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# On the integrability of Birkhoff billiards

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In this survey, we provide a concise introduction to convex billiards and describe some recent results, obtained by the authors and collaborators, on the classification of integrable billiards, namely the so-called Birkhoff conjecture.

This article is part of the theme issue 'Finite dimensional integrable systems: new trends and methods'.

## 1. Introduction

In this survey, we provide a concise introduction to convex billiards and describe some recent results, obtained by the authors and collaborators, on the classification of integrable billiards, namely the so-called *Birkhoff conjecture*.

These conceptually simple models of dynamical systems—yet dynamically very rich and interesting—were first introduced by Birkhoff [1] as paradigmatic examples of Hamiltonian systems, that could be used as a 'playground' to shed light, with as little technicality as possible, on some interesting dynamical features and phenomena appearing in the study of their dynamics.[1]

Since then billiards have captured much attention in many different contexts, becoming a very popular subject

[1]'[...]This example is very illuminating for the following reason: Any dynamical system with two degrees of freedom is isomorphic with the motion of a particle on a smooth surface rotating uniformly about a fixed axis and carrying a conservative field of force with it.[3] In particular if the surface is not rotating and if the field of force is lacking, the paths of the particles will be geodesics. If the surface is conceived of as convex to begin with and then gradually to be flattened to the form of a plane convex curve *C*, the "billiard ball" problems results. But in this problem the formal side, usually so formidable in dynamics, almost completely disappears, and only the interesting qualitative questions need to be considered.[...]' (G. D. Birkhoff, [1, pp. 155–156]).

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of investigation. Not only are their laws of motion very physical and intuitive, but also billiard-type dynamics are ubiquitous. Mathematically, they offer models in every subclass of dynamical system (integrable, regular, chaotic, etc.); more importantly, techniques initially devised for billiards have often been applied and adapted to other systems, becoming standard tools and having ripple effects beyond the field.

More remarkably, the dynamics of these systems is profoundly intertwined with their geometric properties (e.g. the *shape* of the billiard table): while it is evident how the shape completely determines the billiard dynamics, a more subtle and intriguing question is to what extent the knowledge of the dynamics allows one to reconstruct the shape of the billiard domain. This translates into many intriguing unanswered questions and conjectures that have been the focus of very active research over recent decades. Hereafter, we shall address some of them and describe recent advances towards their solutions.

## 2. The billiard map

Let us first recall the definition of the billiard map and its main properties. We refer to [2–4] for a more comprehensive introduction to the study of billiards.

Let $\Omega$ be a strictly convex domain in $\mathbb{R}^2$ with $C^r$ boundary $\partial\Omega$, with $r \geq 3$. The phase space $M$ of the billiard map consists of unit vectors $(x, v)$ whose foot points $x$ are on $\partial\Omega$ and which have inward directions. The billiard ball map $B_\Omega : M \longrightarrow M$ takes $(x, v)$ to $(x', v')$, where $x'$ represents the point where the trajectory starting at $x$ with velocity $v$ hits the boundary $\partial\Omega$ again, and $v'$ is the *reflected velocity*, according to the standard reflection law: angle of incidence is equal to the angle of reflection (figure 1).

**Remark 2.1.**

(i) The dynamical properties of billiards are strongly related to the geometric properties of its shape. Besides the study of Birkhoff billiards, very active areas of research focus on the study of polygonal billiards (in particular, *rational billiards*, whose dynamics can be related to geodesic flows on *translation surfaces* and *Teichmüller theory* (e.g. [5]) or billiards with concave boundary (so-called *dispersive billiards*) of particular interest as models in statistical mechanics and mathematical physics [6].

(ii) More generally, one could consider a Riemannian metric with smooth boundary $(M, \partial M, g)$: the trajectory starting at $x \in \partial M$ with (inward) unit velocity $v$ will follow the corresponding geodesic until it hits the boundary at $x' \in \partial M$; the *reflected* (unit) inward velocity $v'$ is obtained in the following way: the normal component of the hitting velocity instantaneously changes sign, while the tangential one stays unchanged. Observe that in the Birkhoff billiard case (Euclidean planar case), this gives exactly the standard reflection law that we have described above.

**Remark 2.2.**

(i) Observe that if $\Omega$ is not convex, then the billiard map is not continuous; moreover, in this article we shall be interested only in strictly convex domains, namely the curvature at each point is strictly positive (see remark 3.5).

(ii) As pointed out by Halpern [7], if the boundary is not at least $C^3$ (actually, $C^2$ plus a bounded third derivative is enough), then 'strange' phenomena can occur; for example, there might be infinite orbits of finite total length (since we are considering unit velocities, then this could be interpreted as a sort of incompleteness of the billiard flow, namely the velocity becomes tangent to the boundary in finite time).

Let us introduce coordinates on $M$. We suppose that $\partial\Omega$ is parametrized by arc-length $s$ and let $\gamma : \mathbb{R}/\ell\mathbb{Z} \longrightarrow \mathbb{R}^2$ denote such a parametrization, where $\ell$ denotes the length of $\partial\Omega$. Without any

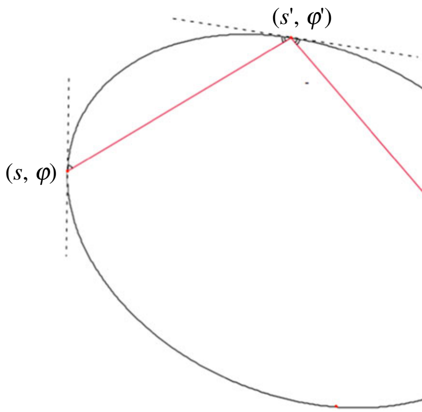


**Figure 1.** Billiard map. (Online version in colour.)

loss of generality, fix an orientation of $\gamma$. Let $\varphi$ be the angle between $v$ and the positive tangent to $\partial\Omega$ at $x$. Hence, $M$ can be identified with the annulus $\mathbb{A}_\ell = \mathbb{R}/\ell\mathbb{Z} \times (0, \pi)$ and the billiard map $B_\Omega$ can be described as

$$\begin{aligned} B_\Omega : \mathbb{A}_\ell &\longrightarrow \mathbb{A}_\ell \\ (s, \varphi) &\longmapsto (s', \varphi'). \end{aligned}$$

Here are some properties of the billiard map:

— $B_\Omega \in C^{r-1}(\mathbb{A}_\ell)$ (e.g. [8, Theorem 4.1]). Moreover, $B_\Omega$ can be continuously extended to $\overline{\mathbb{A}}_\ell = \mathbb{R}/\ell\mathbb{Z} \times [0, \pi]$ by fixing $B_\Omega(s, 0) = B_\Omega(s, \pi) = \mathrm{Id}$ for all $s \in \mathbb{R}/\ell\mathbb{Z}$.

— $B_\Omega$ is a symplectic map, namely it preserves the exact symplectic form $\omega = \sin\varphi\, \mathrm{d}\varphi \wedge \mathrm{d}s = -\mathrm{d}(\cos\varphi\, \mathrm{d}s) =: -\mathrm{d}\alpha$ (observe that this form becomes degenerate on $\partial\mathbb{A}_\ell$), namely $B^*\omega = \omega$ (where $B^*$ denotes the pull-back) (e.g. [4, Theorem 3.1]).
Moreover, $B_\Omega$ is an *exact symplectic map*, namely $B^*\alpha - \alpha = \mathrm{d}h$ is an exact 1-form; the corresponding generating function is given by

$$h(s, s') := -\|\gamma(s) - \gamma(s')\|,$$

namely minus the Euclidean distance between two points on $\partial\Omega$. It is easy to check that

$$\begin{cases} \partial_1 h(s, s') = \cos\varphi \quad \text{and} \\ \partial_2 h(s, s') = -\cos\varphi', \end{cases} \tag{2.1}$$

where $\partial_i$ denotes the derivative with respect to the $i$th variable ($i = 1, 2$).

— If we lift $B_\Omega$ to the universal cover of $\mathbb{A}_\ell$ and introduce new coordinates $(x, y) = (s, -\cos\varphi) \in \mathbb{R} \times (-1, 1)$, then the billiard map becomes a monotone twist map with $h$ as generating function and it preserves the area form $\mathrm{d}x \wedge \mathrm{d}y$ [2–4].

**Remark 2.3.** It follows from (2.1), $\{(s_i, \varphi_i)\}_{i\in\mathbb{Z}}$ is an orbit of $B_\Omega$ if and only if $\{s_i\}_{i\in\mathbb{Z}}$ is a 'critical configuration' for the *action functional*

$$\{s_i\}_{i\in\mathbb{Z}} \longmapsto \sum_{i\in\mathbb{Z}} h(s_i, s_{i+1}),$$

in the usual sense of statistical mechanics; in fact, while this latter sum is infinite, its derivatives are well defined:

$$\frac{\partial}{\partial s_n}\left(\sum_{i\in\mathbb{Z}} h(s_i, s_{i+1})\right) = \partial_1 h(s_n, s_{n+1}) + \partial_2 h(s_{n-1}, s_n).$$

These critical configurations are not necessarily (global) minima. One could be wondering whether (global) minima exist and if they have special dynamical features. *Aubry-Mather theory*—developed independently by Serge Aubry and John Mather at the beginning of the 1980s—focuses exactly on these questions. More specifically, it is concerned with the study of orbits that are *global minimizers* of the action-functional, i.e. every finite segment minimizes the action functional among all configurations with the same number of elements and the same end-points (see [2,9,10] for more details).

Observe that in the billiard case, since the generating function (and hence the action) is given by minus the Euclidean length, then *action minimization* can be rephrased in terms of *length maximization*.

Despite the apparently simple (local) dynamics, the qualitative dynamical properties of billiard maps are extremely non-local. This global influence on the dynamics translates into several intriguing *rigidity phenomena*, which are the basis of several unanswered questions and conjectures (e.g. [2–4,11–21]). Among many, in the following sections we shall address the problem of classifying *integrable billiards*, also known as *Birkhoff conjecture*.

## 3. Caustics

In this section, we would like to introduce the concept of *caustic* of a billiard. Let us first start to introduce the concepts of caustic and integrability by means of two examples.

### (a) Circular billiard

The easiest example of billiard is given by a billiard in a disc $\mathcal{D}$ (for example of radius $R$). It is easy to check in this case that the angle of reflection remains constant at each reflection (see also [4, ch. 2]). If we denote by $s$ the arc-length parameter (i.e. $s \in \mathbb{R}/2\pi R\mathbb{Z}$) and by $\varphi \in (0, \pi/2]$ the angle of reflection, then the billiard map has a very simple form (figure 2):

$$f(s, \theta) = (s + 2R\varphi, \varphi).$$

In particular, $\varphi$ stays constant along the orbit and it represents an *integral of motion* for the map; hence, the property of the orbits are determined by the corresponding angle $\varphi = \pi\omega$, with $\omega \in (0, 1)$. First, observe that the orbits corresponding to $\omega' = 1 - \omega$ are geometrically the same, but with reversed orientation. Hence, one could limit him/herself to $\omega \in (0, \frac{1}{2}]$. Then:

- If $\omega = p/q \in (0, \frac{1}{2}] \cap \mathbb{Q}$, in lowest terms, then the orbit is periodic with minimal period $q$. In particular, it closes after $q$ rebounces and it winds $p$ around the disc before closing.
- If $\omega \in (0, \frac{1}{2}] \setminus \mathbb{Q}$, then the orbit is not periodic and it hits the boundary $\partial\mathcal{D}$ on a dense set of points.

Moreover, this billiard enjoys the peculiar property that all orbits with $\varphi = \pi\omega$ are tangent to the same concentric circle of radius $R\cos\pi\omega$ (figure 2); this concentric circle is an example of *caustics* (see definition 3.2), and it is related to the existence of a homotopically non-trivial invariant curve for the corresponding billiard map, namely the $\mathcal{C}_\omega = \mathbb{R}/2\pi R\mathbb{Z} \times \{\pi\omega\}$ (this relationship between caustics and invariant curves is more subtle, see remark 3.3; figure 3).

**Remark 3.1.** Observe that the whole phase space of the circular billiard map—which is topologically a cylinder—is completely foliated by homotopically non-trivial invariant curves $\mathcal{C}_\omega = \mathbb{R}/2\pi R\mathbb{Z} \times \{\pi\omega\}$. Looking at the billiard table, this corresponds to saying that the billiard table is completely foliated by caustics (the centre of the disc corresponds to a degenerate caustic for orbits with $\varphi = \pi/2$, i.e. diameters). In this regard, circular billiards are an example of *integrable billiards*.

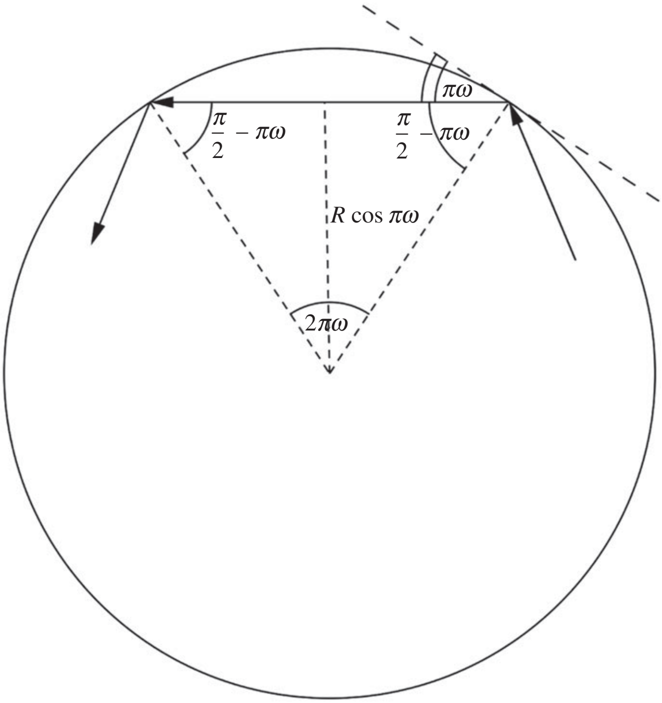


**Figure 2.** Billiard in a disc.

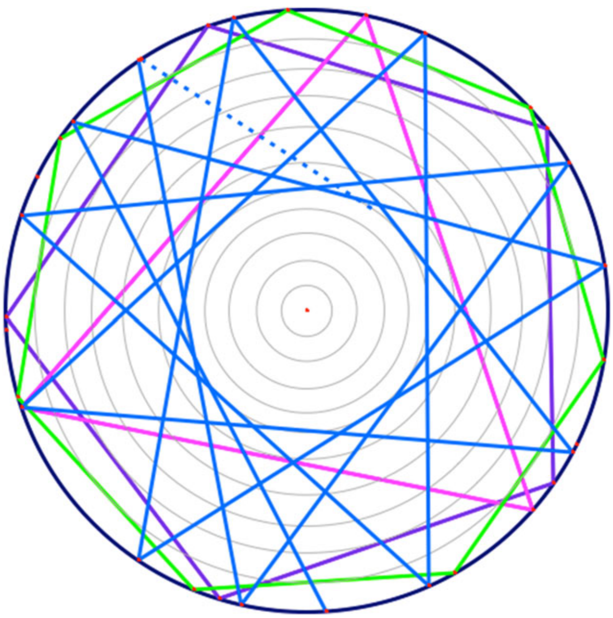

**Figure 3.** Foliation by caustics.

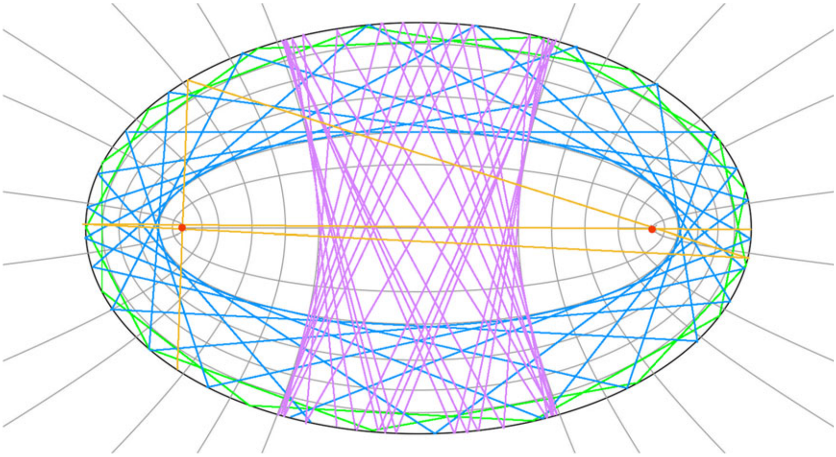

**Figure 4.** Billiard in an ellipse.

## (b) Elliptic billiard

As a second example, let us look at the billiard inside an ellipse

$$\mathcal{E} = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\},$$

with $0 < b \leq a$. Up to rescaling, we can assume that $a = 1$ (clearly, the billiard dynamics remains unchanged under the action of homotheties) and therefore the eccentricity of the ellipse is given by $0 \leq h = \sqrt{1 - b^2} < 1$ and the two foci by $F_{\pm} = (\pm h, 0)$.

Optical properties of conics (an alternative way to consider the billiard ball motion inside a conic) were already well known to ancient Greeks. We refer to [4] for a more detailed discussion (see also [2]). In particular, each trajectory which does not pass through a focal point is always tangent to precisely one confocal conic section. More specifically, billiard trajectories can be classified in the following way:

(i) trajectories that always intersect the open segment between the two foci,
(ii) trajectories that never intersect the closed segment between the two foci, and
(iii) trajectories that alternatively pass through one of the two foci.

In particular, each trajectory in (i) is tangent to a confocal hyperbola, each trajectory in (ii) is tangent to a confocal ellipse, while trajectories of kind (iii) tend asymptotically to the major semiaxis. Confocal ellipses are therefore examples of caustics (also hyperbolae can be considered caustics) which foliate everything but the closed segment between the two foci (figure 4). Hence, this could also be considered as an example of integrable billiards.

Analytic descriptions of the dynamics and the integral of motion are not as easy as in the circular case, yet they can be done by means of elliptic functions and elliptic integrals; we refer the reader to [22,23] for more details.

## (c) Caustics and their existence

Let us give a more precise definition of a *caustic*[2] and discuss some results and questions about their existence.

[2]*Caustic* comes from the greek word καυστικός (kaustikós), meaning 'burning'; this terminology is related to optics and refers to the envelope of reflected or refracted rays of light, namely concentration of lights that can potentially lead to burns.

**Definition 3.2.** We say that a curve $\Gamma$ is a *caustic* for the billiard in $\Omega$ if every time a trajectory is tangent to it, then it remains tangent after every reflection. Moreover, a $C^1$ *convex caustic* is a caustic consisting of a closed $C^1$ curve bounding a strictly convex region inside $\Omega$.

**Remark 3.3.**

(i) Observe that every convex caustic has a well-defined rotation number. In fact, the dynamics tangent to it induces a circle homeomorphism from the boundary to itself; the rotation number of the caustic corresponds to the Poincaré rotation number of this circle homeomorphism.
(ii) One could wonder about the relation between caustics for the billiard in $\Omega$ and invariant curves for the corresponding billiard map $B_\Omega$. One can show that a $C^1$ convex caustic in $\Omega$ corresponds to a homotopically non-trivial invariant curve for the billiard map; however, the converse is not entirely true. In fact, homotopically non-trivial invariant curves of $B_\Omega$ (which are graphs by Birkhoff's theorem) do give rise to caustics, but these caustics need neither be convex nor differentiable.

**Remark 3.4.** The notion of caustics is often connected to the so-called *whispering gallery*, a phenomenon that can be detected under some particular domes, in which whispers can be clearly transmitted and received from distant parts of the gallery.

A natural question that one could ask is whether the existence of caustics is a common or a rare phenomenon. As we have seen before, circular and elliptic billiards possess many caustics.

**Question.** *Are there other Birkhoff billiards with caustics?* And in the case of an affirmative answer: *How many caustics is it reasonable to expect?*

Constructing a Birkhoff billiard with at least one caustic is easy: it is enough to perform the so-called *string construction*, similar to the well-known one to draw a circle as the set of points equidistant from a fixed centre, or to construct an ellipse as the locus of points whose distances from two fixed points have a constant sum. More specifically (see, e.g. [4, ch. 5] for more details), given a curve $\gamma$, one could wrap a closed non-stretchable string around it (of length longer than the one of $\gamma$), pull it tight at a point and move this point around $\gamma$: the curve that one obtains corresponds to a billiard domain that has $\gamma$ as a caustic.

Are there other billiards with infinitely many caustics? Quite surprisingly, the answer is affirmative: all (sufficiently smooth) Birkhoff billiards have infinitely many smooth convex caustics that accumulate to the boundary of the billiard domain. In fact, Lazutkin [24] introduced a very special change of coordinates that reduces the billiard map $B_\Omega$ to a very simple form. Let $L_\Omega : \mathbb{R}/\ell\mathbb{Z} \times [0,\pi] \to \mathbb{R}/\mathbb{Z} \times [0,\delta]$ with small $\delta > 0$ be given by

$$(x,y) = L_\Omega(s,\varphi) := \left( C_\Omega^{-1} \int_0^s \rho^{-2/3}(\tau)\,\mathrm{d}\tau, 4C_\Omega^{-1}\rho^{1/3}(s)\sin\frac{\varphi}{2} \right),$$

where $\rho$ denotes the radius of curvature of $\partial\Omega$, and $C_\Omega := \int_0^\ell \rho^{-2/3}(s)\,\mathrm{d}s$ (sometimes called the *Lazutkin perimeter*). In these new coordinates, the billiard map becomes a more simple expression:

$$B_\Omega^L(x,y) = (x + y + O(y^3), y + O(y^4)).$$

In particular, near the boundary $\{y = 0\}$, this map can be seen as a small perturbation of the integrable map $(x,y) \longmapsto (x + y, y)$, and hence, under suitable regularity assumptions, KAM theorem can be applied (it is sufficient, for example, that $\partial\Omega$ is $C^6$, so that the map is at least $C^5$). Hence, there exists a positive measure Cantor set of smooth homotopically non-trivial invariant curves for the map which accumulates on $\{y = 0\}$ and on which the motion is smoothly conjugate to a rigid rotation with Diophantine rotation number (see [17,24] for a refined version); this translates into the existence of a positive measure set of caustics, accumulating to the boundary of the billiard table.

**Remark 3.5.** Observe that it is extremely important that $\Omega$ is strictly convex. In fact, Mather [25] proved the non-existence of caustics if the curvature of the boundary vanishes at one point. An alternative proof of this result has been provided by Gutkin & Katok [26], where the authors also investigate how the shape of the domain determines the location of caustics, establishing the existence of open regions which are free of caustics and estimating (from below) the size of these regions.

The next step consists then in asking in which cases these caustics foliate the whole billiard table or an open dense subset of it, as it happens in the circular and elliptic cases. In other words: *Are there other examples of integrable billiards?*

This apparently naive question turns out to be much more difficult to extricate, and it has given rise to one of the most famous (and somehow impenetrable) open problems in dynamical systems: the so-called *Birkhoff conjecture*.

## 4. Integrable billiards and the Birkhoff conjecture

As we have seen in the previous section, billiards in a disc or in an ellipse are examples of *integrable billiards*. There are different ways to define integrability (the relation between these notions is an interesting problem itself):

— Through the existence of an integral of motion, globally or locally in the phase space; in the case of circular billiards, for example, an integral of motion is given by $I(s,\varphi) := \varphi$.
— Through the existence of a (smooth or $C^0$) foliation of the phase space (globally or locally), consisting of invariant curves of the billiard map; for example, in the case of circular billiards, this foliation is smooth and it consists of invariant curves $\{\varphi \equiv \varphi_0\}$, for any $\varphi_0 \in (0,\pi)$. As we have remarked above, under suitable conditions, this property translates into the existence of a global/local foliation of the billiard table, consisting of (smooth) convex caustics (in the circular case, these are all concentric circles, whereas in the elliptic one, these are all confocal ellipses).

A natural question follows: Which Birkhoff billiards are integrable?

**Conjecture (Birkhoff).** Circular and elliptic billiards are the only examples of integrable Birkhoff billiards.

**Remark 4.1.** Although some vague indications of this question can be found in [1], to the best of our knowledge, its first appearance as a conjecture was in a paper by Poritsky [18].[3] Thereafter, references to this conjecture (either as *Birkhoff conjecture* or *Birkhoff–Poritsky conjecture*) repeatedly appeared in the literature (e.g. Gutkin [14, Section 1], Moser [27, Appendix A] and Tabachnikov [3, Section 2.4]).

This conjecture assumes very different connotations and levels of complexity, according to the notion of integrability that one takes into account. Despite its long history and the amount of attention that it has captured over recent decades, many interesting formulations of this conjecture still remain unanswered.

### (a) Global integrability

Bialy [12] proved the following result under the assumption of full global integrability.

**Theorem (Bialy).** *If the phase space of the billiard ball map is fully foliated by continuous invariant curves which are not null-homotopic, then it is a circular billiard.*

[3] In [18, Footnote 1] Poritsky acknowledged that the results in the paper were obtained in 1927–29 while he was National Research Fellow in Mathematics at Harvard University, presumably under the supervision of Birkhoff. Although the author does not attribute this conjecture explicitly to Birkhoff, he cites many of his papers on the topic; hence it is reasonable to surmise Birkhoff's influence behind it.

**Remark 4.2.** An integral-geometric approach to prove Bialy's result was proposed by Wojtkowski [28], by means of the so-called mirror formula. This approach was later exploited by Bialy [29] for billiards on the sphere and the hyperbolic plane, as well as for magnetic billiards.

Observe that Bialy and Wojtkowski's result is not in contrast with what we have discussed in the case of elliptic billiards. In fact, in that case the family of convex caustics represented by confocal ellipses do not foliate the whole domain (the segment between the two foci is left out), and neither set of homotopically non-trivial invariant curves has full $\omega$-measure in the phase space: the homotopically trivial invariant curves corresponding to orbits tangent to confocal hyperbolae, foliate a positive $\omega$-measure set (in the phase portrait (figure 4) this set corresponds to the area below the separatrix, i.e. the stable/unstable manifold of the hyperbolic 2-periodic orbit corresponding to the major semi-axis of the ellipse).

What about other notions of integrability? In the study of integrable systems, in fact, in most of the cases integrals of motion are non-degenerate not everywhere, but either on an open-dense subset of the phase space (we shall refer to this as *global integrability*) or just a proper (non-trivial) open subset (we shall refer to this as *local integrability*).

**Remark 4.3.** An interesting result[4] by Innami [30] shows that the existence of caustics with rotation numbers accumulating to $\frac{1}{2}$ implies that the billiard must be an ellipse. This regime of integrability is somehow opposite to the one we are interested in, which is concerned with caustics near the boundary of the billiard table, i.e. with small rotation numbers. Innami's proof is based on Aubry–Mather theory; a simpler and more geometric proof of Innami's result has been recently given in [31].

**Remark 4.4.** Very interestingly, Treschev [21] gives numerical indication that there might exist analytic billiards, different from ellipses, for which the dynamics in a neighbourhood of the elliptic period-2 orbit is conjugate to a rigid rotation. These billiards could be seen as an instance of *local integrability*; however, this regime is somehow complementary to the one usually considered for Birkhoff conjecture. Here one has local integrability in a neighbourhood of an elliptic periodic orbit of period 2 (leading to homotopically trivial invariant curves for the billiard map), while Birkhoff conjecture is usually concerned with integrability in a neighbourhood of the boundary of the billiard table. However, this fact—if verified—would provide an extremely interesting indication that these two regimes of integrability do differ.

**Remark 4.5.** Birkhoff conjecture can also be thought of as an analogue, in the case of billiards, in the following task: classifying integrable (Riemannian) geodesic flows on $\mathbb{T}^2$. The complexity of this question, of course, depends on the notion of integrability that one considers. If one assumes that the whole space is foliated by invariant Lagrangian graphs (i.e. the system is $C^0$-*integrable*), then it follows from Hopf conjecture [32] (see also [33] for the proof in dimensions greater than 2) that the associated metric must be flat. Bialy and Wojtkowski's results in the billiard setting can be considered as the analogues of this result.

However, the question becomes more challenging—and it is still open—if one considers integrability only on an open and dense set (global integrability), or assumes the existence of an open set foliated by invariant Lagrangian graphs (local integrability). Examples of globally integrable (non-flat) geodesic flows on $\mathbb{T}^2$ are those associated to *Liouville-type metrics*, namely metrics of the form

$$ds^2 = (f_1(x_1) + f_2(x_2))(dx_1^2 + dx_2^2).$$

A folklore conjecture states that these metrics are the only globally (resp. locally) integrable metrics on $\mathbb{T}^2$, which, in some sense, can be interpreted as the analogue of Birkhoff conjecture, in the realm of integrable geodesic flows on $\mathbb{T}^2$.

A partial answer to this conjecture (global case) is provided in [34], where the authors prove it under the assumption that the system admits an integral of motion which is quadratic in the momenta. Observe that while the case of quadratic integral of motion reduces to a

[4]We are grateful to M. Bialy for pointing out this reference.

system of linear PDEs, the case of higher degree integrals of motions is very challenging and it turns out to be equivalent to delicate questions on nonlinear PDEs of hydrodynamic type (e.g. [35,36]).

This notion of integrability is related to the so-called *algebraic integrability*, namely the existence of integrals of motion that are polynomial in the velocity. The relationship between this notion of integrability and Birkhoff conjecture (*algebraic Birkhoff conjecture*) has been studied and has led to interesting results [37,38]. Recently, using previous results of [37], Glutsyuk [39] proved the algebraic Birkhoff conjecture.

Finally, we point out that the topological structure of the torus plays a fundamental role in the afore-mentioned conjectures and results. For example, on the two-dimensional sphere there are plenty of non-trivial integrable metrics: the so-called *Zoll surfaces*. A Zoll surface is a surface homeomorphic to the 2-sphere, equipped with a Riemannian metric, all of whose geodesics are closed and of equal length (the first example was a non-trivial example discovered by Zoll in [40]). While the usual unit-sphere metric on $\mathbb{S}^2$ obviously has this property, there also exists an infinite-dimensional family of geometrically distinct deformations that are still Zoll surfaces. In particular, most Zoll surfaces do not have constant curvature. See [41] for more details.

### (i) Perturbative Birkhoff conjecture

Instead of considering all possible Birkhoff billiards, one could restrict the analysis to what happens for domains that are sufficiently close to ellipses and try to study the Birkhoff conjecture in this class of domain, which can be considered as *perturbations* of ellipses. More specifically, we can state the following perturbative version of the Birkhoff conjecture.

**Birkhoff conjecture (perturbative version).** *A smooth strictly convex domain that is sufficiently close (w.r.t. some topology) to an ellipse and whose corresponding billiard map is integrable, is necessarily an ellipse.*

A first result in this direction was obtained by Delshams & Ramírez-Ros [42], who studied entire perturbations of elliptic domains and proved that any nontrivial symmetric perturbation breaks integrability near homoclinic solutions.

More recently, Avila *et al.* proved in [11] that the claim of the perturbative version of the Birkhoff conjecture is true for domains that are sufficiently close to a circular billiard. The complete proof for domains sufficiently close to an ellipse of any eccentricity has been provided in [43]. See §5a for more details.

Let us describe this result more precisely, starting with the following definition.

**Definition 4.6.** Let $\Omega$ be a strictly convex domain.

(i) We say $\Gamma$ is an integrable rational caustic for the billiard map in $\Omega$, if the corresponding (non-contractible) invariant curve $\Gamma$ consists of periodic points; in particular, the corresponding rotation number is rational.

(ii) Let $q_0 \geq 2$ be a positive integer. If the billiard map inside $\Omega$ admits integrable rational caustics for all rotation numbers $0 < \frac{p}{q} < \frac{1}{q_0}$, we say that $\Omega$ is $q_0$-rationally integrable.

The main result proved in [43] is as follows.

**Theorem 4.7 (perturbative Birkhoff conjecture).** *Let $\mathcal{E}_0$ be an ellipse of eccentricity $0 \leq e_0 < 1$ and semi-focal distance c; let $k \geq 39$. For every $K > 0$, there exists $\varepsilon = \varepsilon(e_0, c, K)$ such that the following holds: if $\Omega$ is a 2-rationally integrable $C^k$-smooth domain, whose boundary $\partial\Omega$ is*

— *K-close to $\mathcal{E}_0$, with respect to the $C^k$-norm,*
— *$\varepsilon$-close to $\mathcal{E}_0$, with respect to the $C^1$-norm,*

*then $\Omega$ is an ellipse.*

**Remark 4.8.** Actually, it is sufficient to ask only the existence of rational integral caustics of rotation number $\frac{1}{q}$, for all $q \geq 3$.

Some ideas on the proof will be outlined in §5a.

### (ii) Non-perturbative results?

A possible strategy to extend our results to a non-perturbative version of this conjecture involves the use of some *geometric flow* to transform the domain into a small perturbation of an ellipse. Roughly speaking, the most important features of this flow should be:

(i) preservation of strict convexity and smoothness of the boundary;
(ii) convergence (possibly, up to some renormalization of the length or the area) to the set of elliptic domains, which must be an invariant set for the flow; and
(iii) preservation of integrability.

It is clear that if such a flow exists, then (i)–(iii) imply that any integrable Birkhoff billiard $\Omega$ can be mapped into an integrable Birkhoff billiard $\Omega'$ close to an ellipse; using our perturbative result, we can deduce that $\Omega'$ must be an ellipse; since the set of ellipses is invariant under the (backward) flow, then also $\Omega$ must be an ellipse.

In [43, Appendix G] we suggested as a possible candidate the so-called *affine length shortening (ALS) flow* (see, for instance [44] for more details). More specifically, a flow describing the evolution of plane curves in the direction of the *affine* normal, with speed proportional to the *affine* curvature. This flow satisfies property (i) and (ii) [44]; the main obstacle consists in proving that property (iii) holds (if one believes in Birkhoff conjecture, then it should hold, since ellipses are an invariant set for the flow). In [43, Appendix G], we proposed to prove this by introducing a family of functions, measuring the *non-integrability* of the domains, and conjecturing that they behave as Lyapunov functions for the ALS flow.

We remark that property (iii) for the classical *Euclidean curve shortening flow* (namely, the evolution is in the direction of the Euclidean normal with speed proportional to the Euclidean curvature) does not hold in general, as proved in [45].

## (b) Local integrability and the Birkhoff conjecture

What can be said for *locally integrable* Birkhoff billiards? As we have noted in remark 4.4, the correct regime that one should consider seems to be integrability in a neighbourhood of the boundary of the billiard table, i.e. for small rotation numbers.

Let us denote with $\mathcal{E}_{e,c} \subset \mathbb{R}^2$ an ellipse of eccentricity $e$ and semifocal distance $c$. We state the following local version of the Birkhoff conjecture.

**Local Birkhoff conjecture.** *For any integer $q_0 \geq 3$, there exist $e_0 = e_0(q_0) \in (0,1)$, $m_0 = m_0(q_0)$, $n_0 = n_0(q_0) \in \mathbb{N}$ such that the following holds. For each $0 < e \leq e_0$ and $c \geq 0$, there exists $\varepsilon = \varepsilon(e, c, q_0) > 0$ such that the following holds.*
*If $\mathcal{E}_{e,c}$ is an ellipse of eccentricity $e$ and semi-focal distance $c$, and $\Omega$ is a $q_0$-rationally integrable $C^{m_0}$-smooth domain, whose boundary $\partial\Omega$ is $\varepsilon$-close to $\mathcal{E}_0$, with respect to the $C^{n_0}$-norm, then $\Omega$ must be an ellipse.*

*This conjecture has been studied in a recent work [15]. More precisely, the following results have been proved.*

**Theorem 4.9.**

(i) *The Local Birkhoff Conjecture holds true for $q_0 = 2, 3, 4, 5$, with $m_0 = 40q_0$ and $n_0 = 3q_0$.*
(ii) *The Local Birkhoff Conjecture holds true for $q_0 > 5$ with $m_0 = 40q_0$ and $n_0 = 3q_0$, subject to checking that $q_0 - 2$ matrices (which are explicitly described) are invertible.*

**Remark 4.10.**

(i) Case $q_0 = 2$ was proven in [11] (see also [30,43]).
(ii) Smoothness exponents are probably not optimal.
(iii) Notice that in the proof we actually need only the existence of rationally integrable caustics of rotation numbers, less than $1/q_0$, of the form $j/q$ for $j = 1, 2, 3$.
(iv) The invertibility condition on finitely many matrices, to which the claim of part (ii) of theorem 4.9 is subject, is explicit and computable. In [16], it is described how to implement an algorithm to verify it by means of symbolic computations. The coefficients of these matrices are completely determined by the $e$-expansion of the action-angle parametrization of the ellipse, which, in turn, is explicitly given by elliptic integrals; it turns out that the entries of these matrices are either 0, 1 or of the form $\xi \cos^{-2j}(w\pi)e^{2j}$, where $\xi \in \mathbb{Q}$, $j \in \mathbb{N}$, $w \in \{1/(2k+1), 2/(2k+1), 1/2k, 3/2k : k > j\}$.

Some ideas on the proof will be outlined in §5b.

# 5. Some ideas on the proof of theorems 4.7 and 4.9

## (a) Perturbative Birkhoff conjecture (theorem 4.7)

Let us provide a description of the strategy that we adopted in [43] to prove theorem 4.7.

For small eccentricities, theorem 4.7 was proven in [11]. Let us start by describing the simplified setting of integrable infinitesimal deformations of a circle. This provides an insight into the strategy of the proof in the general case.

Let $\Omega_0$ be a circle centred at the origin and radius $\rho_0 > 0$. Let $\Omega_\varepsilon$ be a one-parameter family of smooth deformations given in the polar coordinates $(\rho, \varphi)$ by

$$\partial\Omega_\varepsilon = \{(\rho, \varphi) = (\rho_0 + \varepsilon\rho(\varphi) + O(\varepsilon^2), \varphi)\}.$$

Consider the Fourier expansion of $\rho$:

$$\rho(\varphi) = \rho'_0 + \sum_{k>0} \rho_k \sin(k\varphi) + \rho_{-k}\cos(k\varphi).$$

**Theorem 5.1 (Ramírez-Ros [19]).** *If $\Omega_\varepsilon$ has an integrable rational caustic $\Gamma_{1/q}$ of rotation number $1/q$, for any $\varepsilon$ sufficiently small, then we have $\rho_{kq} = \rho_{-kq} = 0$ for any integer $k$.*

Let us now assume that the domains $\Omega_\varepsilon$ are 2-rationally integrable for all sufficiently small $\varepsilon$ and ignore for a moment the dependence on the parametrization: then the above theorem implies that $\rho'_k = \rho''_k = 0$ for $k > 2$, i.e.

$$\begin{aligned}\rho(\varphi) &= \rho'_0 + \rho'_1\cos\varphi + \rho''_1\sin\varphi + \rho'_2\cos 2\varphi + \rho''_2\sin 2\varphi\\ &= \rho'_0 + \rho^*_1\cos(\varphi - \varphi_1) + \rho^*_2\cos 2(\varphi - \varphi_2),\end{aligned}$$

where $\varphi_1$ and $\varphi_2$ are appropriately chosen phases.

**Remark 5.2.** Observe that

— $\rho_0$ corresponds to an homothety;
— $\rho^*_1$ corresponds to a translation in the direction forming an angle $\varphi_1$ with the polar axis $\{\varphi = 0\}$;
— $\rho^*_2$ corresponds to a deformation of the circle into an ellipse of small eccentricity, whose major axis forms an angle $\varphi_2$ with the polar axis.

This implies that, infinitesimally (as $\varepsilon \to 0$), rationally integrable deformations of a circle are tangent to the 5-parameter family of ellipses.

In order to extend these ideas to the case of an integrable perturbation (not necessarily a deformation) of an ellipse, a more elaborate strategy is needed, involving more quantitative estimates and approximation procedure (we refer to [11,43] for more technical details). In particular, Fourier modes are replaced by new functions determined by the dynamics inside the approximating ellipse, that we call *dynamical modes* $\{c_q, s_q\}_{q\geq 3}$, which are given by

$$c_q(\varphi) := \frac{\cos((2\pi q/4K(k_q))F(\varphi;k_q))}{\sqrt{1-k_q^2\sin^2\varphi}} \quad \text{and}$$

$$s_q(\varphi) := \frac{\sin((2\pi q/4K(k_q))F(\varphi;k_q))}{\sqrt{1-k_q^2\sin^2\varphi}},$$

where $k_q$ denotes the eccentricity of the confocal ellipse corresponding to the caustic of rotation number $1/q$, while

$$F(\varphi;k) := \int_0^{\varphi} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\tau}} \quad \text{and} \quad K(k) := F\left(\frac{\pi}{2};k\right)$$

are the elliptic integrals of first kind (see, for example, [46] for more details on these functions and their properties).

The core of the proof consists in showing that these dynamical modes together with the infinitesimal generators of homotheties, translations, rotations and hyperbolic rotations (i.e. those transformations preserving the set of ellipses), form a basis of $L^2(\mathbb{R}/2\pi\mathbb{Z})$. This is one of the main difficulties (maybe the hardest one) involved in the extension of the perturbative result in [11] to the case of perturbations of any ellipse, as studied in [43]. While in the former case, one can take advantage of the fact that these functions can be considered small perturbations of the Fourier modes, in the latter new strategies need to be exploited.

In [43] we consider analytic extensions of the action-angle coordinates of the elliptic billiard, more specifically, of the boundary parametrizations induced by each integrable caustic (these functions can be explicitly expressed in terms of elliptic integrals and Jacobi elliptic functions). A detailed study of their complex singularities and the size of their maximal strips of analyticity allowed us to deduce their linear independence (both for finite and infinite combinations) and, by a suitable codimension argument, to show that they form a complete set of generators, thus completing the proof that they are a basis of $L^2(\mathbb{R}/2\pi\mathbb{Z})$.

## (b) Local Birkhoff conjecture for nearly circular domains (theorem 4.9)

The main difficulty in this case—in comparison with the one discussed in theorem 4.7 and §5a—is that we cannot use the preservation of integrable rational caustics for all rotation number $1/q$, with $q \geq 3$; hence, we need to recover the missing conditions on the corresponding Fourier coefficients of the perturbation.

Our key idea is the following: for ellipses of small eccentricity $e > 0$, we study the Taylor expansion, with respect to $e$, of the corresponding action-angle coordinates. Using this expansion, we derive the necessary condition for the preservation of integrable rational caustics, in terms of the Fourier coefficients of the perturbation, up to the precision of order $e^{2N}$, for some positive integer $N = N(q_0)$.

Let us outline our strategy, starting from some special cases.

— *Case* $q_0 = 3$: We lose a pair of conditions corresponding to Fourier coefficients of order 3. We exploit the conditions obtained from the existence of integrable rational caustics of rotation numbers $\frac{1}{5}, \frac{1}{7}, \frac{2}{7}$: we use the corresponding expansions, with respect to $e$, up to the precision $O(e^6)$, to derive a system of linear equations for the 3rd, 5th and 7th Fourier coefficients. Solving this linear system will provide us with the estimates needed for Fourier coefficients of order 3.

— *Case* $q_0 = 4$: In this case, we lose two pairs of conditions corresponding to Fourier coefficients of order $q = 3, 4$. These will be recovered in two steps:

(i) To recover the one corresponding to Fourier coefficients of order 3, we study the necessary conditions for the existence of integrable rational caustics of rotation numbers $\frac{1}{5}, \frac{1}{7}, \frac{1}{9}$ and $\frac{2}{9}$, written in terms of the Fourier coefficients of the perturbation, and consider their expansions, with respect to $e$, up to order $O(e^8)$. We then derive a linear system for the 3rd, 5th, 7th and 9th Fourier coefficients, whose solution will provide us with the estimates needed for the Fourier coefficients of order 3.

(ii) To recover the one corresponding to Fourier coefficients of order 4, we study the necessary conditions for the existence of integrable rational caustics of rotation numbers $\frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}$ and $\frac{3}{14}$, which give rise to a system of linear equation for the 4th, 6th, 8th, 10th, 12th and 14th Fourier coefficients; as before, the solution of this linear system will give us the estimates needed for the Fourier coefficients of order 4.

— *The general case*: Along the same lines described in the previous two items, we outlined in [16] a general (conditional) procedure to deal with this problem for any $q_0 \geq 3$; the implementation of this scheme is based on the assumption that certain explicit non-degeneracy conditions for the corresponding linear systems hold. We remark, however, that all of these conditions are very explicit and the algorithm is explicitly described, so needs to be implemented on a computer.

Data accessibility. This article has no additional data.

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