

Differentiability of Mather's average action and integrability on closed surfaces

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Abstract

In this paper we study the differentiability of Mather's average action, or β -function, on closed surfaces, with particular attention to its relation to the integrability of the system.

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1. Introduction

In the study of Tonelli Lagrangian and Hamiltonian systems, a central role in understanding the dynamical and topological properties of the action-minimizing sets (also called *Mather* and *Aubry sets*), is played by the so-called *Mather's average action* (sometimes referred to as β -function or *effective Lagrangian*), with particular attention to its differentiability and non-differentiability properties. Roughly speaking, this is a convex superlinear function on the first homology group of the base manifold, which represents the minimal action of invariant probability measures within a prescribed *homology class*, or *rotation vector* (see (1) for a more precise definition). Understanding whether or not this function is differentiable, or even smoother, and what are the implications of its regularity to the dynamics of the system is a formidable problem, which is still far from being completely understood. Examples of Lagrangians admitting a smooth β -function are easy to construct; trivially, if the base manifold M is such that $\dim H_1(M; \mathbb{R}) = 0$ then β is a function defined on a single point and it is therefore smooth. Furthermore, if $\dim H_1(M; \mathbb{R}) = 1$ then a result by Carneiro [8] allows one to conclude that β is differentiable everywhere, except possibly at the origin. As soon as $\dim H_1(M; \mathbb{R}) \geq 2$ the situation becomes definitely less clear and the smoothness of β becomes a more 'untypical' phenomenon. Nevertheless, it is still possible to find some interesting examples in which it is smooth. For instance, let $H : T^*\mathbb{T}^n \rightarrow \mathbb{R}$ be a completely integrable Tonelli Hamiltonian system, given by $H(x, p) = h(p)$, and consider the associated

Lagrangian $L(x, v) = \ell(v)$ on $T\mathbb{T}^m$. It is easy to check that in this case, up to identifying $H_1(\mathbb{T}^m; \mathbb{R})$ with \mathbb{R}^m , one has $\beta(h) = \ell(h)$ and therefore β is as smooth as the Lagrangian and the Hamiltonian are. One can weaken the assumption on the complete integrability of the system and consider C^0 -integrable systems, i.e. Hamiltonian systems that admit a foliation of the phase space by disjoint invariant continuous Lagrangian graphs, one for each possible cohomology class (see definition 1). It is then possible to prove that also in this case the associated β function is C^1 (see lemma 5). These observations raise the following question: with the exception of the mentioned trivial cases, does the regularity of β imply the integrability of the system?

In this paper we address the above problem in the case of Tonelli Lagrangians on closed surfaces, not necessarily orientable. The main results can be summarized as follows.

Main results. *Let M be a closed surface and $L : TM \rightarrow \mathbb{R}$ a Tonelli Lagrangian.*

- (i) *If M is not the sphere, the projective plane, the Klein bottle or the torus, then β cannot be C^1 on the whole of $H_1(M; \mathbb{R})$ (corollaries 1 and 2).*
- (ii) *If M is not the torus then the Lagrangian cannot be C^0 -integrable (proposition 4).*
- (iii) *If M is the torus, then β is C^1 if and only if the system is C^0 -integrable (theorem 3).*

In particular, in corollary 3 we deduce several information on the dynamics of C^0 -integrable systems. Moreover in section 4.1 we discuss the case of mechanical Lagrangians on the two-torus and show that in this case: β is C^1 if and only if the system is the geodesic flow associated with a flat metric on \mathbb{T}^2 (proposition 6).

The paper is organized as follows. In section 2 we provide a brief overview of Mather's theory for Tonelli Lagrangian systems and introduce the minimal average action, or β function. In section 3 we discuss several properties of the β function associated with Lagrangians on closed surfaces, with particular attention to the implications of its differentiability, or the lack thereof, to the structure of the Mather and Aubry sets. We consider both orientable and non-orientable surfaces, pointing out the differences between the two cases. Finally in section 4 we introduce the concept of C^0 -integrability and relate it to the regularity of β .

2. Mather's average action and Aubry–Mather theory

Let us start by recalling some basic facts about Mather's theory for Tonelli Lagrangians. We refer the reader to [18, 19] for the proofs of these results (see also [24]). Let M be a compact and connected smooth n -dimensional manifold without boundary. Denote by TM its tangent bundle and T^*M the cotangent one and denote points in TM and T^*M respectively by (x, v) and (x, p) . We will also assume that the cotangent bundle T^*M is equipped with the canonical symplectic structure, which we will denote ω . A *Tonelli Lagrangian* is a C^2 function $L : TM \rightarrow \mathbb{R}$, which is strictly convex and uniformly superlinear in the fibres; in particular this Lagrangian defines a flow on TM , known as *Euler–Lagrange flow* Φ_t^L , whose integral curves are solutions of $\frac{d}{dt} \frac{\partial L}{\partial v}(x, v) = \frac{\partial L}{\partial x}(x, v)$. Let $\mathfrak{M}(L)$ be the space of compactly supported probability measures on TM invariant under the Euler–Lagrange flow of L . To every $\mu \in \mathfrak{M}(L)$, we may associate its *average action*

$$A_L(\mu) = \int_{TM} L \, d\mu.$$

It is quite easy to check that since μ is invariant by the Euler–Lagrange flow, then for each $f \in C^1(M)$ we have $\int df(x) \cdot v \, d\mu = 0$ (see for instance [18, lemma p 176]). Therefore we

can define a linear functional

$$\begin{aligned} \mathbf{H}^1(M; \mathbb{R}) &\longrightarrow \mathbb{R} \\ c &\longmapsto \int_{TM} \eta_c(x) \cdot v \, d\mu, \end{aligned}$$

where η_c is any closed 1-form on M , whose cohomology class is c . By duality, there exists $\rho(\mu) \in \mathbf{H}_1(M; \mathbb{R})$ such that

$$\int_{TM} \eta_c(x) \cdot v \, d\mu = \langle c, \rho(\mu) \rangle \quad \forall c \in \mathbf{H}^1(M; \mathbb{R})$$

(the bracket on the right-hand side denotes the canonical pairing between cohomology and homology). We call $\rho(\mu)$ the *rotation vector* of μ . It is possible to show [18, proposition p 178] that the map $\rho : \mathfrak{M}(L) \longrightarrow \mathbf{H}_1(M; \mathbb{R})$ is surjective and hence there exist invariant probability measures for each rotation vector. Let us consider the minimal value of the average action A_L over the set of probability measures with rotation vector h (this minimum exists because of the lower semicontinuity of the action functional [18, lemma p 176]):

$$\begin{aligned} \beta : \mathbf{H}_1(M; \mathbb{R}) &\longrightarrow \mathbb{R} \\ h &\longmapsto \min_{\mu \in \mathfrak{M}(L): \rho(\mu)=h} A_L(\mu). \end{aligned} \tag{1}$$

This function β is what is generally known as *Mather’s β -function*. A measure $\mu \in \mathfrak{M}(L)$ realizing such a minimum amongst all invariant probability measures with the same rotation vector, i.e. $A_L(\mu) = \beta(\rho(\mu))$, is called an *action minimizing measure with rotation vector $\rho(\mu)$* or, equivalently, a $(L, \rho(\mu))$ -*minimizing measure*. The β -function is convex, and therefore one can consider its *conjugate* function (given by Fenchel’s duality) $\alpha : \mathbf{H}^1(M; \mathbb{R}) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \alpha(c) &:= \max_{h \in \mathbf{H}_1(M; \mathbb{R})} (\langle c, h \rangle - \beta(h)) = - \min_{h \in \mathbf{H}_1(M; \mathbb{R})} (\beta(h) - \langle c, h \rangle) \\ &= - \min_{\mu \in \mathfrak{M}(L)} (A_L(\mu) - \langle c, \rho(\mu) \rangle) = - \min_{\mu \in \mathfrak{M}(L)} A_{L-\eta_c}(\mu), \end{aligned}$$

where η_c is any smooth closed 1-form on M with cohomology class c . Observe that the modified Lagrangian $L - \eta_c$ (η_c can be also seen as a function on TM : $\eta_c(x, v) := \eta_c(x) \cdot v$) is still of Tonelli type and has the same Euler–Lagrange flow as L (it follows easily from the closedness of η_c). Nevertheless, invariant probability measures that minimize the action functional $A_{L-\eta_c}$ change accordingly to the chosen cohomology class. A measure $\mu \in \mathfrak{M}(L)$ realizing the minimum of $A_{L-\eta_c}$ (withouth any constraint on the rotation vector) is called c -*action minimizing* or, equivalently, $(L - c)$ -*minimizing measure*. This leads to the definition of a first important family of invariant subsets of TM :

- for a homology class $h \in \mathbf{H}_1(M; \mathbb{R})$, we define the *Mather set of rotation vector h* as:

$$\tilde{\mathcal{M}}^h := \bigcup \{ \text{supp } \mu : \mu \text{ is action minimizing with rotation vector } h \};$$

- for a cohomology class $c \in \mathbf{H}^1(M; \mathbb{R})$, we define the *Mather set of cohomology class c* as:

$$\tilde{\mathcal{M}}_c := \bigcup \{ \text{supp } \mu : \mu \text{ is } c\text{-action minimizing} \}.$$

The relation between these sets is described in the following lemma. To state it, let us recall that, like any convex function on a finite-dimensional space, Mather’s β function admits a subderivative at any point $h \in \mathbf{H}_1(M; \mathbb{R})$, i.e. we can find $c \in \mathbf{H}^1(M; \mathbb{R})$ such that

$$\forall h' \in \mathbf{H}_1(M; \mathbb{R}), \quad \beta(h') - \beta(h) \geq \langle c, h' - h \rangle.$$

As it is usually done, we will denote by $\partial\beta(h)$ the set of $c \in H^1(M; \mathbb{R})$ which are subderivatives of β at h , i.e. the set of c 's which satisfy the above inequality (this set is also called the *Legendre transform of h*). By Fenchel's duality, we have

$$c \in \partial\beta(h) \iff \langle c, h \rangle = \alpha(c) + \beta(h). \quad (2)$$

Lemma 1. *Let h, c be, respectively, an arbitrary homology class in $H_1(M; \mathbb{R})$ and an arbitrary cohomology class $H^1(M; \mathbb{R})$. We have*

$$(1) \tilde{\mathcal{M}}^h \subseteq \tilde{\mathcal{M}}_c \iff (2) c \in \partial\beta(h).$$

Proof. Let us prove that (1) \implies (2). If $\tilde{\mathcal{M}}^h \subseteq \tilde{\mathcal{M}}_c$, then there exists a c -action minimizing invariant measure μ with rotation vector h . Let η_c be a closed 1-form with $[\eta_c] = c$. From the definitions of α, β and rotation vector:

$$-\alpha(c) = \int_{TM} (L - \eta_c) d\mu = \int_{TM} L d\mu - \langle c, h \rangle = \beta(h) - \langle c, h \rangle;$$

since β and α are convex conjugated, then c is a subderivative of β at h .

Finally, in order to show (2) \implies (1), let us prove that any action minimizing measure with rotation vector h is c -action minimizing. In fact, if $c \in \partial\beta(h)$ then $\beta(h) = \langle c, h \rangle - \alpha(c)$; therefore for any $\mu \in \mathfrak{M}(L)$ with $\rho(\mu) = h$ and η_c as above:

$$-\alpha(c) = \beta(h) - \langle c, h \rangle = \int_{TM} (L - \eta_c) d\mu.$$

This proves that μ is c -action minimizing and it concludes the proof. \square

Remark 1. One can also find a relation between the Mather sets corresponding to different cohomology classes, in terms of the strict convexity of the α -function, or better the lack thereof (see also section 2). The regions where α is not strictly convex are called *flats*: for instance the Legendre transform of α at c , denoted $\partial\alpha(c)$, which is the set of homology classes for which equality (2) holds, is an example of flat. It is possible to check that if two cohomology classes are in the relative interior of the same flat F of α , then their Mather sets coincide (see [15, 18]). We denote by $\tilde{\mathcal{M}}(F)$ the common Mather set to all the cohomologies in the relative interior of F .

In addition to the Mather sets, one can also construct another interesting family of compact invariant sets. We define the *Aubry set $\tilde{\mathcal{A}}_c$ of cohomology class c* by looking at a special kind of global minimizers: we say that $(x, v) \in TM$ lies in $\tilde{\mathcal{A}}_c$ if there exists a sequence of C^1 curves $\gamma_n : [0, T_n] \rightarrow M$, such that

- $\gamma_n(0) = \gamma_n(T_n) = x$ for all n
- $\dot{\gamma}_n(0)$ and $\dot{\gamma}_n(T_n)$ tend to v as n tends to infinity
- T_n goes to infinity with n
- $\int_0^{T_n} (L - \eta_c + \alpha(c))(\gamma_n(t), \dot{\gamma}_n(t)) dt$ tends to 0 as n tends to infinity.

One can prove [19, 12] that if $(x, v) \in \tilde{\mathcal{A}}_c$, then the curve $\gamma(t) := \pi \Phi_t^L$, where $\pi : TM \rightarrow M$ denotes the canonical projection, is a c -global minimizer. However, not all c -global minimizers can be obtained in this way.

These action-minimizing sets that we have defined, are such that $\tilde{\mathcal{M}}_c \subseteq \tilde{\mathcal{A}}_c$ for all $c \in H^1(M; \mathbb{R})$ and moreover one of their most important features is that they are graphs over M (*Mather's graph theorem* [18, 19]), i.e. the projection along the fibres $\pi|_{\tilde{\mathcal{A}}_c(L)}$ is injective and its inverse $(\pi|_{\tilde{\mathcal{A}}_c(L)})^{-1} : \pi(\tilde{\mathcal{A}}_c(L)) \rightarrow \tilde{\mathcal{A}}_c(L)$ is Lipschitz. Hereafter we will denote by \mathcal{M}^h , \mathcal{M}_c and \mathcal{A}_c the corresponding projected sets.

Another interesting characterization of the Aubry set is provided by *weak KAM theory* [12] in terms of *critical subsolutions* of Hamilton–Jacobi equation or, in a more geometric way, in terms of some special *Lipschitz Lagrangian graphs*. Let us consider the Hamiltonian system associated with our Tonelli Lagrangian. In fact, if one considers the Legendre transform associated with L , i.e. the diffeomorphism $\mathcal{L}_L : TM \rightarrow T^*M$ defined by $\mathcal{L}_L(x, v) = (x, \frac{\partial L}{\partial v}(x, v))$, then it is possible to introduce a Hamiltonian system $H : T^*M \rightarrow \mathbb{R}$, where $H(x, p) = \sup_{v \in T_x M} (\langle p, v \rangle - L(x, v))$. It is easy to check that H is also C^2 , strictly convex and uniformly superlinear in each fibre: H is also said to be *Tonelli* (or sometimes *optical*). Recall that the flow on T^*M associated with this Hamiltonian, known as the *Hamiltonian flow* of H , is conjugated—via the Legendre transform—to the Euler–Lagrange flow of L . Therefore one can define the analogue of the Mather and Aubry sets in the cotangent space, simply considering $\mathcal{M}_c^* := \mathcal{L}_L(\widetilde{\mathcal{M}}_c)$ and $\mathcal{A}_c^* := \mathcal{L}_L(\widetilde{\mathcal{A}}_c)$. These sets still satisfy all the properties mentioned above, including the graph theorem. Moreover, they will be contained in the energy level $\{H(x, p) = \alpha(c)\}$, where $\alpha : H^1(M; \mathbb{R}) \rightarrow \mathbb{R}$ is exactly *Mather’s α -function* introduced before.

Using Fathi’s weak KAM theory [12] it is possible to obtain a nice characterization of the Aubry set in terms of critical subsolutions of Hamilton–Jacobi equation. As above, let η_c be a closed 1-form with cohomology class c ; we will say that $u \in C^{1,1}(M)$ is an η_c -critical subsolution if it satisfies $H(x, \eta_c + d_x u) \leq \alpha(c)$ for all $x \in M$. The existence of such functions has been showed by Bernard [4]. If one denotes by \mathcal{S}_{η_c} the set of η_c critical subsolutions, then:

$$\mathcal{A}_c^* = \bigcap_{u \in \mathcal{S}_{\eta_c}} \{(x, \eta_c(x) + d_x u) : x \in M\}. \tag{3}$$

This set does not depend on the particular choice of η_c , but only on its cohomology class. Observe that since in T^*M , with the standard symplectic form, there is a 1-1 correspondence between Lagrangian graphs and closed 1-forms, then we can interpret the graphs of these critical subsolutions as Lipschitz Lagrangian graphs in T^*M and the Aubry set may be defined as their *non-removable* intersection (see also [22]).

3. Differentiability of β on closed surfaces

In this section we have gathered all the material we need on the differentiability of β . Let M be a closed surface, not necessarily orientable, $L : TM \rightarrow \mathbb{R}$ a Tonelli Lagrangian on M and $H : T^*M \rightarrow \mathbb{R}$ the associated Hamiltonian.

Let us recall that a homology class h is said to be k -irrational, if k is the dimension of the smallest subspace of $H_1(M; \mathbb{R})$ generated by integer classes and containing h . In particular, *1-irrational* means ‘on a line with rational slope’, while *completely irrational* stands for $\dim H_1(M; \mathbb{R})$ -irrational. Moreover, a homology h is said to be *singular* if its Legendre transform $\partial\beta(h)$ is a *singular flat*, i.e. its Mather set $\mathcal{M}(\partial\beta(h))$ —see remark 1—contains fixed points. Observe that the set of singular classes, unless it is empty, contains the zero class and is compact.

For $h \in H_1(M; \mathbb{R}) \setminus \{0\}$, we define the *maximal radial flat* R_h of β containing h as the largest subset of the half-line $\{th : t \in [0, +\infty)\}$ containing h (not necessarily in its relative interior) in restriction to which β is affine. The possibility of radial flats is the most obvious difference between the β functions of Riemannian metrics (see [3, 14]) and those of general Lagrangians. An instance of radial flat is found in [9].

Let h be a homology class. Assume h is 1-irrational. Then for all t such that $th \in R_h$, th is also 1-irrational. So every (L, th) -minimizing measure is supported on periodic orbits (see [3, proposition 5.6]). Furthermore R_h is a flat of β , so there exists some cohomology

class c such that for every t such that $th \in R_h$, for every (L, th) -minimizing measure μ , the support of μ is contained in the Mather set $\widetilde{\mathcal{M}}_c$. Then Mather's graph theorem says that the projections of the periodic orbits that support the (L, th) -minimizing measures, for every t such that $th \in R_h$, are pairwise disjoint. We usually denote these periodic orbits by $(\gamma_i, \dot{\gamma}_i)_{i \in I}$, where I is some (possibly infinite) set. Recall that [16, theorem 1.3 and proposition 2.4] combine to say the following.

Theorem 1. *Assume*

- M is a closed surface
- L is an autonomous Tonelli Lagrangian on M
- h is a 1-irrational, non-singular homology class
- c lies in the relative interior of $\partial\beta(h)$
- $(\gamma_i, \dot{\gamma}_i)_{i \in I}$ are the periodic orbits which comprise the supports of all (L, th) -minimizing measures, for all th in R_h .

Then, the Mather set $\widetilde{\mathcal{M}}_c$ is precisely the union of the $(\gamma_i, \dot{\gamma}_i)$, and $\widetilde{\mathcal{A}}_c = \widetilde{\mathcal{M}}_c$.

From this we deduce this lemma.

Lemma 2. *Assume*

- M is a closed surface
- L is an autonomous Tonelli Lagrangian on M
- h_0 is a 1-irrational, non-singular homology class
- c lies in the relative interior of $\partial\beta(h_0)$
- $(\gamma_i, \dot{\gamma}_i)_{i \in I}$ are the periodic orbits which comprise the supports of all (L, th) -minimizing measures, for all th in R_h
- μ_i is the probability measure equidistributed on $(\gamma_i, \dot{\gamma}_i)$, for any $i \in I$
- h_i is the homology class of μ_i , for any $i \in I$.

Then, there exists a finite subset J of I such that $\partial\alpha(c)$ is the convex hull of the h_i , $i \in J$.

Proof. For any $i \in I$, μ_i is $(L - c)$ -minimizing, so $h_i \in \partial\alpha(c)$. By the convexity of $\partial\alpha(c)$, it follows that the convex hull of the h_i , $i \in I$, is contained in $\partial\alpha(c)$.

Conversely, take $h \in \partial\alpha(c)$ and an (L, h) -minimizing measure μ . Then μ is $(L - c)$ -minimizing because $h \in \partial\alpha(c)$, therefore the support of μ is contained in the Mather set $\widetilde{\mathcal{M}}_c$, that is, μ is supported by some or all of the $(\gamma_i, \dot{\gamma}_i)$. Hence the homology class of μ , which is h , is a convex combination of the h_i . So $\partial\alpha(c)$ is contained in the convex hull of the h_i , $i \in I$.

We still have to find the finite subset J . Recall that if α_j , $j \in J$, is a collection of pairwise disjoint closed curves on a compact surface M , then the collection of homology classes $[\alpha_j]$, $j \in J$, is finite. Moreover its cardinality is at most $3/2(\dim H_1(M; \mathbb{R}) - 1)$ if M is orientable. See, for instance, [3] for an upper bound on non-orientable surfaces.

Since the closed curves γ_i are pairwise disjoint, their homology classes form a finite subset h_1, \dots, h_n of $H_1(M; \mathbb{R})$. Now, for every $i \in I$, the homology class h_i of μ_i is $T_i^{-1}[\gamma_i]$, where T_i is the period of the periodic orbit $(\gamma_i, \dot{\gamma}_i)$. So all the homology classes h_i , $i \in I$, lie in the finite union of straight lines $\mathbb{R}h_1 \cup \dots \cup \mathbb{R}h_n$. Thus the convex hull of the h_i , $i \in I$, has at most $2n$ extremal points, two in each straight line $\mathbb{R}h_1, \dots, \mathbb{R}h_n$. Therefore $\partial\alpha(c)$, which is the convex hull of the h_i , $i \in I$, is a finite polytope. \square

Now we can state the main results of this section. First let us get the Klein bottle case out of the way. We will use the following lemma from [8]:

Lemma 3. *If L is an autonomous Tonelli Lagrangian on a closed manifold M , then at every $h \in H_1(M; \mathbb{R}) \setminus \{0\}$, β is differentiable in the radial direction, that is, the map*

$$\begin{aligned} B_h : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \beta(th) \end{aligned}$$

is C^1 on $\mathbb{R} \setminus \{0\}$.

Lemma 4. *If L is an autonomous Tonelli Lagrangian on the Klein bottle, then β is C^1 , except possibly at 0.*

Proof. The first Betti number of the Klein bottle is one, that is, there exists $h_0 \in H_1(\mathbb{K}; \mathbb{R}) \setminus \{0\}$ such that for all $h \in H_1(\mathbb{K}; \mathbb{R})$, there exists $t \in \mathbb{R}$ such that $h = th_0$. Then, we use lemma 3. □

The meaning of the next theorem is that for an autonomous Lagrangian on a closed surface, at a 1-irrational, non-singular homology class, β is differentiable only in the directions where it is flat, and in the radial direction. Indeed, in the statement below, $\mathcal{V}(h_0)$ may be viewed as a measure of the non-differentiability of β at h_0 , while $\partial\alpha(c_0)$ is the largest flat containing h_0 in its relative interior.

Theorem 2. *Let*

- M be a closed surface
- L be an autonomous Tonelli Lagrangian on M
- h_0 be a 1-irrational, non-singular homology class
- $(\gamma_i, \dot{\gamma}_i)_{i \in I}$ be the periodic orbits which comprise the supports of all (L, th) -minimizing measures, for all th in R_{h_0}
- c_0 be a cohomology class in the relative interior of $\partial\beta(h_0)$
- $\mathcal{V}(h_0)$ be the vector subspace of $H^1(M; \mathbb{R})$ generated by the differences $c_1 - c_2$, where c_1, c_2 are elements of $\partial\beta(h_0)$.

Then we have

- either

$$\mathcal{V}(h_0) = \partial\alpha(c_0)^\perp = \bigcap_{i \in I} h_i^\perp$$

where orthogonality is meant with respect to the duality between $H_1(M; \mathbb{R})$ and $H^1(M; \mathbb{R})$

- or $M = \mathbb{T}^2$ and the closed curves $\gamma_i, i \in I$, foliate M ; in this case $\mathcal{V}(h_0) = \{0\}$.

Proof. We begin by proving that we always have

$$\mathcal{V}(h_0) \subseteq \partial\alpha(c_0)^\perp, \tag{4}$$

regardless of the irrationality or singularity of h_0 . Take c in $\partial\beta(h_0)$ and h in $\partial\alpha(c_0)$. Recall that by [16, lemma A.1], since c_0 lies in the relative interior of $\partial\beta(h_0)$, and $c \in \partial\beta(h_0)$, we have $\partial\alpha(c_0) \subseteq \partial\alpha(c)$, whence $h \in \partial\alpha(c)$. Therefore we have

$$\begin{aligned} \alpha(c_0) + \beta(h) &= \langle c_0, h \rangle, \\ \alpha(c) + \beta(h) &= \langle c, h \rangle. \end{aligned}$$

On the other hand, since $h \in \partial\alpha(c_0)$, by [8], $\alpha(c_0)$ is the energy level of any (L, h) -minimizing measure. Since we also have $h \in \partial\alpha(c)$, it follows that $\alpha(c) = \alpha(c_0)$, whence

$$\langle c_0 - c, h \rangle = 0$$

that is, $c_0 - c$ lies in h^\perp . Since h is arbitrary in $\partial\alpha(c_0)$, this yields $c_0 - c \in \partial\alpha(c_0)^\perp$, whence (4).

Now we prove that under the hypothesis of the theorem, we have

$$\mathcal{V}(h_0) \supseteq \partial\alpha(c_0)^\perp, \tag{5}$$

unless $M = \mathbb{T}^2$ and the γ_i foliate \mathbb{T}^2 . First observe that by lemma 2,

$$\partial\alpha(c_0)^\perp = \bigcap_{i \in J} h_i^\perp.$$

Since c_0 lies in the relative interior of $\partial\beta(h_0)$, [16, lemma A.4] says that the largest flat of α containing c_0 in its relative interior is $\partial\beta(h_0)$.

Recall [15, theorem 1]: *Let L be an autonomous Tonelli Lagrangian on a compact surface M , for any cohomology class c . If*

- F_c is the largest flat of α containing c in its relative interior
- V_c is the vector space generated by differences $c' - c''$, for all c', c'' in F_c
- E_c is the space of cohomology classes of 1-forms supported away from \mathcal{A}_c , then

$$E_c = V_c.$$

In our case, $F_{c_0} = \partial\beta(h_0)$, so $V_{c_0} = \mathcal{V}(h_0)$, hence

$$E_{c_0} = \mathcal{V}(h_0). \tag{6}$$

Now we will prove that either the γ_i foliate M , or $E_{c_0} = \bigcap_{i \in J} h_i^\perp$. First recall that by theorem 1, the projected Aubry set \mathcal{A}_{c_0} is precisely the union of the γ_i .

Denote by A_ϵ the ϵ -neighbourhood of $\mathcal{A}_{c_0} = \bigcup_{i \in J} \gamma_i$ in M . Since \mathcal{A}_{c_0} consists of pairwise disjoint, simple closed curves, either

- $\mathcal{A}_{c_0} = M$,
- or there exists $\epsilon_1 > 0$ such that for any $0 < \epsilon \leq \epsilon_1$, A_ϵ is a disjoint union of annuli and Möbius strips.

In the first case, M has a non-singular vector field. We claim that this implies M to be the torus or the Klein bottle. Recall from [21] that if M is an orientable surface and X is a vector field on M , then the sum of the indices of the zeroes of X equals the Euler characteristic of M . Therefore, if X is non-singular, then M is the torus. If M is non-orientable and X is a non-singular vector field on M , by considering the orientation cover of M we conclude that M is the Klein bottle.

If M is the Klein bottle, and the γ_i foliate M , then the homology classes h_i are non-zero and pairwise proportional. Since the first Betti number of the Klein bottle is one, we have $\bigcap_{i \in J} h_i^\perp = \{0\}$. On the other hand, by lemma 4, β is C^1 so $\mathcal{V}(h_0) = \{0\}$. Thus the conclusion of the theorem holds when $\mathcal{A}_{c_0} = M$.

So assume we are in the second case. Observe that since the dimension of $H^1(M; \mathbb{R})$ is finite, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$, E_c is the set of cohomology classes of 1-forms supported at least ϵ -away from \mathcal{A}_c . Take $0 < \epsilon \leq \epsilon_0$. Set

$$\mathcal{H}_\epsilon := i_*(H_1(A_\epsilon; \mathbb{R})),$$

where $i : A_\epsilon \rightarrow M$ is the canonical inclusion. Then \mathcal{H}_ϵ is generated by the homology classes h_i , where i lies in the finite set J . Thus, denoting

$$\mathcal{H}_\epsilon^\perp := \{c \in H^1(M; \mathbb{R}) : \forall h \in \mathcal{H}_\epsilon, \langle c, h \rangle = 0\},$$

we have

$$\mathcal{H}_\epsilon^\perp = \bigcap_{i \in J} h_i^\perp. \tag{7}$$

So, in order to prove the theorem, we just need to show that $\mathcal{H}_\epsilon^\perp = E_{c_0}$. Take c in $\mathcal{H}_\epsilon^\perp$ and a closed 1-form η with cohomology c . Then η vanishes on any cycle contained in A_ϵ , so there exists a C^1 function f on M such that $\eta_x = d_x f$ for every x in A_ϵ . Then, $\eta - df$ is supported outside A_ϵ , so $[\eta - df] = c$ lies in E_{c_0} . This proves $\mathcal{H}_\epsilon^\perp \subseteq E_{c_0}$. The other inclusion is obvious, so $\mathcal{H}_\epsilon^\perp = E_{c_0}$. By equations (6) and (7), this yields equation (5), and finishes the proof of the theorem. \square

From theorem 2 we deduce that when M is oriented, β is never differentiable at any 1-irrational, non-singular homology class, unless $M = \mathbb{T}^2$ and M is foliated by periodic orbits.

Corollary 1. *Let*

- M be a closed, oriented surface
- L be an autonomous Tonelli Lagrangian on M
- h_0 be a 1-irrational, non-singular homology class
- $(\gamma_i, \dot{\gamma}_i)_{i \in I}$ be the periodic orbits which comprise the supports of all (L, th) -minimizing measures, for all th in R_{h_0} .

Then we have

- either the dimension of $\partial\beta(h_0)$ is at least the genus of M
- or $M = \mathbb{T}^2$ and the closed curves $\gamma_i, i \in I$, foliate M ; in this case β is differentiable at h_0 .

Proof. We use the notations of the proof of theorem 2. Recall that, since M is oriented, the first homology of M is endowed with a symplectic form, induced by the algebraic intersection of oriented cycles. For the reader who is not familiar with surfaces, we quickly explain the idea: if γ_1 and γ_2 are two closed curves on M , and they have the same homology class, then the complement in M of the union of γ_1 and γ_2 is disconnected. Let X be one of its connected components, and let α be another closed curve. Then, if α enters and leaves X through γ_1 , then the two intersections, counted with sign, cancel out, so the algebraic intersection of α with γ_1 equals the algebraic intersection of α with γ_2 . If α enters X through γ_1 and leaves X through γ_2 , again the algebraic intersection of α with γ_1 equals the algebraic intersection of α with γ_2 . Thus the algebraic intersection induces a skew-symmetric bilinear form on the first homology of M . The non-degeneracy of this bilinear form is simply the fact that given any simple closed curve, one may find a closed curve which intersects it exactly once. See, for instance, [10].

Since \mathcal{H}_ϵ is generated by the homology classes of disjoint closed curves, \mathcal{H}_ϵ is isotropic with respect to the symplectic intersection form, hence the dimension of \mathcal{H}_ϵ is at most half that of $H_1(M; \mathbb{R})$. Recall that half the dimension of $H_1(M; \mathbb{R})$ is precisely the genus of M . So the dimension of $\mathcal{H}_\epsilon^\perp = E_c$ is at least the genus of M . Now, we have seen in the proof of theorem 2 that either the $\gamma_i, i \in I$, foliate M , or $\mathcal{H}_\epsilon^\perp$ is precisely the vector subspace of $H^1(M; \mathbb{R})$ generated by the differences of elements of $\partial\beta(h_0)$, thus proving that the dimension of $\partial\beta(h_0)$ is at least the genus of M . \square

Remark 2. Observe that the above result is not true if h is singular. Indeed, take a vector field X on a closed surface Σ_2 of genus 2, which consists of periodic orbits, except for a singular graph with two fixed points and four hetero/homoclinic orbits between the two fixed points (see figure 1). One can embed it into the Euler–Lagrange flow of a Tonelli Lagrangian, given by $L_X(x, v) = \frac{1}{2} \|v - X(x)\|_x^2$. It is possible to show (see for instance [13]) that $\text{Graph}(X)$ is invariant and that in this case $\tilde{\mathcal{A}}_0 = \text{Graph}(X)$; therefore, the projected Aubry set \mathcal{A}_0 is the whole surface Σ_2 , so β is differentiable at all homology classes in $\partial\alpha(0)$ (see [15, theorem 1]). However, $\mathcal{A}_0 = \Sigma_2$ is not foliated by periodic orbits.

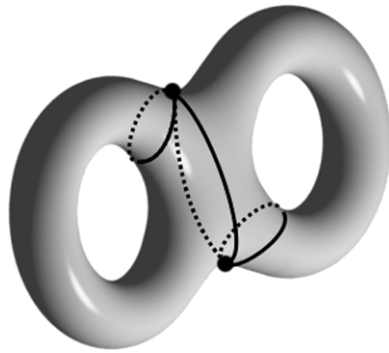


Figure 1. A surface Σ_2 of genus two, consisting of periodic orbits, except for two fixed points and four heteroclinics between them.

When M is not orientable, the situation is different because β may have flats of maximal dimension (that is, of dimension equal to the first Betti number of M). So β may well be differentiable at some 1-irrational, non-singular homology class. For instance, by [3, theorem 1.3], there exists a Riemannian metric on M , whose stable norm has a (finite) polyhedron as its unit ball. Let L be the Lagrangian induced by this Riemannian metric, then the β -function of L is half the square of the stable norm. In particular β is differentiable everywhere but along a finite number of straight lines. So β is differentiable at most 1-irrational classes. Furthermore, since the Lagrangian is a Riemannian metric:

- no homology class other than zero is singular
- the β -function is quadratic (i.e. 2-homogeneous) so radial faces are trivial, i.e. $\forall h, R_h = \{h\}$.

On the other hand, for every homology class the Mather set is a finite union of closed geodesics, so we never have $\mathcal{M}(R_h) = M$.

In general all we have is the following.

Corollary 2. *Let M be a closed, non-orientable surface other than the projective plane or the Klein bottle, and let $L : TM \rightarrow \mathbb{R}$ be a Tonelli Lagrangian. Then, there exists some homology class h such that β is not differentiable at h .*

Proof. Let

- $h_0 \in H_1(M; \mathbb{R})$ be 1-irrational and non-singular
- c_0 be a cohomology class in the relative interior of $\partial\beta(h_0)$
- $(\gamma_i, \dot{\gamma}_i)_{i \in I}$ be the periodic orbits which comprise the supports of all (L, th_0) -minimizing measures, for all th_0 in R_{h_0}
- μ_i be the probability measure equidistributed on $(\gamma_i, \dot{\gamma}_i)$
- h_i be the homology class of μ_i
- J be the finite subset of I , given by lemma 2, such that $\partial\alpha(c_0)$ is the convex hull of the $h_i, i \in J$.

We are going to show that β is not differentiable at h_j , for some $j \in J$. Because of theorem 2 we need to choose h_j carefully, so that R_{h_j} does not meet the relative interior of $\partial\alpha(c_0)$. This can be done as follows. By the Hahn–Banach theorem, there exists an affine hyperplane \mathcal{H} in $H_1(M; \mathbb{R})$, which separates $\partial\alpha(c_0)$ from zero. Define

$$\begin{aligned} \Psi : \partial\alpha(c_0) &\longrightarrow \mathcal{H} \\ h &\longmapsto \mathbb{R}h \cap \mathcal{H}. \end{aligned}$$

Then $\Psi(\partial\alpha(c_0))$ is convex because Ψ sends straight lines to straight lines. Take an extremal point x in $\Psi(\partial\alpha(c_0))$. Then for all $h \in \Psi^{-1}(x)$, h is an extremal point of $\partial\alpha(c_0)$, so there exists some j in J such that $\Psi(h_j) = x$.

First note that h_j is 1-irrational because $h_j = [\mu_j] = T_j^{-1}[\gamma_j]$, where T_j is the least period of the periodic orbit $(\gamma_j, \dot{\gamma}_j)$. Furthermore h_j is non-singular, as we now prove. Take c in the relative interior of $\partial\beta(h_j)$. Since $h_j \in \partial\alpha(c_0)$, we have $c_0 \in \partial\beta(h_j)$. Now by [15], since $c_0 \in \partial\beta(h_j)$ and c lies in the relative interior of $\partial\beta(h_j)$, the Aubry set of c is contained in the Aubry set of c_0 , which is the union of the $(\gamma_i, \dot{\gamma}_i)$ over all $i \in I$ by theorem 1. Therefore the Aubry set of c does not contain any fixed point. Since c lies in the relative interior of $\partial\beta(h_j)$, this entails that h_j is non-singular.

We want to apply theorem 2 to h_j . Take t such that th_j lies in R_{h_j} . By [16, lemma 2.2], we have $\partial\beta(h_j) \subseteq \partial\beta(th_j)$, so $c_0 \in \partial\beta(th_j)$, that is, $th_j \in \partial\alpha(c_0)$. Now recall that $\partial\alpha(c_0)$ is the convex hull of the $h_i, i \in I$, so any (L, th_j) -minimizing measure is supported on some of the $(\gamma_i, \dot{\gamma}_i)$. Let K be the subset of I such that the union of the supports of all (L, th_j) -minimizing measures, for all t such that th_j lies in R_{h_j} , is the union over $k \in K$ of $(\gamma_k, \dot{\gamma}_k)$. Now we prove that for all $k \in K, h_k \in \mathbb{R}h_j$.

Assume it is not so, i.e. there exists some k in K such that $h_k = [\mu_k] \notin \mathbb{R}h_j$. Since $k \in K$, there exists t such that $th_j \in R_{h_j}$, and μ_k is an ergodic component of some (L, th_j) -minimizing measure. Thus we may write

$$th_j = \lambda h_k + (1 - \lambda)h,$$

where $\lambda \in (0, 1)$ and $h \in \partial\alpha(c_0)$. Then $\Psi(th_j)$, which is x , is a convex combination of $\Psi(h_k)$, which is not x , and $\Psi(h)$. This contradicts the extremality of x . We have proved that for all $k \in K, h_k \in \mathbb{R}h_j$. Now we apply theorem 2 to h_j : since M is not the Klein bottle, we have

$$\mathcal{V}(h_j) = \bigcap_{k \in K} h_k^\perp.$$

Now since for all $k \in K, h_k \in \mathbb{R}h_j$, we have

$$\bigcap_{k \in K} h_k^\perp = h_j^\perp.$$

Since M is neither the Klein bottle nor the projective plane, the dimension of h_j^\perp is greater than zero, that is, β is not differentiable at h_j . □

For the sake of completeness, let us recall that by [15, corollary 3] we also have:

Proposition 1. *For any autonomous Lagrangian L on a closed surface M , β is differentiable at every completely irrational homology class.*

3.1. The case when β is C^1 and $M = \mathbb{T}^2$

When M is the two-torus and β is C^1 everywhere, we can rule out radial flats of β .

Proposition 2. *Let L be an autonomous Tonelli Lagrangian on the two-torus such that β is differentiable at every point of $H_1(\mathbb{T}^2; \mathbb{R})$. Then for all $h \in H_1(\mathbb{T}^2; \mathbb{R}) \setminus \{0\}$, we have $R_h = \{h\}$. As a consequence, β is strictly convex.*

Proof. Consider a radial flat $R_h = [\lambda h, h]$ of β . We want to prove that $\lambda = 1$.

First case: h is 1-irrational. Let

- h_0 be an integer homology class such that $h = \tau^{-1}h_0$, where τ is the minimal period of a periodic orbit which carries an (L, h) -minimizing measure

- (x, v) be an element of $\mathbb{T}\mathbb{T}^2$ such that the probability measure carried by the orbit $\Phi_t^L(x, v)$ has homology λh
- T be the minimal period of $\Phi_t^L(x, v)$ (so we have $T = \tau/\lambda$). If (x, v) is a fixed point we just set $T := +\infty$.

Pick

- a sequence of real numbers $t_n > 1$ such that t_n converges to 1
- for each $n \in \mathbb{N}$, $c_n \in H^1(\mathbb{T}^2; \mathbb{R})$ such that $\partial\beta(t_n h) = c_n$ (recall that β is differentiable at $t_n h$).

First note that $t_n h$ is not singular, for any n . Indeed if it were, then zero would lie in the radial flat of $t_n h$, that is, β would be affine along the segment $[0, t_n h]$. But $t_n > 1$, so the segment $[0, t_n h]$ properly contains $[\lambda h, h]$, which contradicts $R_h = [\lambda h, h]$. Then by corollary 1, since β is differentiable at $t_n h$, which is 1-irrational, we have $\mathcal{M}_{c_n} = M$, thus there exists, for every n , a $v_n \in T_x M$ such that $(x, v_n) \in \widetilde{\mathcal{M}}_{c_n}$, and moreover the orbit $\Phi_t^L(x, v_n)$ is periodic. By semicontinuity of the Aubry set, and by the graph theorem, v_n converges to v as n goes to infinity. Let T_n be the minimal period of $\Phi_t^L(x, v_n)$. We now prove that $\liminf T_n \geq T$.

Indeed, if some subsequence $T_{n_k} := T_k$ converged to $T' < T$, we would have $\Phi_{T'}^L(x, v) = (x, v)$, contradicting the minimality of T . If (x, v) is a fixed point, we have $v = 0$, so v_n converges to zero, so T_n tends to infinity.

Now if λ were < 1 , since $\liminf T_n \geq T$, for n large enough there would exist a c_n -minimizing measure with homology in $[\lambda h, h]$. This means that the radial flats R_h and $R_{t_n h}$ overlap, in other words, $t_n h \in R_h$. This contradicts the fact that $R_h = [\lambda h, h]$, hence $\lambda = 1$.

Second case. h is 2-irrational, that is, completely irrational. Then any (L, h) -minimizing measure is supported on a lamination of the torus without closed leaves. Any such lamination is uniquely ergodic, in particular h is not contained in any non-trivial flat of β , radial or not, regardless of the Lagrangian.

The statement about the differentiability of β implying its strict convexity is now just a consequence of the fact, observed by Carneiro in [8], that for an autonomous Lagrangian on a closed, orientable surface M , the flats of β are contained in isotropic subspaces of $H_1(M; \mathbb{R})$ with respect to the intersection symplectic form on $H_1(M; \mathbb{R})$. In particular, when $M = \mathbb{T}^2$, all flats are radial. \square

Proposition 3. *Let $L : \mathbb{T}\mathbb{T}^2 \rightarrow \mathbb{R}$ be a Tonelli Lagrangian. Assume that β is C^1 everywhere. Then zero is the only (possibly) singular class, and for every 1-irrational, non-zero homology class h , $\mathcal{M}^h = \mathbb{T}^2$, and \mathcal{M}^h is foliated by periodic orbits with the same homology class and minimal period.*

Proof. First note that if a non-zero class h is singular, then there exists c in $\partial\beta(h)$, such that there is a fixed point in the Mather set of c . Thus R_h contains the homology of the Dirac measure on the fixed point, which is zero. This contradicts proposition 2, which says that $R_h = \{h\}$.

Now take a non-zero, 1-irrational homology class h , so h is non-singular, and by corollary 1, $\mathcal{M}(R_h) = \mathbb{T}^2$, and $\mathcal{M}(R_h)$ is foliated by periodic orbits. Since $R_h = \{h\}$ by proposition 2, we have $\mathcal{M}^h = \mathbb{T}^2$. It is well known that all leaves of a foliation of the two-torus by homologically non-trivial closed curves are homologous. So there exists $h_0 \in H_1(\mathbb{T}^2; \mathbb{Z})$ such that the projection to \mathbb{T}^2 of any orbit contained in \mathcal{M}^h is homologous to h_0 . Besides, for each $x \in \mathbb{T}^2$, let $T(x)$ be the minimal period of the periodic orbit γ_x in \mathcal{M}^h through x . The

homology class of the probability measure carried by γ_x is $T^{-1}h_0$. Now the fact that there are no non-trivial radial flats of β implies that $T(x)$ is independent of x . \square

4. Integrability

Let us recall the definition of C^0 -integrability (see also [1]).

Definition 1. A Tonelli Hamiltonian $H : T^*M \rightarrow \mathbb{R}$ is said to be C^0 -integrable, if there exists a foliation of T^*M made by invariant Lipschitz Lagrangian graphs, one for each cohomology class.

Lemma 5. Let M be a compact manifold of any dimension, $L : TM \rightarrow \mathbb{R}$ a Tonelli Lagrangian and $H : T^*M \rightarrow \mathbb{R}$ the associated Hamiltonian. If H is C^0 -integrable, then β is C^1 .

Proof. Suppose that H is C^0 -integrable. This means that the cotangent space T^*M is foliated by disjoint invariant Lipschitz Lagrangian graphs. Let us denote by Λ_c the invariant Lagrangian graph of such a foliation, corresponding to the cohomology class c . It is easy to check that each Λ_c is the graph of a solution of Hamilton–Jacobi equation $H(x, \eta_c + du) = \alpha(c)$, where η_c is a closed 1-form on M with cohomology class c , and therefore from weak KAM theory [12] it follows that for each $c \in H^1(M; \mathbb{R})$ the Aubry set $\mathcal{A}_c^* \subseteq \Lambda_c$. If for some $h \in H_1(M; \mathbb{R})$ there exist $c \neq c'$ such that $c, c' \in \partial\beta(h)$, then $\widetilde{\mathcal{M}}^h \subseteq \widetilde{\mathcal{M}}_c \cap \widetilde{\mathcal{M}}_{c'}$ (lemma 1). But this implies that $\widetilde{\mathcal{M}}_c \cap \widetilde{\mathcal{M}}_{c'} \neq \emptyset$ and therefore $\Lambda_c \cap \Lambda_{c'} \supseteq \mathcal{A}_c^* \cap \mathcal{A}_{c'}^* \supseteq \mathcal{M}_c^* \cap \mathcal{M}_{c'}^* \neq \emptyset$, which contradicts the disjointness of the Lagrangian graphs forming the foliation. Hence, β must be differentiable everywhere. \square

We can now prove the main results stated in section 1.

Proposition 4. The torus is the only closed surface which admits a C^0 -integrable Hamiltonian.

Proof. First, no Hamiltonian on the sphere can be C^0 -integrable. Indeed, any Lagrangian graph is exact since the sphere is simply connected, and any two exact Lagrangian graphs intersect, because any C^1 function on the sphere has critical points. Likewise, no Hamiltonian on the projective plane can be C^0 -integrable, otherwise its lift to the sphere would be C^0 -integrable.

For the Klein bottle \mathbb{K} , we need to work a little bit more. For each $x \in \mathbb{K}$, let us define

$$\begin{aligned} F_x : H^1(\mathbb{K}; \mathbb{R}) &\simeq \mathbb{R} &\longrightarrow & T_x^*\mathbb{K} \simeq \mathbb{R}^2 \\ c &\longmapsto &\Lambda_c \cap T_x^*\mathbb{K}, \end{aligned}$$

where Λ_c , for $c \in H^1(\mathbb{K}; \mathbb{R})$, are the Lagrangian graphs foliating $T^*\mathbb{K}$. It is possible to check that F_x is continuous (see [1, lemme 4.22] or the proof of the continuity of (8) in theorem 3) and injective (as it follows from the disjointness of the Λ_c ’s). Moreover, if the Hamiltonian is C^0 -integrable, the map F_x is surjective. Now there is no such thing as a continuous bijection from \mathbb{R} to \mathbb{R}^2 , so there is no C^0 -integrable Hamiltonian on the Klein bottle.

The same argument can be used for any surface with first Betti number > 2 . Finally, no Hamiltonian on the connected sum of three projective planes can be C^0 -integrable, otherwise it would lift to a C^0 -integrable Hamiltonian on a surface of genus two. \square

Theorem 3. Let $L : T\mathbb{T}^2 \rightarrow \mathbb{R}$ be a Tonelli Lagrangian on the two-torus. Then, β is C^1 if and only if the system is C^0 -integrable.

Proof. [\Leftarrow] It follows from lemma 5. [\Rightarrow] For each homology class h , let us denote by $c_h := \partial\beta(h)$. If h is non-singular and 1-irrational, then proposition 3 says that $\Lambda_{c_h} := \mathcal{A}_{c_h}^*$ is

an invariant Lipschitz Lagrangian graph of cohomology class c_h , which is foliated by periodic orbits of homology h and same minimal period T_h . Since the dynamics on these Lipschitz Lagrangian graph is totally periodic, i.e. $\Phi_{T_h}^H|_{\Lambda_{c_h}} = \text{Id}|_{\Lambda_{c_h}}$, then it follows from the result in [1] (see for instance the proof of Théorème 4) that this graph is in fact C^1 .

Observe that such cohomology classes c_h are dense in $H^1(\mathbb{T}^2; \mathbb{R})$. Indeed, 1-irrational non-singular homology classes are dense in $H_1(\mathbb{T}^2; \mathbb{R})$ because 1-irrational homology classes are dense in $H_1(\mathbb{T}^2; \mathbb{R})$ and, by proposition 3, zero is the only singular class. On the other hand, since β is differentiable, the Legendre transform is a homeomorphism from $H_1(\mathbb{T}^2; \mathbb{R})$ to $H^1(\mathbb{T}^2; \mathbb{R})$.

Using the semicontinuity of the Aubry set in dimension 2 (see [5]), we can deduce that for each $c \in H^1(\mathbb{T}^2; \mathbb{R})$ the Aubry set \mathcal{A}_c^* projects over the whole manifold and therefore it is still an invariant Lipschitz Lagrangian graph, which we will denote Λ_c . Observe that all these Λ_c 's are disjoint. This is a straightforward consequence of the differentiability of β . In fact if for some $c \neq c'$ we had that $\Lambda_c \cap \Lambda_{c'} \neq \emptyset$, then $\mathcal{M}_c^* \cap \mathcal{M}_{c'}^* \neq \emptyset$; but this would contradict the differentiability of β at some homology class. It only remains to prove that the union is the whole $\mathbb{T}^*\mathbb{T}^2$. Observe that for each $c \in H^1(\mathbb{T}^2; \mathbb{R})$, Λ_c is the graph of the unique solution of the Hamilton–Jacobi equation $H(x, \eta_c + du_c) = \alpha(c)$, where $u_c \in C^{1,1}(\mathbb{T}^2)$ and η_c is a closed 1-form on \mathbb{T}^2 with cohomology class c . For each $x_0 \in \mathbb{T}^2$, let us define

$$\begin{aligned} F_{x_0} : H^1(\mathbb{T}^2; \mathbb{R}) &\simeq \mathbb{R}^2 &\longrightarrow & \mathbb{T}_{x_0}^*\mathbb{T}^2 \simeq \mathbb{R}^2 \\ c &\longmapsto & \eta_c(x_0) + d_{x_0}u_c. \end{aligned} \tag{8}$$

This map is injective (as it follows from the disjointness of the Λ_c 's). Moreover, F_{x_0} is also continuous. In fact, let $c_n \rightarrow c$ and consider the sequence of Lipschitz functions $\lambda_{c_n} := \eta_{c_n} + du_{c_n}$. This sequence is equibounded and equilipschitz (it follows from the results in [12] or from Mather's graph theorem for the Aubry sets [19]). Therefore, applying Ascoli-Arzelà's theorem, we can conclude that—up to selecting a subsequence— λ_{c_n} converge uniformly to some $\tilde{\lambda} = \eta_c + du$, for some $u \in C^{1,1}(\mathbb{T}^2)$. Observe that since $H(x, \lambda_{c_n}(x)) = \alpha(c_n)$ for all $x \in \mathbb{T}^2$ and all n , and α is continuous, then $H(x, \tilde{\lambda}(x)) = \alpha(c)$ for all x . Therefore, u is a solution of Hamilton–Jacobi equation $H(x, \eta_c + du) = \alpha(c)$. But, as we noticed above, for any given $c \in H^1(\mathbb{T}^2; \mathbb{R})$ there exists a unique solution of this equation (essentially because $\mathcal{A}_c = \mathbb{T}^2$). Therefore, $\tilde{\lambda} = \lambda_c$ and hence $F_{x_0}(c_n) \rightarrow F_{x_0}(c)$.

Using that the above map is continuous, we can conclude that $F_{x_0}(\mathbb{R}^2)$ is open (see for instance [6]). In a similar way, one can show that this image is also closed. In fact, let $y_k = F_{x_0}(c_k)$ be a sequence in $\mathbb{T}_{x_0}^*\mathbb{T}^2$ converging to some y_0 . Since each family of classical solutions of Hamilton–Jacobi on which the α -function (i.e. the energy) is bounded gives rise to a family of equi- $C^{1,1}$ functions (see [12]), then there exist $\tilde{u} \in C^{1,1}(\mathbb{T}^2)$ and $\tilde{c} \in H^1(\mathbb{T}^2; \mathbb{R})$, such that $c_k + du_{c_k} \rightarrow \tilde{c} + d\tilde{u}$ uniformly. From the continuity of the α -function, it follows that $H(x, \tilde{c} + d_x\tilde{u}) = \alpha(\tilde{c})$. As before, since $\Lambda_{\tilde{c}}$ is the unique invariant Lagrangian graph with cohomology \tilde{c} , then $\Lambda_{\tilde{c}} = \{(x, \tilde{c} + d_x\tilde{u} : x \in \mathbb{T}^2\}$ and therefore $y_0 = \tilde{c} + d_{x_0}\tilde{u} = F_{x_0}(\tilde{c}) \in F_{x_0}(\mathbb{R}^2)$. This shows that $F_{x_0}(\mathbb{R}^2)$ is also closed and therefore it is all of \mathbb{R}^2 . Since this holds for all $x_0 \in \mathbb{T}^2$, we can conclude that $\bigcup_c \Lambda_c = \mathbb{T}^*\mathbb{T}^2$, that is, the system is C^0 -integrable. \square

We can deduce more information about the dynamics on a C^0 -integrable system (see also [1]).

Corollary 3. *Let $H : \mathbb{T}^*M \rightarrow \mathbb{R}$ be a C^0 -integrable Hamiltonian on a two-dimensional closed manifold M . Then, M is diffeomorphic to \mathbb{T}^2 . Moreover:*

- (i) *for each 1-irrational homology class h , there exists an invariant Lagrangian graph foliated by periodic orbits with homology h and the same minimal period;*

- (ii) for each completely irrational homology class, there exists an invariant Lagrangian graph on which the motion is conjugated to an irrational rotation on the torus or to a Denjoy type homeomorphism;
- (iii) there exists a dense G_δ set of (co)homology classes, for which the motion on the corresponding invariant torus is conjugated to a rotation;
- (iv) as for the 0-homology class, there exists a C^1 invariant torus $\Lambda_{c(0)} = \{(x, \frac{\partial L}{\partial v}(x, 0) : x \in \mathbb{T}^2\}$ consisting of fixed points.

Proof. If H is C^0 integrable, then β is C^1 (lemma 5). By proposition 4, M is diffeomorphic to the 2-torus and it follows from theorem 3 that for each $c \in H^1(\mathbb{T}^2; \mathbb{R})$ the Aubry set \mathcal{A}_c^* is a Lipschitz Lagrangian graph and all these sets provide a partition of the cotangent space \mathbb{T}^*M .

(i) It is a consequence of proposition 3.

(ii) Let h be a completely irrational homology class and let $c_h = \partial\beta(h)$. We will follow the discussions in [8, p 1084] (see also [20, p 66 et seqq.] for more technical details). The Hamiltonian flow on Λ_{c_h} induces a flow on \mathbb{T}^2 without fixed points (otherwise the homology class would be singular and this would contradict proposition 3) or closed trajectories (otherwise the homology class would be 1-irrational). Thus, one can construct a non-contractible closed curve Γ , which is transverse to the flow. Using an argument *à la* Poincaré-Bendixon, it follows that all trajectories starting on Γ must return to it. Hence, one can define a continuous map from a compact subset of Γ to itself, which is order preserving and with irrational rotation number. Therefore, it is either conjugate to an irrational rotation or a Denjoy type homeomorphism.

(iii) We know from (i) that for all 1-irrational homology classes, the motion on the corresponding graph is conjugated to a rotation. Observe now that because of (ii), it is sufficient to show that the set of homology classes \mathcal{H}_r for which the Mather set projects over the whole torus is a dense G_δ set. In fact, if this is the case, then the flow cannot be conjugated to a Denjoy type homeomorphism and must be therefore conjugated to a rotation. Let us start by observing that this set \mathcal{H}_r is clearly dense since it contains all 1-irrational homology classes. The fact that it is also G_δ follows from [11, corollaire 4.5], stating that the set of strictly ergodic flows on a compact set is G_δ in the C^0 topology. In fact, let us consider $\mathcal{F}(M)$ the set of flows on M , endowed with the C^0 topology. One can consider the continuous and injective map given by

$$\begin{aligned} \psi : H_1(M; \mathbb{R}) &\longrightarrow \mathcal{F}(M) \\ h &\longmapsto \pi_M \circ \Phi^H|_{\Lambda_{c_h}}, \end{aligned}$$

i.e. we associate with each $h \in H_1(M; \mathbb{R})$, the projection of the Hamiltonian flow restricted to the invariant Lagrangian graph Λ_{c_h} . Observe that injectivity easily follows from the fact that they have different rotation vectors, while continuity is a consequence of the continuity of the Hamiltonian flow. Now, let us define $\mathcal{H}_r = \psi^{-1}(S(M) \cap \psi(H_1(M; \mathbb{R})))$, where $S(M)$ denotes the set of strictly ergodic flows on M . Using that $S(M)$ is a G_δ in $\mathcal{F}(M)$ [11, corollaire 4.5] and that ψ is a homeomorphism on its image, we can then conclude that \mathcal{H}_r is also a G_δ set in $H_1(M; \mathbb{R})$.

(iv) Since the union of the Aubry sets foliates all the phase space, then any point $(x, 0)$ will be contained in some Aubry set and therefore, using [13, proposition 3.2], it follows that it is a fixed point of the Euler–Lagrange flow. Hence, the Dirac-measure $\delta_{(x,0)}$ is invariant and action minimizing (since its support is contained in some Aubry set); clearly, such a measure has rotation vector equal to 0. Therefore, $\widetilde{\mathcal{M}}^0 \supseteq \mathbb{T}^2 \times \{0\}$ and from the graph property it follows that they coincide. Then, $\Lambda_{c(0)} := \mathcal{A}_{c(0)}^* = \{(x, \frac{\partial L}{\partial v}(x, 0) : x \in \mathbb{T}^2\}$. □

Remark 3. Although we believe that the motions on all invariant Lagrangian tori must be conjugated to rotations, we have not been able to show more than (iii). It remains an open question whether it is possible or not that a Denjoy type homeomorphism is embedded into a C^0 -integrable Hamiltonian system.

4.1. The case of mechanical Lagrangians

Recall that a mechanical Lagrangian is $L(x, v) = 1/2 g_x(v, v) + f(x)$, where g is a Riemannian metric and f is a C^2 function on \mathbb{T}^2 . In this case we can bridge the gap between C^0 and C^1 integrability, using Burago and Ivanov's theorem on metrics without conjugate points [7].

Proposition 5. *Let L be a C^0 -integrable mechanical Lagrangian on an n -dimensional torus. Then the potential f is identically constant and the metric g is flat. In particular, L is C^1 -integrable.*

Proposition 6. *Let L be a mechanical Lagrangian on an 2-dimensional torus, whose β -function is C^1 . Then the potential f is identically constant and the metric g is flat. In particular, L is C^1 -integrable.*

First we need the following lemma (see also [23, lemma 1]), which will be needed for the proof of proposition 5. Observe that one could also prove proposition 6 by using corollary 3 (iv), which is however valid only for $n = 2$.

Lemma 6. *Let*

- M be a closed manifold of any dimension
- L be an autonomous Tonelli Lagrangian on M , such that $L(x, v) = L(x, -v)$ for all (x, v) in TM .

Then the Aubry set $\tilde{\mathcal{A}}_0$ consists of fixed points of the Euler–Lagrange flow.

Proof. Take $(x, v) \in \tilde{\mathcal{A}}_0$. Then by [12] there exists a sequence of C^1 curves $\gamma_n : [0, T_n] \rightarrow M$, such that

- $\gamma_n(0) = \gamma_n(T_n) = x$ for all $n \in \mathbb{N}$
- $T_n \rightarrow +\infty$ as $n \rightarrow +\infty$
- $\dot{\gamma}_n(0) \rightarrow v$ and $\dot{\gamma}_n(T_n) \rightarrow v$ as $n \rightarrow +\infty$
- $\int_0^{T_n} (L(\gamma_n, \dot{\gamma}_n) + \alpha(0)) dt \rightarrow 0$ as $n \rightarrow +\infty$.

Consider the sequence of curves $\delta_n : [0, T_n] \rightarrow M$, such that $\delta_n(t) := \gamma_n(T_n - t)$ for all $n \in \mathbb{N}, t \in [0, T_n]$. Then since L is symmetrical, $\int_0^{T_n} (L(\delta_n, \dot{\delta}_n) + \alpha(0)) dt \rightarrow 0$ as $n \rightarrow +\infty$, which proves that $(x, -v) \in \tilde{\mathcal{A}}_0$. Therefore, by Mather's graph theorem, $v = 0$, so (x, v) lies in the zero section of TM . Now, by [13, proposition 3.2], for any Lagrangian, the intersection of $\tilde{\mathcal{A}}_0$ with the zero section of TM consists of fixed points of the Euler–Lagrange flow. This proves the lemma. \square

Proof (Propositions 5 and 6). Now let us assume that the Lagrangian is mechanical. Then the only fixed points of the Euler–Lagrange flow are the critical points of the potential f , and the only minimizing fixed points are the minima of f . So if f is not constant, the Lagrangian cannot be C^0 -integrable. Furthermore, since the Lagrangian is C^0 -integrable, every orbit is minimizing, in particular, there are no conjugate points. So by Burago and Ivanov's proof of the Hopf conjecture [7], the metric g is flat. This proves proposition 5. Proposition 6 is now just a consequence of theorem 3.

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