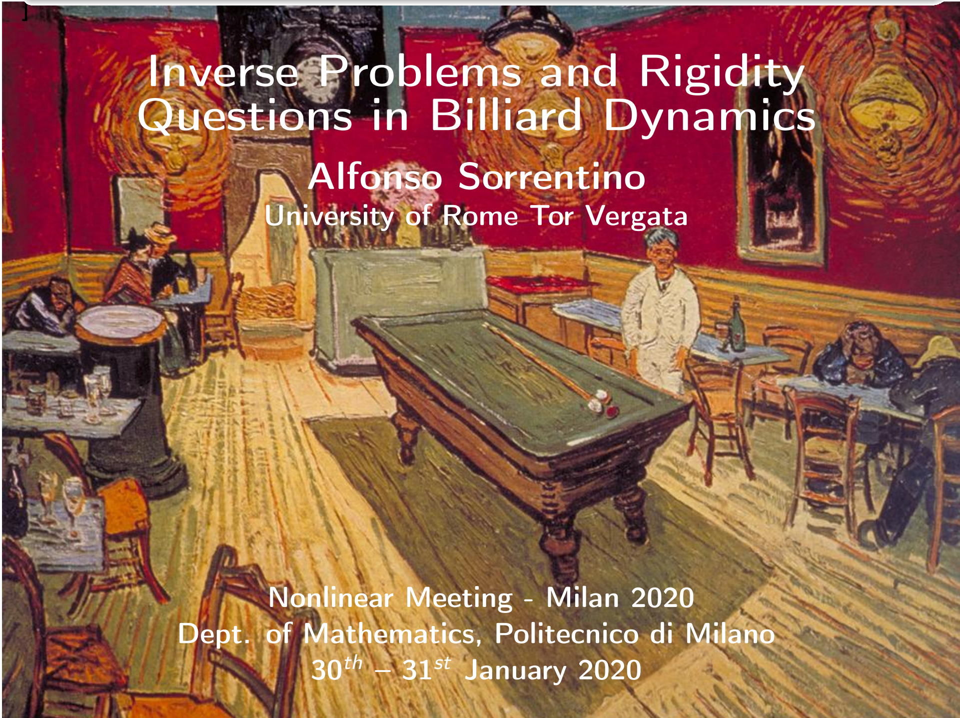

Inverse Problems and Rigidity Questions in Billiard Dynamics

Alfonso Sorrentino
University of Rome Tor Vergata

Nonlinear Meeting - Milan 2020
Dept. of Mathematics, Politecnico di Milano
30^{th} – 31^{st} January 2020

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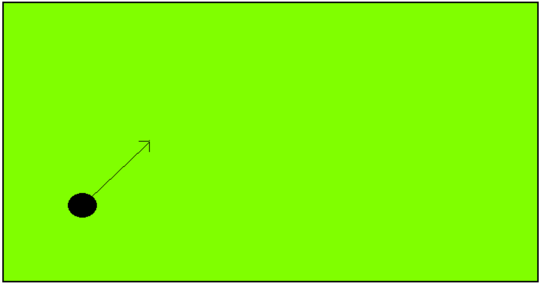

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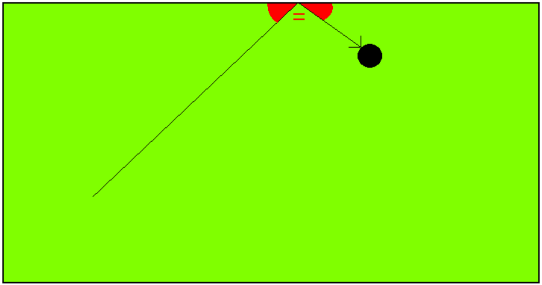

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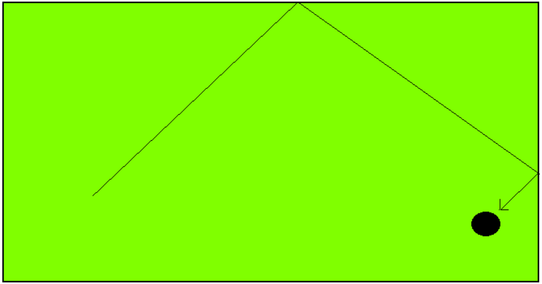

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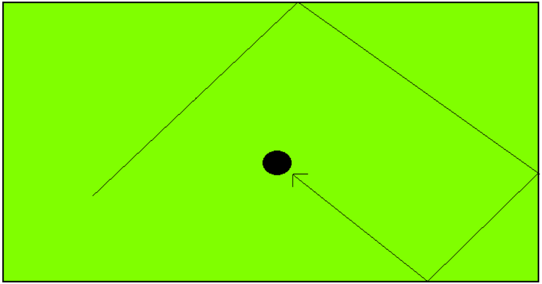

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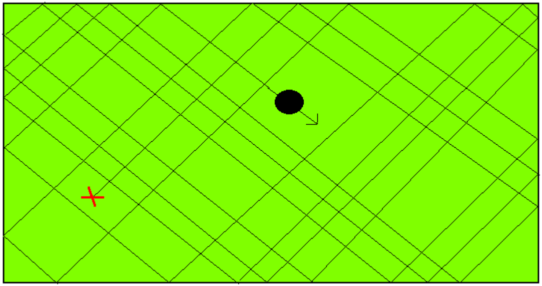

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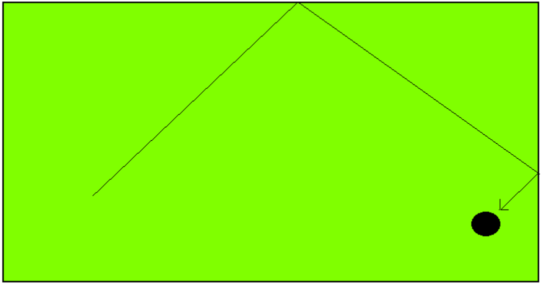
Observation: Between two consecutive bounces, the ball moves along a segment with constant velocity (nothing interesting happens!).
It suffices to know the points where the ball hits the boundary to reconstruct the whole dynamics!



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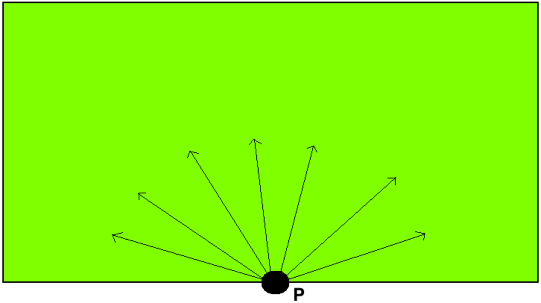

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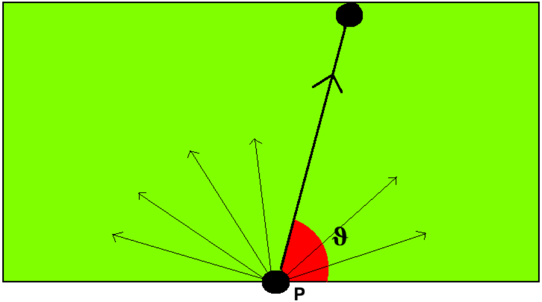
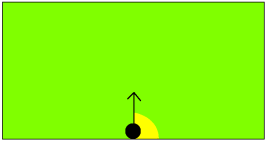
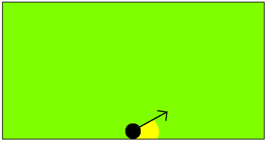
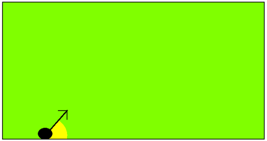
Where will the ball hit the boundary next?
It depends on the initial angle $\vartheta \in (0, \pi)$!



Examples of orbits in a rectangular billiard







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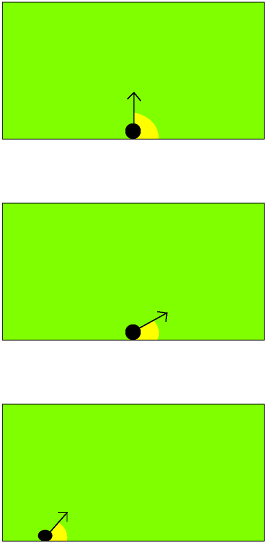
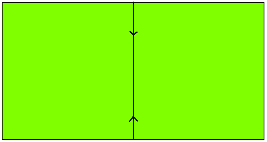


Periodic orbit

Number of bounces (period) = 2



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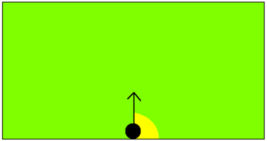

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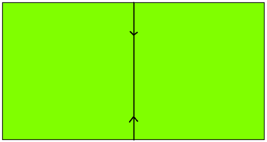

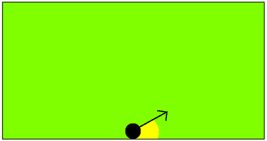

Periodic orbit

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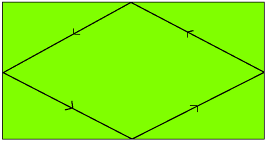
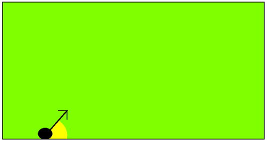
= 4





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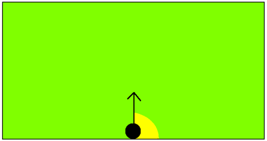

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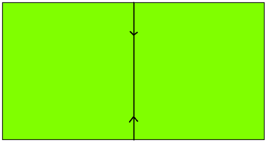

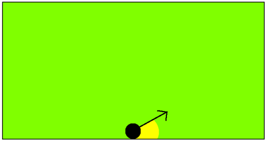

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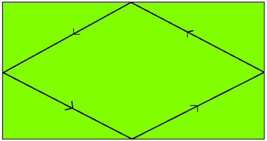
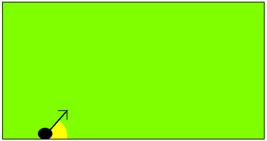
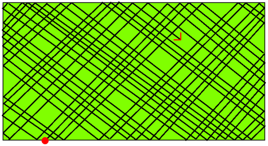




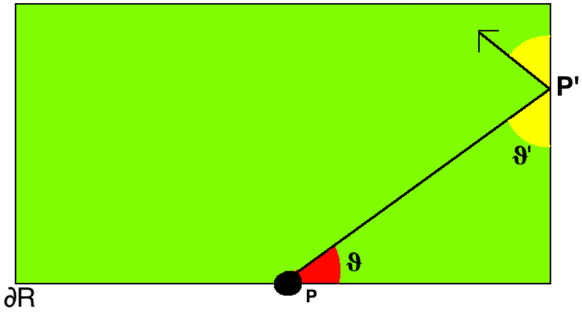
Non-periodic orbit



The Billiard Map

The billiard map is a map that to each initial pair (P, ϑ) associates the point at which the ball will hit the boundary next and the corresponding angle of incidence:

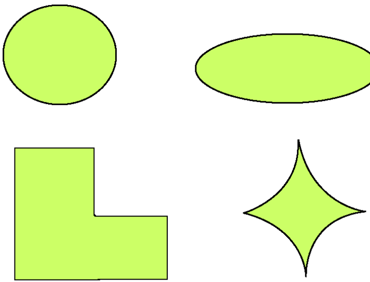
$$\begin{array}{rcl} B : \partial R \times (0, \pi) & \longrightarrow & \partial R \times (0, \pi) \\ (P, \vartheta) & \longrightarrow & (P', \vartheta') \end{array}$$



Why do we consider only rectangular billiards?

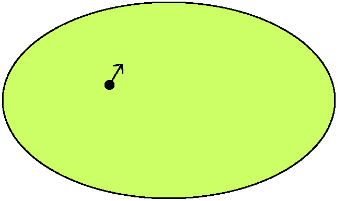
The dynamics inside a billiard is completely determined by its geometry (*i.e.*, its shape)!

One could choose billiard tables with different shapes:



One could also assume that the domain lies inside a Riemannian manifold rather than in the Euclidean plane.

Dynamics inside a general (Euclidean) table



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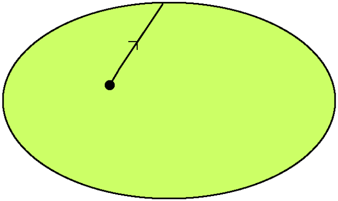

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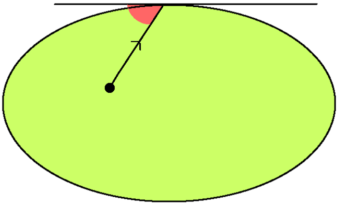

Reflection law: One considers the angle formed with the tangent line

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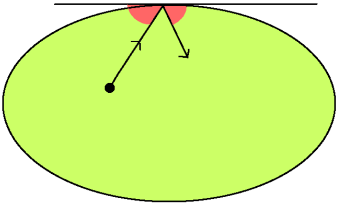

Reflection law: One considers the angle formed with the tangent line

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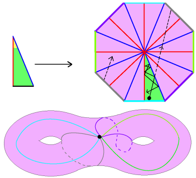
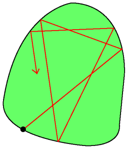
In the case of a table lying in a Riemannian manifold, the ball moves along geodesics instead of straight lines.

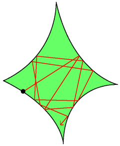
The study of the dynamics of billiards is a very active area of research. Dynamical behaviours and properties are strongly related to the shape of the table.



Polygonal billiards:
- Related to the study of the geodesic flow on a translation surface (with singular points);
- Teichmüller theory.



(Strictly) Convex Billiards:
- Birkhoff billiards (G. Birkhoff, 1927: a paradigm of Hamiltonian systems).
- The billiard map is a twist map.
- Coexistence of regular (KAM, Aubry-Mather) and chaotic dynamics.



Concave Billiards (or dispersive):
- Nearby Orbits tend to move apart (exponentially).
- Hyperbolicity and chaotic behaviour (Y. Sinai, 1970).
- Study of statistical properties of orbits.

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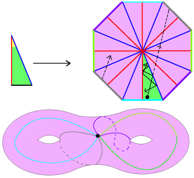

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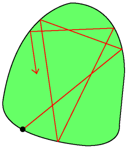

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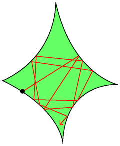

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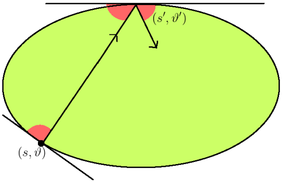
Birkhoff Billiards

Let Ω be a strictly convex domain in $\mathbb{R}^2$ with C^r boundary $\partial\Omega$, with $r \geq 3$. Let $\partial\Omega$ be parametrized by arc-length s (fix an orientation and denote by ℓ its length) and ϑ "shooting" angle (w.r.t. the positive tangent to $\partial\Omega$). The Billiard map is:

$$\begin{aligned} B : \mathbb{R}/\ell\mathbb{Z} \times (0,\pi) &\longrightarrow \mathbb{R}/\ell\mathbb{Z} \times (0,\pi) \\ (s,\vartheta) &\longmapsto (s',\vartheta'). \end{aligned}$$



This simple model has been first proposed by G.D. Birkhoff (1927) as a mathematical playground where "*the formal side, usually so formidable in dynamics, almost completely disappears and only the interesting qualitative questions need to be considered*".

Properties of Birkhoff billiard maps

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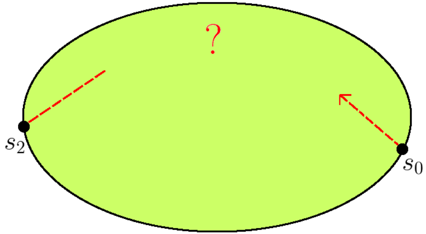
- B is a twist map $\leftarrow$ (Aubry-Mather theory, KAM theory, etc.);
- B has a generating function:

$$h(s,s') := \|\gamma(s) - \gamma(s')\|,$$

i.e., the Euclidean distance between two points on $\partial\Omega$. In particular if $B(s,\vartheta) = (s',\vartheta')$, then:

$$\begin{cases} \partial_1 h(s,s') = -\cos\vartheta \\ \partial_2 h(s,s') = \cos\vartheta' . \end{cases}$$

Dynamics and Length



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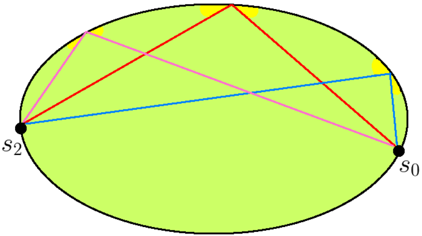

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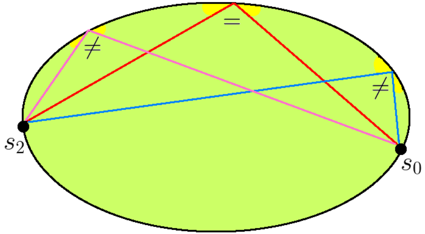

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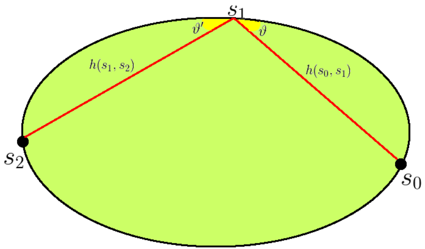


Let us consider the length functional:

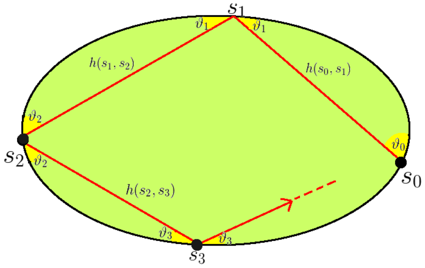
$$\mathcal{L}(s_1) := h(s_0, s_1) + h(s_1, s_2) \quad s_1 \in (s_0, s_2).$$

Then:

$$\frac{d}{ds}\mathcal{L}(s_1) \;=\; \partial_2 h(s_0, s_1) + \partial_1 h(s_1, s_2) = \cos\vartheta - \cos\vartheta'.$$

The real orbit (i.e., $\vartheta = \vartheta'$) correspond to $s_1 \in (s_0, s_2)$ such that $\frac{d}{ds}\mathcal{L}(s_1) = 0$ (i.e., s_1 is a critical point).

Dynamics and Length



$\{(s_n, \vartheta_n)\}_{n\in\mathbb{Z}}$ is an orbit $\iff$ $\{s_n\}_{n\in\mathbb{Z}}$ is a "critical configuration"

of the Length functional:

$$\mathcal{L}(\{s_n\}_n) := \sum_{n\in\mathbb{Z}} h(s_n, s_{n+1}).$$

Relation between the Dynamics and the length of trajectories (Geometry).

Dynamics $\longleftrightarrow$ Geometry

Study of Dynamics: understand the properties of orbits (periodicity, symmetries, chaos, etc...)

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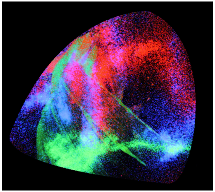
While the dependence of the dynamics on the geometry of the domain is well perceptible, an intriguing challenge is:

To what extent dynamical information can be used
to reconstruct the shape of the domain.



This apparently naïve question is at the core of different intriguing conjectures, among the most difficult to tackle in the study of dynamical systems!

Example I: Circular billiard

Digression: A Mad Tea-Party

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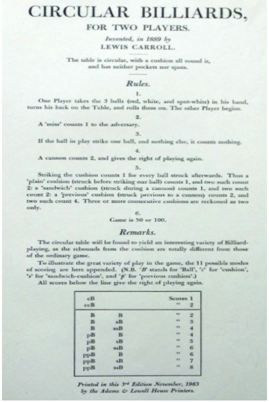
Charles Lutwidge Dodgson (1832-1898)
(better known as Lewis Carroll).

'But I don't want to go among mad people', Alice remarked. 'Oh, you can't help that', said the Cat: 'we're all mad here. You're mad.' 'How do you know I'm mad?', said Alice. 'You must be', said the Cat, 'or you wouldn't have come here.'



CIRCULAR BILLIARDS,
FOR TWO PLAYERS.
Invented, in 1889 by
LEWIS CARROLL.

The table is circular, with a cushion all round it, and has neither pockets nor spots.

Rules.

1.

One Player takes the 3 balls (red, white, and spot-white) in his hand, turns his back on the Table, and rolls them on. The other Player begins.

2.

A 'miss' counts 1 to the adversary.

3.

If the ball in play strike one ball, and nothing else, it counts nothing.

4.

A cannon counts 2, and gives the right of playing again.

5.

Striking the cushion counts 1 for every ball struck afterwards. Thus a 'plain' cushion (struck before striking one ball) counts 1, and two such count 2: a 'sandwich' cushion (struck during a cannon) counts 1, and two such count 2: a 'previous' cushion (struck previous to a cannon) counts 2, and two such count 4. Three or more consecutive cushions are reckoned as two only.

6.

Game is 50 or 100.

Remarks.

The circular table will be found to yield an interesting variety of Billiard-playing, as the rebounds from the cushion are totally different from those of the ordinary game.

To illustrate the great variety of play in the game, the 11 possible modes of scoring are here appended. (N.B. '*B*' stands for 'Ball', '*c*' for 'cushion', '*s*' for 'sandwich-cushion', and '*p*' for 'previous cushion'.)

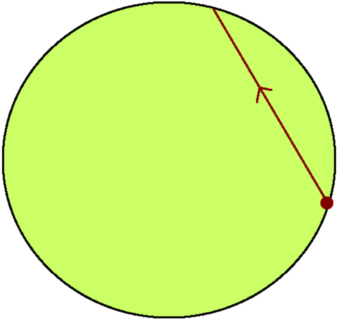
All scores below the line give the right of playing again.

cB		Scores	1
ccB		"	2
B	B	"	2
B	sB	"	3
B	ssB	"	4
pB	B	"	4
pB	sB	"	5
pB	ssB	"	6
ppB	B	"	6
ppB	sB	"	7
ppB	ssB	"	8

Printed in this 3rd Edition November, 1963
by the Adams & Lowell House Printers.

Lewis Carroll thought of playing billiards on a circular table in 1889 and first published its rules the following year (and a circular billiard table was actually made for him!)

Example I: Circular billiard



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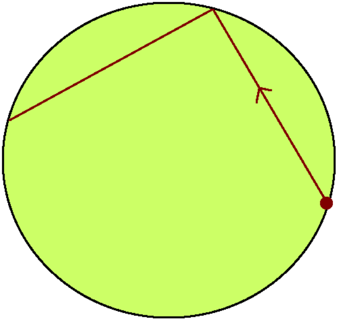

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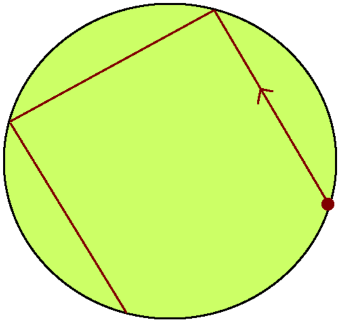

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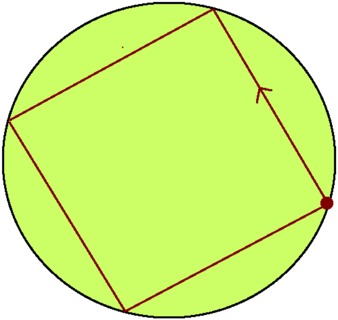

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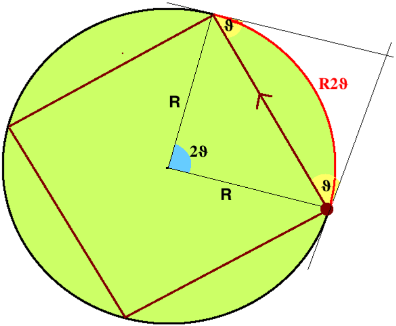
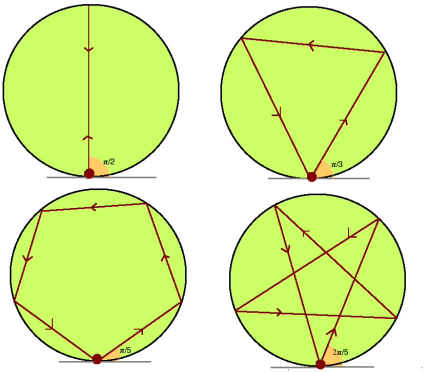


The angle remains constant at each bounce: it is an Integral of motion.
This is an example of integrable dynamical system.

Example I: Circular billiard

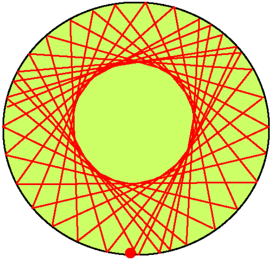
If ϑ is a rational multiple of π, then the resulting orbit is periodic:



For every rational $\frac{p}{q} \in (0, \frac{1}{2}]$ there exist infinitely many periodic orbits with q bounces (period) and which turn p times around before closing (winding number). $\frac{p}{q}$ is called rotation number.

Example I: Circular billiard

If ϑ is NOT a rational multiple of π, then the orbit hits the boundary on a dense set of points (Kroenecker's theorem):



The trajectory does not fill in the table: there is a region (a disc) which is never crossed by the ball!

Observe that the trajectory is always tangent to a circle (this is an example of caustic).

What is true for general Birkhoff billiards?

- Do there always exist periodic orbits? How many?

- How often does the existence of caustics occur? Are there other integrable billiards?

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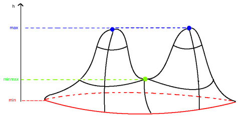
A variation proof exploits the relation between orbits and lengths: one of the two orbits maximizes the length among all configurations with that rotation number, while the other is obtained via a min-max procedure.



(Mountain pass lemma)

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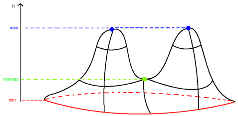

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Q1 - Do the collection of their lenghts encode any information on Ω?

- How often does the existence of caustics occur? Are there other integrable billiards? $\longrightarrow$ Birkhoff conjecture

Q2 - What does integrability say about the geometry of the table?

Integrability of billiards

There are several ways to define integrability for Hamiltonian systems:

- Liouville-Arnol'd integrability (existence of integrals of motion);
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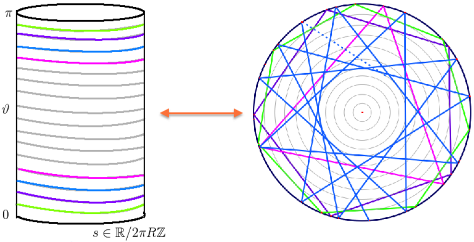


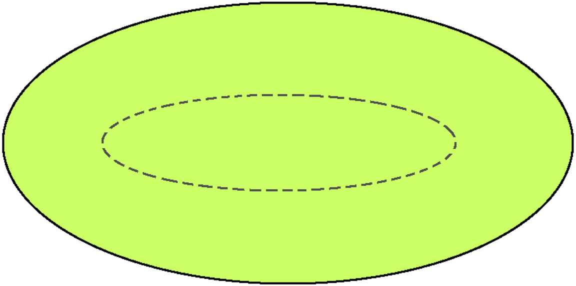
Phase space of the circular billiard map

Foliation of the circular billiard table by caustics

Integrability $\longleftrightarrow$ (Part of) the billiard table is foliated by caustics

Caustics

A convex caustic is a closed C^1 curve in the interior of Ω, bounding itself a strictly convex domain, with the property that each trajectory that is tangent to it stays tangent after each reflection.



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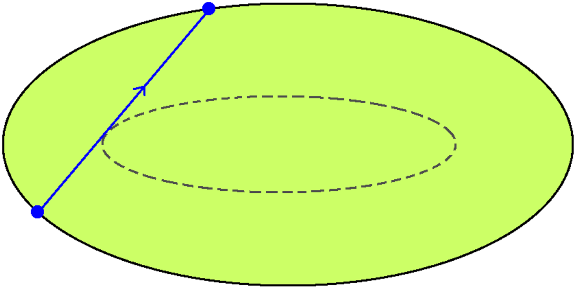

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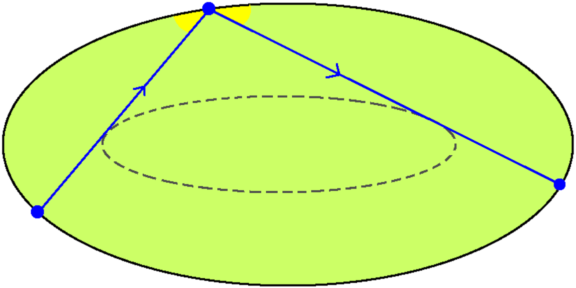

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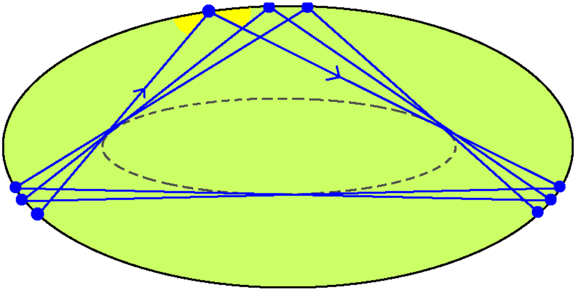

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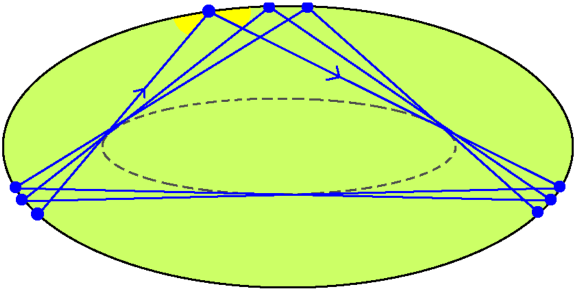
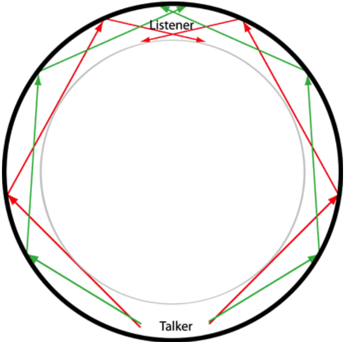


To a convex caustic in Ω corresponds an invariant circle for the billiard map.

(The converse is not entirely true: invariant curves give rise to caustics, but they might not be convex, nor differentiable).

Digression: Caustics and Whispering Galleries

Whispering Gallery in St. Paul Cathedral in London (Lord Rayleigh, 1878 ca.)



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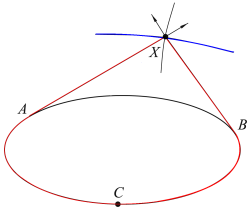

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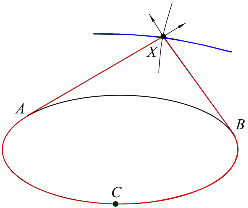

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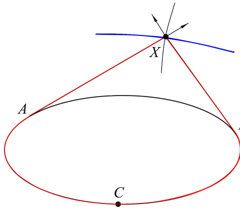

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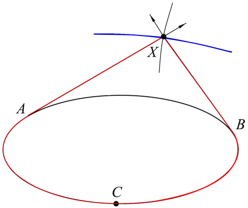

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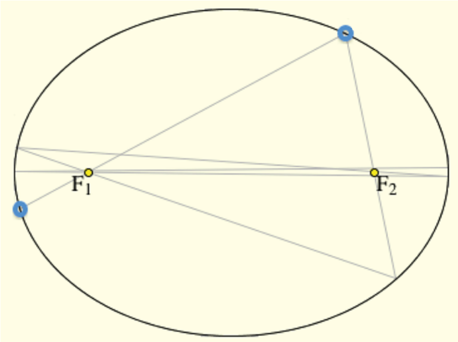
- Do there exist other examples of billiards admitting a foliation by caustics?

Example II: Elliptic billiard

Curiosity: The New York Times (1st July 1964) ran a full-page ad for Elliptipool, played on an elliptical table with a single pocket at one of the two foci. The ad said that on the following day the game would be demonstrated at Stern's department store by movie stars Paul Newman and Joanne Woodward.

Example II: Elliptic billiard



If the trajectory passes through one of the foci, then it always passes through them, alternatively.

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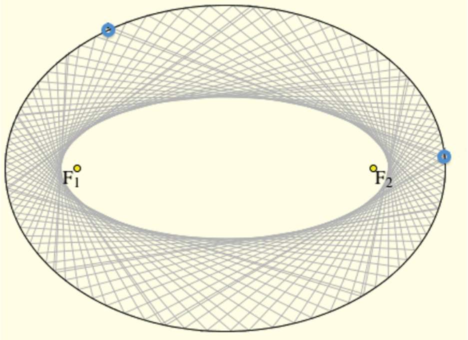

If the trajectory does not intersect the segment between the foci, then it never does and it is tangent to a confocal ellipse (a convex caustic).

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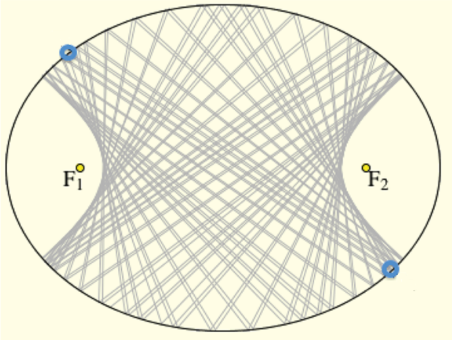

If the trajectory intersects the segment between the foci, then it always does and it is tangent to a confocal hyperbola (a non-convex caustic).

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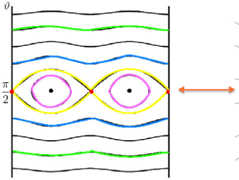
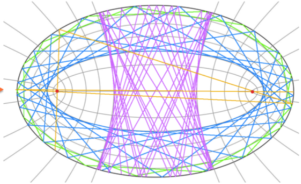


Phase space of an elliptic billiard map


Dynamics inside an elliptic billiard and caustics

Some Properties of Elliptic billiards:

- For every rational $\frac{p}{q} \in (0, \frac{1}{2})$ there exist infinitely many periodic orbits rotation number $\frac{p}{q}$.
- There exist only two periodic orbits of period 2 (i.e., rotation number $\frac{1}{2}$): the two semi-axes.
- There exist infinitely many convex caustics (and also non-convex ones).

The ellipse, with the exception of the closed segment between the foci, is foliated by convex caustics. It is an Integrable billiard.

Birkhoff conjecture

Conjecture (Birkhoff-Poritsky)

The only integrable billiard maps correspond to billiards inside ellipses.

Although some vague indications of this question can be found in Birkhoff's works (1920's-30's), its first appearance was in a paper by Poritsky (1950), who was a National Research Fellow in Mathematics at Harvard University, presumably under the supervision of Birkhoff.

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It is important to consider strictly convex domains!
Mather (1982) proved the non-existence of caustics (hence, some sort of non-integrability) if the curvature of the boundary vanishes at (at least) one point. See also Gutkin-Katok (1995).

Previous contributions

Despite its long history and the amount of attention that this conjecture has captured, it remains still open. Important contributions are the following:

- Bialy (1993): If the phase space of the billiard map is completely foliated by continuous invariant curves which are not null-homotopic, then it is a circular billiard.

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- In a different setting, when there exists an integral of motion that is polynomial in the velocity (Algebraic Birkhoff conjecture), the fact that the billiard is an ellipse has been recently proved by Glutsyuk (2018), based on previous results by Bialy-Mironov (2017).

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Birkhoff Conjecture (Perturbative version)

A smooth strictly convex domain that is sufficiently close (w.r.t. some topology) to an ellipse and whose corresponding billiard map is integrable, is necessarily an ellipse.

- First results in this direction were obtained by:
  - Levallois (1993): Non-integrability of algebraic perturbations of elliptic billiards.
  - Delshams and Ramírez-Ros (1996): Non-integrability of entire symmetric perturbations of ellipses (these perturbations break integrability near the homoclinic solutions).

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- Avila, De Simoi and Kaloshin (2016) proved that perturbative version of Birkhoff conjecture holds true for domains that are nearly circular.

Main Result: the Perturbative Birkhoff Conjecture

Our main result is that the Perturbative Birkhoff conjecture holds true for any ellipse.

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Theorem [Kaloshin - S., *Annals of Math.* (2018)]

Let $\mathcal{E}_0$ be an ellipse of eccentricity $0 \leq e_0 < 1$ and semi-focal distance c; let $k \geq 39$. For every $K > 0$, there exists $\varepsilon = \varepsilon(e_0, c, K)$ such that the following holds.
Let Ω be a C^k domain such that:

- Ω admits integrable rational caustics$^{(*)}$ of rotation number $1/q$, $\forall\, q \geq 3$,
- $\partial\Omega$ is K-close to $\mathcal{E}_0$, with respect to the C^k-norm,
- $\partial\Omega$ is ε-close to $\mathcal{E}_0$, with respect to the C^1-norm,

then Ω must be an ellipse.

(*) An integrable rational caustic corresponds to a (non-contractible) invariant curve of the billiard map foliated by periodic points.

Local integrability and Birkhoff conjecture

One could consider weaker notions of integrability.

For example: what can be said for locally integrable Birkhoff billiards? Namely, consider integrability in a neighborhood of the boundary of the billiard table, i.e., for sufficiently small rotation numbers.

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The analogous conjecture would be:

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If Ω is a Birkhoff billiard admitting a foliation by caustics with rotation numbers in $(0, \delta)$, for some $0 < \delta \leq 1/2$, then Ω must be an ellipse.

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For $\delta = 1/2$ it follows from a result by Innami (2002).
For $\delta = 1/3$ from [Kaloshin-S., 2018].

Local Perturbative Birkhoff conjecture (LPBC)

Let us consider a perturbative version of this conjecture.

Theorem(*) [Huang, Kaloshin, S., *GAFA* (2018)]

For any integer $q_0 \geq 3$, there exist $e_0 = e_0(q_0) \in (0, 1)$, $m_0 = m_0(q_0)$, $n_0 = n_0(q_0) \in \mathbb{N}$ such that the following holds.

For each $0 < e \leq e_0$ and $c \geq 0$, there exists $\varepsilon = \varepsilon(e, c, q_0) > 0$ such that if

- $\mathcal{E}_0$ is an ellipse of eccentricity e and semi-focal distance c,
- Ω admits integrable rational caustics for all $0 < \frac{p}{q} \leq \frac{1}{q_0}$,
- $\partial\Omega$ is C^{m_0} domain,
- $\partial\Omega$ is ε-close (in the C^{n_0} topology) to $\mathcal{E}_0$,

$\Longrightarrow$ Ω itself is an ellipse.

(*) For $q_0 \geq 6$, the proof is conditional to checking that $q_0 - 2$ matrices (which are explicitely described) are invertible.

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Candidates: curvature flow (NO!, it does not preserve integrability, Damasceno, Dias Carneir, Ramírez-Ros (2017)), affine curvature flow (maybe?), ... Any other suggestion?

From billiards to Integrable geodesic flows on the Torus

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Example of globally integrable (non-flat) geodesic flows on $\mathbb{T}^2$ are those associated to Liouville-type metrics:

$$ds^2 = (f_1(x_1) + f_2(x_2))\,(dx_1^2 + dx_2^2).$$

Folklore conjecture: these metrics are the only globally (resp. locally) integrable metrics on $\mathbb{T}^2$.

IDEA: apply similar ideas to prove a perturbative version of this conjecture.

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We define the Length spectrum of Ω:

$$\mathcal{L}(\Omega) := \mathbb{N}^{+} \cdot \{\text{lengths of billiard periodic orbits in } \Omega\} \cup \ell \cdot \mathbb{N}^{+}.$$

Periodic orbits and the (Marked) Length spectrum

What can of information does the set of periodic orbits encode?

We define the Length spectrum of Ω:

$$\mathcal{L}(\Omega) := \mathbb{N}^+ \cdot \{\text{lengths of billiard periodic orbits in } \Omega\} \cup \ell \cdot \mathbb{N}^+.$$

One could also refine $\mathcal{L}(\Omega)$. Consider pairs (length, rotation number) and define the Marked Length spectrum $\mathcal{ML}(\Omega)$.
In particular, for every $p/q \in (0, 1/2]$ define:

$$\mathcal{ML}(\Omega)(p/q) := \max\{\text{lenghts of per. orbits of rot. number } p/q\}.$$

This is also related to Mather's β-function for billiards:

$$\beta(p/q) := -\frac{1}{q}\mathcal{ML}(\Omega)(p/q).$$

From the spectrum to the dynamics

What dynamical information does $\mathcal{M}\mathcal{L}(\Omega)$ encode?

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Theorem [Huang, Kaloshin, S., *Duke Math. Journal* (2018)]

For generic billiard domain, it is possible to recover from the (maximal) marked length spectrum, the Lyapunov exponents of its Aubry-Mather (A-M) orbits), i.e., the periodic orbits with maximal length in their rotation number class.

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- In Riemannian geometry, similar questions have been studied in the case of negatively curved surfaces (Guillemin, Kazhdan,Croke, Otal, Fathi, etc...) and some higher dimensional case (Guillarmou-Lefeuvre, 2019).

Can you hear the shape of a drum?

Let $\Omega \subset \mathbb{R}^2$ and consider the problem of finding $u \not\equiv 0$ and $\lambda \in [0, +\infty)$ such that:

$$\begin{cases} \Delta u + \lambda^2 u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We define the Laplace Spectrum as: $\mathrm{Spec}(\Omega) := \{0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$.

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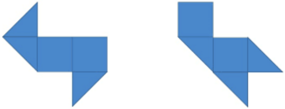
- The answer is well-known to be negative (all known examples are not convex and they are bounded by curves that are only piecewise analytic).
- (Osgood-Phillips-Sarnak) A C^∞ isospectral set is compact. Conjecture (Sarnak): A C^∞ isospectral set consists of isolated points.
- (Zelditch, 2009) positive answer for generic analytic axial-symmetric convex domains.



Counterexample by Gordon-Webb-Wolpert (1992)

Laplace Spectrum and Length Spectrum

An easy example:

If $\Omega = (0,\pi) \times (0,\pi)$, then

$$\mathrm{Spec}(\Omega) = \{\sqrt{n^2 + m^2} : \; (n,m) \in \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}\}$$

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Theorem (Andersson and Melrose, 1977)

The wave trace $w(t) := \mathrm{Re}\left(\sum_{\lambda_n \in \mathrm{Spec}(\Omega)} e^{i\lambda_n t}\right)$ is well-defined as a distribution and it is smooth away from the length spectrum:

$$\text{sing. supp.}\left(w(t)\right) \subseteq \pm\mathcal{L}(\Omega) \cup \{0\}.$$

Generically, equality holds.

Hence, at least for generic domains, one can recover the length spectrum from the Laplace one.

Can you hear the shape of a billiard?

Ω is called length spectrally rigid if any smooth one-parameter isospectral deformation $\{\Omega_\varepsilon\}_{|\varepsilon|<\varepsilon_0}$ with $\Omega_0 = \Omega$ is an isometry.

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Question: Which Birkhoff billiard domains are Length spectrally rigid?

Work in Progress [Callis, De Simoi, Kaloshin, S.]

For any $r \geq 9$, there is a C^r-generic set (open and dense) of strictly convex axial symmetric domains that are length spectrally rigid.

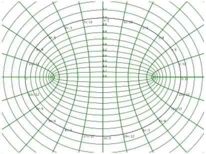
For axial symmetric domains close to a disk, length spectral rigidity was proven by De Simoi, Kaloshin and Wei in 2016.

Thank you for your attention

Sketch of the proof of Theorem [Kaloshin-S.] 1/5

- Consider elliptic coordinates (μ, φ):

$$\begin{cases} x = c \cosh \mu \cos \varphi \\ y = c \sinh \mu \sin \varphi \end{cases}$$



describing confocal ellipses ($\mu = \mu_0$) and hyperbolae ($\varphi = \varphi_0$); $c > 0$ represents the semifocal distance.

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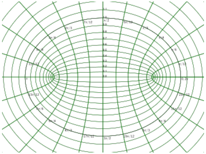
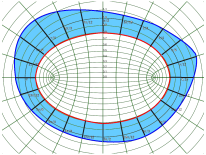


describing confocal ellipses ($\mu = \mu_0$) and hyperbolae ($\varphi = \varphi_0$); $c > 0$ represents the semifocal distance.

- We express a perturbation of a given ellipse $\{\mu_= \mu_0\}$ as:



$$\mu_\varepsilon(\varphi) = \mu_0 + \varepsilon \mu_1(\varphi) + O(\varepsilon^2).$$

(Observe that the coordinate frame depends on the unperturbed ellipse)

Sketch of the proof of Theorem [Kaloshin-S.] 2/5

Let us start by considering a rationally integrable deformation Ω_ε of $\Omega_0 = \mathcal{E}_0$.

Action-angle coordinates for the billiard map in the ellipse $\mathcal{E}_0$. For $q \geq 3$, let $\varphi_q(\theta)$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, denote the parametrization of the boundary induced by the dynamics on the caustic of rotation number $1/q$:

$$B_{\mathcal{E}_0}(\mu_0, \varphi_q(\theta)) = (\mu_0, \varphi_q(\theta + 2\pi/q)).$$

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Lemma [Pinto-de-Carvalho, Ramírez-Ros (2013)]

Let Ω_ε admit a rationally integrable caustic of rotation number $1/q$ for all ε. We denote by $\{\varphi_q^k\}_{k=0}^q$ the periodic orbit of the billiard map in $\mathcal{E}_0$ with rotation number $1/q$ and starting at φ; then $L_1(\varphi) = \sum_{k=1}^q \mu_1(\varphi_q^k) \equiv c_q$, where c_q is a constant independent of φ.

$L_1(\varphi)$ represents the subharmonic Melnikov potential of the elliptic caustic of rotation number $1/q$ under the deformation.

Sketch of the proof of Theorem [Kaloshin-S.] 3/5

Therefore, with respect to the action-angle variables we have that for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$:

$$\sum_{k=1}^{q} \mu_1(\varphi_q(\theta + 2\pi k/q)) \equiv \mathrm{c_q}.$$

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If $u(x)$ denotes either $\cos x$ and $\sin x$, then

$$\int_0^{2\pi} \mu_1(\varphi_q(\theta))\, u(q\,\theta)\, d\theta = 0,$$

which, using the expression for φ_q and by some change of variables, implies:

$$\int_0^{2\pi} \mu_1(\varphi)\, \frac{u\left(\frac{2\pi\, q}{4K(k_q)} F(\varphi; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2 \varphi}}\, d\varphi = 0.$$

- k_q is the eccentricity of the elliptic caustic of rotation number $1/q$
- $F(\varphi, k)$ the incomplete elliptic integral of the first kind;
- $K(k)$ the complete elliptic integral of the first kind, i.e. $K(k) = F(\pi/2, k)$.

Sketch of the proof of Theorem [Kaloshin-S.] 4/5

We define a family of dynamical modes $\{c_q, s_q\}_{q\geq 3}$ given by

$$c_q(\varphi) := \frac{\cos\left(\frac{2\pi\, q}{4K(k_q)}F(\varphi;k_q)\right)}{\sqrt{1-k_q^2\sin^2\varphi}} \qquad s_q(\varphi) := \frac{\sin\left(\frac{2\pi\, q}{4K(k_q)}F(\varphi;k_q)\right)}{\sqrt{1-k_q^2\sin^2\varphi}}.$$

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Consider also five extra functions related to elliptic motions: $e_1, \ldots, e_5$: they correspond to infinitesimal generators of motions that transform ellipses into ellipses (translations, rotations, homotheties, hyperbolic rotations).

Sketch of the proof of Theorem [Kaloshin-S.] 5/5

Key result: Basis property

$\{e_j\}_{j=1}^5 \cup \{c_q, s_q\}_{q\geq 3}$ form a basis of $L^2(\mathbb{T})$.

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- Consider complex analytic extensions of these functions.
- A detailed study of their complex singularities and the size of their maximal strips of analiticity, allow us to deduce their linear independence (both for finite and infinite combinations).
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From Deformative to Perturbative Setting:

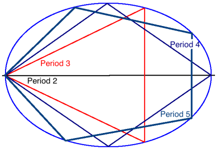
- Annihilation conditions are replaced by smallness condition;
- Approximate $\partial\Omega$ with its "best" approximating ellipse:

$$\partial\Omega = \{(\mu_0^* + \mu_{\rm pert}(\varphi), \varphi): \ \varphi \in [0, 2\pi)\};$$

- Using smallness conditions and Basis property, deduce that $\|\mu_{\rm pert}\|_{L^2}$ must be zero.

Some (vague) ideas on the proof of spectral rigidity 1/2

- Look at simple part of the length spectrum: q-gons (periodic orbits of rotation number $1/q$).
- If the domain is axial symmetric, then symmetric q-gons exist (Birkhoff): $S_q(\Omega) = \{(x_q^{(k)}, \varphi_q^{(k)})\}_{k=0}^{q-1}$



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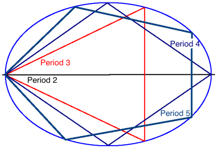

- Consider isospectral deformation: $\partial\Omega_t := \partial\Omega_0 + tn(s) + O(t^2)$. Then:

$$\ell_q(n) := \sum_{k=0}^{q-1} n(x_q^{(k)}) \sin \varphi_q^{(k)} = 0 \qquad \forall q \geq 2.$$

Some (vague) ideas on the proof of spectral rigidity 2/2

- Define a linearized isospectral operator:

$$\begin{aligned} \mathcal{L}_\Omega : C^r_{\mathrm{sym}}(\mathbb{T}) &\longrightarrow \ell^\infty \\ n &\longmapsto \{\ell_q(n)\}_q \end{aligned}$$

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- For symmetric perturbations of the disk, this was done by De Simoi, Kaloshin, Wei (2016). The linearized operator for the disk is upper triangular with units on the diagonal + Perturbation analysis.
- For generic domains, we need to introduce new ingredients:
  - For large q's, it is still a perturbative regime: similar to the circular case (it follows from Lazutkin's result).
  - For small q's, we need to find good substitutes.

  Candidates: some non-perturbative invariants that we call Marvizi-Melrose-Lazutkin's invariants (see [S., DCDS 2015]) and define a mixed linearized isospectral operator, that we hope to prove it is injective! (Work in progress...)