then, using homogeneity, we extend it to $H_1(\mathbb{T}^2; \mathbb{Q})$ by

$$\|\alpha h\|_s := |\alpha| \|h\|_s \quad \forall h \in H_1(\mathbb{T}^2; \mathbb{Z}) \text{ and } \forall \alpha \in \mathbb{Q}.$$

Thus, we have extended ℓ_g to $H_1(\mathbb{T}^2; \mathbb{Q})$ and this new function is positively homogeneous and satisfies the triangle inequality:

$$\ell_g(h_1 + h_2) \leq \ell_g(h_1) + \ell_g(h_2) \quad \forall h_1, h_2 \in H_1(\mathbb{T}^2; \mathbb{Q}).$$

Moreover it is a Lipschitz function on $H_1(\mathbb{T}^2; \mathbb{Q}) \simeq \mathbb{Q}^2 \subset \mathbb{R}^2$ with respect to the Euclidean norm $\|\cdot\|$; indeed, if $h = (x_1, x_2) \in \mathbb{Q}^2$ we have

$$\ell_g(h) \leq \ell_g((x_1, 0)) + \ell_g((0, x_2)) \leq K(|x_1| + |x_2|) \leq 2K\|x\|,$$

where $K := \max\{\ell_g((1, 0)), \ell_g((0, 1))\}$.

Since $H_1(\mathbb{T}^2; \mathbb{Q})$ is dense in $H_1(\mathbb{T}^2; \mathbb{R})$, this function has a unique continuous extension to a semi-norm $\|\cdot\|_s$ on $H_1(\mathbb{T}^2; \mathbb{R})$, which turns out to be a norm.

The unit ball $\mathcal{B}_1 := \{h \in H_1(\mathbb{T}^2; \mathbb{R}) : \|h\|_s \leq 1\}$ is strictly convex, namely its boundary does not contain straight line segments. Equivalently, if h_1 and h_2 are linearly independent, then

$$\|h_1 + h_2\|_s < \|h_1\|_s + \|h_2\|_s$$

(see for example [3, exercise 8.5.15]).

The following theorem was proven by Bangert [2, theorem 5.3] (see also related results by Mather [19, sections 2 and 3] for twist maps by Massart-Sorrentino [18, corollaries 2 and 3] for Lagrangian flows on surfaces and by Klempnauer–Schröder [12, main theorem 1.6] for the case of Finsler metrics).

Theorem 4 (Bangert). *Let $h = (h_1, h_2) \in H_1(\mathbb{T}^2; \mathbb{R}) \setminus \{0\}$, then*

- *If $h_2 \neq 0$ and $h_1/h_2 \in \mathbb{R} \setminus \mathbb{Q}$, then $\|\cdot\|_s$ is differentiable at h .*
- *If $h_2 = 0$ or $h_2 \neq 0$ and $h_1/h_2 \in \mathbb{Q}$, then $\|\cdot\|_s$ is differentiable at h if and only if there*

exists a foliation of \mathbb{T}^2 by shortest closed geodesics in the same homotopy class, which is the primitive element in $H_1(\mathbb{T}^2; \mathbb{Z}) \cap \{\mathbb{R} \cdot h\}$.

For more properties of minimal geodesics at irrational directions and their asymptotic behaviour, see [26, main theorem 1.7].

3.2. Mather's β -function

The stable norm is related to the so-called *Mather's minimal average action* (or *Mather's β function*) [20]. Hereafter we provide a brief introduction; we refer the reader to [29, section 3.3] for a more comprehensive one.

Let us consider the Lagrangian associated to the geodesic flow on (\mathbb{T}^2, g) :

$$L : T\mathbb{T}^2 \longrightarrow \mathbb{R}$$

$$(x, v) \longmapsto \frac{1}{2}g_x(v, v).$$

Let $\mathcal{M}(L)$ be the set of Borel probability measures μ on $T\mathbb{T}^2$ that are invariant under the Euler–Lagrange flow of L (i.e. the geodesic flow). To each element $\mu \in \mathcal{M}(L)$, we can associate its *homology* $\rho(\mu) \in H_1(\mathbb{T}^2; \mathbb{R})$ (also called *Schwartzman asymptotic cycle*) in the following way (see [20, 29] for more details): it is the unique element of $H_1(\mathbb{T}^2; \mathbb{R})$ for which

$$\langle [\eta], \rho(\mu) \rangle = \int_{T\mathbb{T}^2} \eta(x, v) \, d\mu(x, v), \tag{11}$$

where η is any closed 1-form on \mathbb{T}^2 , $\eta(x, v)$ denotes the 1-form η computed at a tangent vector (x, v) , $[\eta] \in H^1(\mathbb{T}^2; \mathbb{R})$ represents its cohomology, and $\langle \cdot, \cdot \rangle$ is the canonical pairing between (real) cohomology and homology groups. Observe that the integral on the right-hand side of (11) only depends on the cohomology class of η (see for example [20, Lemma on p 176]). One can prove that $\rho : \mathcal{M}(L) \longrightarrow H_1(\mathbb{T}^2; \mathbb{R})$ is surjective and continuous [29, proposition 3.2.2].

We define *Mather's β function* as

$$\beta_g : H_1(\mathbb{T}^2; \mathbb{R}) \longrightarrow \mathbb{R}$$

$$h \longmapsto \beta_g(h) := \min_{\mu \in \rho^{-1}(h)} \int_{T\mathbb{T}^2} L_g(x, v) \, d\mu.$$

There is the following relation with the stable norm [16, proposition 1.4.2] (see also [17]):

$$\beta_g(h) = \frac{1}{2} \|h\|_s^2.$$

4. Fock's function and stable norm of punctured and one-holed tori

We want to discuss how to prove theorem 2 by extending the above results on Mather's β function and the stable norm, to hyperbolic punctured and one-holed tori.

Let us consider first a punctured torus \mathcal{T}^2 , namely a surface homeomorphic to $\mathbb{T}^2 \setminus \{\text{point}\}$, equipped with a complete hyperbolic metric g of finite area. The puncture corresponding to the removed point will be called *cusp* and hereafter denoted by C .

The definition of the stable norm can be done as above. For each primitive homology class $h \in H_1(\mathcal{T}^2; \mathbb{Z})$, define $\ell_g(h)$ to be the length of the unique simple geodesic with homology class h (see [22, corollary 1]). For a non-primitive homology class h there exists $n \in \mathbb{N}$ and

\hat{h} primitive such that $h = n\hat{h}$ and we have $\ell_g(h) = n\ell_g(\hat{h})$. One can prove that ℓ_g satisfies the triangle inequality (which is strict when we sum linearly independent cycles). The stable norm $\|\cdot\|_s$ can be then defined exactly as above: firstly, it coincides with ℓ_g on $H_1(\mathcal{T}^2; \mathbb{Z})$; then, it is extended homogeneously along lines of rational slopes, and by continuity to the whole $H_1(\mathcal{T}^2; \mathbb{R})$.

The following theorem has been stated, without proof, in [22] (we slightly rephrase its statement and provide a proof).

Theorem 5 (McShane and Rivin). *Let $\|\cdot\|_s$ be the stable norm of (\mathcal{T}^2, g) and let $h = (h_1, h_2) \in H_1(\mathcal{T}^2; \mathbb{R}) \setminus \{0\}$. Then the following hold true:*

- (i) *If $h_2 \neq 0$ and $h_1/h_2 \in \mathbb{R} \setminus \mathbb{Q}$, then $\|\cdot\|_s$ is differentiable at h*
- (ii) *If $h_2 = 0$ or $h_2 \neq 0$ and $h_1/h_2 \in \mathbb{Q}$, then $\|\cdot\|_s$ is not differentiable.*

Proof. Let us now describe how this result could be obtained from Bangert’s results for \mathbb{T}^2 . The idea is simply to ‘plug the hole’ in a smooth way, which does not affect the minimizing properties of the metric.

Recall that any minimal compact lamination on a punctured torus does not intersect small cusp regions (namely, a neighbourhood of the cusp bounded by a horocycle); this result goes back at least to Poincaré, although a sharper version was proven by McShane [21, lemma 1.3.2] (see also [23, theorems 1.1 and 1.2]).

Proposition 1. Let $\varepsilon > 0$. Any punctured torus has a cusp region with bounding curve of length $4 - \varepsilon$ and this bound is optimal. No simple closed geodesic intersects a cusp region with a boundary curve of length $4 - \varepsilon$.

Therefore, we proceed as follows (see [14], proof of corollary 1.7). First, remove from \mathcal{T}^2 a cusp region of length, for example, equals to 1; then, glue a Euclidean hemisphere of equator length 1. In this way, we obtain a two dimensional torus and the minimum length geodesics on this torus do not enter the added hemisphere. Therefore, the two metrics have the same stable norm.

Now applying Bangert’s theorem (theorem 4) we have a McShane–Rivin result. Indeed, there cannot be a foliation of \mathbb{T}^2 consisting of homologous closed geodesics since due to the hyperbolicity of the flow there exists at most one simple closed geodesic in each homology class. □

Now let us consider a special case of hyperbolic equianharmonic punctured torus \mathcal{T}^2 . In this case the lengths of simple closed geodesics are given by

$$l = 2 \operatorname{arccosh} \frac{3}{2} m,$$

where m is the corresponding Markov number. Comparing the definition of the stable norm with the definition of Fock’s function we have

Theorem 6. *Fock’s function (3) is the half of the restriction of the corresponding stable norm on $H_1(\mathcal{T}^2, \mathbb{R}) = \mathbb{R}^2(p, q)$ to the line $p = x, q = 1$.*

By a general fact, the restriction of the norm to an affine line is a convex function on the line, so we have a proof of Fock’s result (theorem 1) (which is essentially the same as his own).

We can now prove theorem 2, namely that Fock’s function ψ is differentiable at all irrational and not differentiable at any rational.

Proof [Theorem 2]. Applying theorem 6, these properties automatically follow from the above results of Bangert (theorem 4) and McShane–Rivin (theorem 5). \square

The same arguments work for the hyperbolic tori with a hole (see [14]) and we can prove a similar result for the function ψ_a , introduced in (10).

Proof [Theorem 3]. We can cut the hyperbolic torus with a hole along the simple geodesic around the hole and glue the Euclidean hemisphere of the same equator length. It is geometrically evident that any shortest simple closed geodesic on such ‘filled torus’ cannot cross the hemisphere (see [14]), so the stable norm remains the same.

It is known (essentially after Fricke, see Goldman [7], section 3) that the hyperbolic structures on one-holed torus are parametrized by the positive real component of the cubic surface

$$X^2 + Y^2 + Z^2 - XYZ = c, \quad c < 0, \quad (12)$$

where the geodesic length of the hole is given by

$$l = 2 \operatorname{arcosh} \frac{2-c}{2}.$$

A natural action of the corresponding mapping class group $SL_2(\mathbb{Z})$ is generated by the Vieta involutions $(X, Y, Z) \mapsto (Y, X, XY - Z)$ and cyclic permutations. This allows us to compute the corresponding stable norm recursively in the same way as for the Markov numbers.

Applying all this to the specific one-holed torus corresponding to the integer orbit (9), and proceeding as in the proof of theorem 2, we complete the proof. \square

Remark 1. These arguments clearly work for any orbit of $SL_2(\mathbb{Z})$ on the real Markov surface (12), so theorem 3 can be straightforwardly extended to the equation (12) with negative c and arbitrary positive initial data.

5. Concluding remarks

Let us mention some natural questions that would need further investigation.

The picture of the unit ball for the corresponding stable norm from McShane–Rivin [22] suggests that it has ‘corners’ at every rational points. Can we make this more quantitative? What can we say about the left and right derivatives of Fock’s function at rational points, in particular at the symmetry point $x = \frac{1}{2}$? Since the function is convex they do exist. What are the corresponding values? Can we compute the derivatives of Fock’s function at quadratic irrationals x , in particular, at Markov–Hurwitz ‘most irrational’ numbers (see e.g. [1, 30])?

A generalisation to a higher genus case is another natural direction, where one can use Bowen–Series symbolic coding [28]. An interesting example of similar coding for the octagon tessellation of hyperbolic disc was studied in detail by Smillie and Ulcigrai [24].

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