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On the local Birkhoff conjecture for convex billiards

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Dedicated to the memory of our thesis advisor John N. Mather: a great mathematician and a remarkable person

Abstract

The classical Birkhoff conjecture claims that the boundary of a strictly convex integrable billiard table is necessarily an ellipse (or a circle as a special case). In this article we prove a complete local version of this conjecture: a small integrable perturbation of an ellipse must be an ellipse. This extends and completes the result in Avila-De Simoi-Kaloshin, where nearly circular domains were considered. One of the crucial ideas in the proof is to extend action-angle coordinates for elliptic billiards into complex domains (with respect to the angle), and to thoroughly analyze the nature of their complex singularities. As an application, we are able to prove some spectral rigidity results for elliptic domains.

1. Introduction

A *mathematical billiard* is a system describing the inertial motion of a point mass inside a domain, with elastic reflections at the boundary (which is assumed to have infinite mass). This simple model was first proposed by G. D. Birkhoff as a mathematical playground where "*the formal side*, *usually so formidable in dynamics*, *almost completely disappears and only the interesting qualitative questions need to be considered*," [7, pp. 155–156].

Since then billiards have captured much attention in many different contexts, becoming a very popular subject of investigation. Not only is their law of motion very physical and intuitive, but billiard-type dynamics is ubiquitous. Mathematically, they offer models in every subclass of dynamical systems (integrable, regular, chaotic, etc.); more importantly, techniques initially devised

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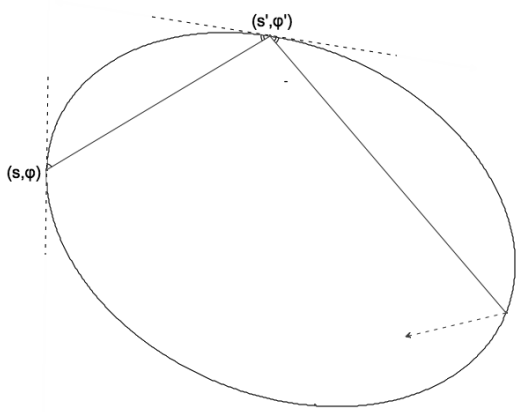

Figure 1.

for billiards have often been applied and adapted to other systems, becoming standard tools and having ripple effects beyond the field.

Let us first recall some properties of the billiard map. We refer to [44], [47], [48] for a more comprehensive introduction to the study of billiards.

Let Ω be a strictly convex domain in $\mathbb{R}^2$ with C^r boundary $\partial\Omega$, with $r \geq 3$. The phase space M of the billiard map consists of unit vectors (x, v) whose foot points x are on $\partial\Omega$ and that have inward directions. The billiard ball map $f : M \longrightarrow M$ takes (x, v) to (x', v'), where x' represents the point where the trajectory starting at x with velocity v hits the boundary $\partial\Omega$ again, and v' is the *reflected velocity*, according to the standard reflection law: angle of incidence is equal to the angle of reflection (Figure 1).

Remark 1. Observe that if Ω is not convex, then the billiard map is not continuous; in this article we will be interested only in strictly convex domains (see Remark 4). Moreover, as pointed out by Halpern [19], if the boundary is not at least C^3, then the flow might not be complete.

Let us introduce coordinates on M. We suppose that $\partial\Omega$ is parametrized by arc-length s and let $\gamma : [0, |\partial\Omega|] \longrightarrow \mathbb{R}^2$ denote such a parametrization, where $|\partial\Omega|$ denotes the length of $\partial\Omega$. Let ϕ be the angle between v and the positive tangent to $\partial\Omega$ at x. Hence, M can be identified with the annulus $\mathbb{A} = [0, |\partial\Omega|] \times (0, \pi)$ and the billiard map f can be described as

$$\begin{aligned} f : [0, |\partial\Omega|) \times (0, \pi) &\longrightarrow [0, |\partial\Omega|) \times (0, \pi) \\ (s, \phi) &\longmapsto (s', \phi'). \end{aligned}$$

In particular, f can be extended to $\bar{\mathbb{A}} = [0, |\partial\Omega|] \times [0, \pi]$ by fixing $f(s, 0) = (s, 0)$ and $f(s, \pi) = (s, \pi)$ for all s. Let us denote by

$$\ell(s, s') := \|\gamma(s) - \gamma(s')\|$$

the Euclidean distance between two points on $\partial\Omega$. It is easy to prove that

$$\begin{cases} \dfrac{\partial \ell}{\partial s}(s, s') = -\cos\phi, \\ \dfrac{\partial \ell}{\partial s'}(s, s') = \cos\phi' . \end{cases} \tag{1}$$

Remark 2. If we lift everything to the universal cover and introduce new coordinates $(x, y) = (s, -\cos\phi) \in \mathbb{R} \times (-1, 1)$, then the billiard map is a twist map with ℓ as generating function, and it preserves the area form $dx \wedge dy$. See [44], [47], [48].

Despite the apparently simple (local) dynamics, the qualitative dynamical properties of billiard maps are extremely non-local. This global influence on the dynamics translates into several intriguing *rigidity phenomena*, which are at the basis of several unanswered questions and conjectures (see, for example, [3], [5], [11], [18], [21], [22], [40], [39], [41], [44], [46], [47], [48], [50]). Amongst many, in this article we will address the question of classifying *integrable billiards*, also known as the *Birkhoff conjecture*. As an application of our main result, in Section 1.2 we will also discuss certain spectral rigidity properties of ellipses.

1.1. *Integrable billiards and the Birkhoff conjecture.* The easiest example of billiards is given by a billiard in a disc $\mathcal{D}$ (for example of radius R). It is easy to check in this case that the angle of reflection remains constant at each reflection (see also [48, Ch. 2]). If we denote by s the arc-length parameter (i.e., $s \in \mathbb{R}/2\pi R\mathbb{Z}$) and by $\theta \in (0, \pi/2]$ the angle of reflection, then the billiard map has a very simple form:

$$f(s, \theta) = (s + 2R\,\theta,\ \theta).$$

In particular, θ stays constant along the orbit, and it represents an *integral of motion* for the map. Moreover, this billiard enjoys the peculiar property of having the phase space — which is topologically a cylinder — completely foliated by homotopically non-trivial invariant curves $\mathcal{C}_{\theta_0} = \{\theta \equiv \theta_0\}$. These curves correspond to concentric circles of radii $\rho_0 = R\cos\theta_0$ and are examples of what are called *caustics*, i.e., (smooth and convex) curves with the property that if a trajectory is tangent to one of them, then it will remain tangent after each reflection (see Figure 2).

A billiard in a disc is an example of an *integrable billiard*. There are different ways to define global/local integrability for billiards (the equivalence of these notions is an interesting problem itself):

- either through the existence of an integral of motion, globally or locally near the boundary (in the circular case an integral of motion is given by $I(s, \theta) = \theta$);

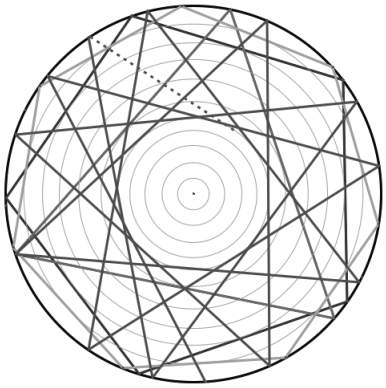

Figure 2. Billiard in a disc

- or through the existence of a (smooth) foliation of the whole phase space (or locally in a neighborhood of the boundary $\{\theta = 0\}$), consisting of invariant curves of the billiard map; for example, in the circular case these are given by $\mathcal{C}_\theta$.

The second property translates (under suitable assumptions) into the existence of a (smooth) family of caustics, globally or locally near the boundary (in the circular case, the concentric circles of radii $R\cos\theta$).

In [5], Misha Bialy proved the following result concerning global integrability (see also [51]).

THEOREM (Bialy). *If the phase space of the billiard ball map is globally foliated by continuous invariant curves that are not null-homotopic, then it is a circular billiard.*

However, while circular billiards are the only examples of global integrable billiards, integrability itself is still an intriguing open question. One could consider a billiard in an ellipse: this is in fact integrable (see Section 2). Yet, the dynamical picture is very distinct from the circular case: as it is showed in Figure 3, each trajectory that does not pass through a focal point, is always tangent to precisely one confocal conic section, either a confocal ellipse or the two branches of a confocal hyperbola (see, for example, [48, Ch. 4]). Thus, the confocal ellipses inside an elliptical billiards are convex caustics, but they do not foliate the whole domain: the segment between the two foci is left out. (Describing the dynamics explicitly is much more complicated: see, for example, [49] and Section 2.)

QUESTION (Birkhoff). *Are there other examples of integrable billiards*?

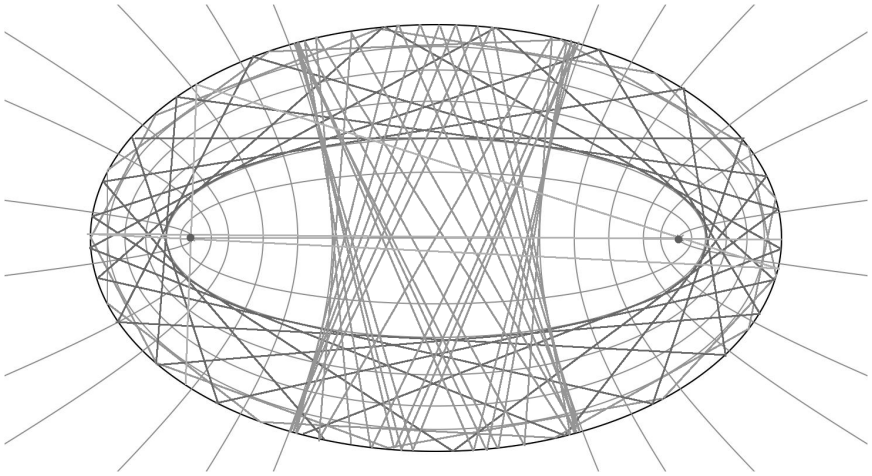

Figure 3. Billiard in an ellipse

Remark 3. Although some vague indications of this question can be found in [7], to the best of our knowledge, its first appearance as a conjecture was in a paper by Poritsky [40], where the author attributes it to Birkhoff himself.[1] Thereafter, references to this conjecture (either as the *Birkhoff conjecture* or the *Birkhoff-Poritsky conjecture*) repeatedly appeared in the literature; see, for example, Gutkin [18, §1], Moser [32, App. A], Tabachnikov [47, §2.4], etc.

Remark 4. In [28] Mather proved the non-existence of caustics (hence, the non-integrability) if the curvature of the boundary vanishes at one point. This observation justifies the restriction of our attention to strictly convex domains.

Remark 5. (i) Interestingly, Treschev in [50] gives indication that there might exist analytic billiards, different from ellipses, for which the dynamics in a neighborhood of the elliptic period-2 orbit is conjugate to a rigid rotation. These billiards can be seen as an instance of *local integrability*; however, this regime is somehow complementary to the one conjectured by Birkhoff. Here one has local integrabilility in a neighborhood of an elliptic periodic orbit of period 2, while the Birkhoff conjecture is related to integrability in a neighborhood of the boundary. This gives an indication that these two notions of integrability do differ.

(ii) An algebraic version of this conjecture states that the only billiards admitting polynomial (in the velocity) integrals are circles and ellipses. For recent results in this direction, see [6].

[1] Poritsky was Birkhoff's doctoral student, and [40] was published several years after Birkhoff's death.

Despite its long history and the amount of attention that this conjecture has captured, it still remains open. As far as our understanding of integrable billiards is concerned, the most important related results are the above–mentioned theorem by Bialy [5] (see also [51]), a result by Innami[2] [23], in which he shows that the existence of caustics with rotation numbers accumulating to $1/2$ implies that the billiard must be an ellipse,[3] a result by Delshams and Ramírez-Ros [12] in which they study entire perturbations of elliptic billiards and prove that any non-trivial symmetric perturbation of the elliptic billiard is not integrable, near homoclinic solutions, and a very recent result by Avila, De Simoi and Kaloshin [3] in which they show a perturbative version of this conjecture for ellipses of small eccentricity.

Let us introduce an important notion for this paper.

Definition 6. (i) We say Γ is an integrable rational caustic for the billiard map in Ω if the corresponding (non-contractible) invariant curve Γ consists of periodic points; in particular, the corresponding rotation number is rational.

(ii) If the billiard map inside Ω admits integrable rational caustics of rotation number $1/q$ for all $q > 2$, we say that Ω is rationally integrable.

Remark 7. A simple sufficient condition for rational integrability is the following (see [3, Lemma 1]). Let $\mathcal{C}_\Omega$ denote the union of all smooth convex caustics of the billiard in Ω; if the interior of $\mathcal{C}_\Omega$ contains caustics of rotation number $1/q$ for any $q > 2$, then Ω is rationally integrable.

Our main result is the following.

Main Theorem (Local Birkhoff conjecture). *Let $\mathcal{E}_0$ be an ellipse of eccentricity $0 \le e_0 < 1$ and semi-focal distance c; let $k \ge 39$. For every $K > 0$, there exists $\varepsilon = \varepsilon(e_0, c, K)$ such that the following holds: if Ω is a rationally integrable C^k-smooth domain so that $\partial\Omega$ is C^k-K-close and C^1-ε-close to $\mathcal{E}_0$, then Ω is an ellipse.*

Remark 8. One could replace the smallness condition in the C^1-norm with a smallness condition with respect to the C^0 topology. (This can be showed by using interpolation inequalities and the convexity of the domains.)[4]

Remark 9. In [21] we prove a similar rigidity statement for a different type of rational integrability. Namely, we describe an algorithm to prove that for any given $q_0 \ge 3$, there exists $e_0 = e(q_0) > 0$ such that every sufficiently smooth perturbation of $\mathcal{E}_e$, with $0 < e < e_0$, having integrable rational caustics of rotation numbers p/q, for all $0 < p/q < 1/q_0$, must be an ellipse. This algorithm

[2]We are grateful to M. Bialy for pointing out this reference.

[3]This regime of integrability is somehow diametrically opposed to ours, since we are interested in integrability near the boundary of the billiard domain.

[4]This remark was suggested to the authors by Camillo De Lellis.

is conditional on checking the invertibility of finitely many explicit matrices, which we prove in the cases $q_0 = 3, 4, 5$. Observe that the analysis in [21] only applies to ellipses of small eccentricity as in [3], since Taylor expansions with respect to e are needed in order to get higher order (integrability) conditions.

One of the crucial ideas to extend the analysis beyond the almost circular case in [3] is to consider analytic extensions of the action-angle coordinates of the elliptic billiard (more specifically, of the boundary parametrizations induced by each integrable caustic) and to study their singularities (see Section 7). These functions can be explicitly expressed in terms of elliptic integrals and Jacobi elliptic functions (see Section 3.1). This analysis will be exploited to define a *dynamically-adapted basis* for $L^2(\mathbb{R}/2\pi\mathbb{Z})$, which will provide the main framework to carry out our analysis. See Section 4.2 for a more detailed description of the scheme of the proof.

In addition to this, in Appendix F we propose a possible strategy to use the *affine length shortening* (ALS) *flow* (see, for instance, [42]) as a potential approach to prove the *global* Birkhoff conjecture. Our proposal is based on the fact that the ALS flow evolves any convex domain with smooth boundary into an ellipse in finite time.

1.2. *Applications for spectral rigidity of ellipses.* In this subsection we describe an interesting application of our Main Theorem to spectral rigidity properties of ellipses.[5]

Let Ω be a smooth strictly convex (planar) domain. While the dependence of the dynamics on the geometry of the domain is well perceptible, an intriguing challenge is to understand to which extent dynamical information can be used to reconstruct the shape of the domain. A particularly interesting problem in this direction is to unravel which information on the geometry of the billiard domain the set of periodic orbits does encode. More ambitiously, one could wonder whether a complete knowledge of this set allows one to reconstruct the shape of the billiard and hence the whole of its dynamics. Several results in this direction (and in related ones) are contained, for instance, in [4], [11], [17], [20], [21], [26], [27], [38], [39], [44], [46], [52].

Let us start by introducing the *length spectrum* of a domain Ω.

Definition 10 (length spectrum). Given a domain Ω, the length spectrum of Ω is given by the set of lengths of its periodic orbits, counted with multiplicity:

$$\mathcal{L}(\Omega) := \mathbb{N}\{\text{ lengths of closed geodesics in } \Omega\} \cup \mathbb{N}\,|\partial\Omega|,$$

where $|\partial\Omega|$ denotes the length of the boundary of Ω.

[5]This was suggested to the authors by Hamid Hezari.

A remarkable relation exists between the length spectrum of a billiard in a convex domain Ω and the spectrum of the Laplace operator in Ω with Dirichlet boundary conditions (similarly for Neumann boundary conditions):

$$\begin{cases} \Delta f = \lambda f \quad \text{in } \Omega, \\ f|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

From the physical point of view, the eigenvalues λ's are the eigenfrequencies of the membrane Ω with a fixed boundary. K. Andersson and R. Melrose [2] proved the following relation between the Laplace spectrum and the length spectrum. Call the function

$$w(t) := \sum_{\lambda_i \in \text{ spec } \Delta} \cos(t\sqrt{-\lambda_i})$$

the wave trace. Then, the wave trace $w(t)$ is a well-defined generalized function (distribution) of t, smooth away from the length spectrum, namely,

$$\text{sing. supp.}\big(w(t)\big) \subseteq \pm\mathcal{L}(\Omega) \cup \{0\}. \tag{3}$$

So if $l > 0$ belongs to the singular support of this distribution, then there exists either a closed billiard trajectory of length l, or a closed geodesic of length l in the boundary of the billiard table.

Generically, equality holds in (3). More precisely, if no two distinct orbits have the same length and the Poincaré map of any periodic orbit is non-degenerate, then the singular support of the wave trace coincides with $\pm\mathcal{L}(\Omega)$ $\cup\{0\}$ (see, e.g., [38]). This theorem implies that, at least for generic domains, one can recover the length spectrum from the Laplace one.

This relation between periodic orbits and spectral properties of the domain immediately recalls a more famous spectral problem (probably the most famous) — *Can one hear the shape of a drum*? — as formulated in a very suggestive way by Mark Kac [24] (although the problem had already been stated by Hermann Weyl). More precisely, is it possible to infer information about the shape of a drumhead (i.e., a domain) from the sound it makes (i.e., the list of basic harmonics/ eigenvalues of the Laplace operator with Dirichlet or Neumann boundary conditions)? This question has not been completely solved yet: there are several negative answers (for instance by Milnor [31] and Gordon, Webb, and Wolpert [14]), as well as some positive ones.

Hezari and Zelditch, going in the affirmative direction, proved in [20] that, given an ellipse $\mathcal{E}$, any 1-parameter C^∞-deformation Ω_ε that preserves the Laplace spectrum (with respect to either Dirichlet or Neumann boundary conditions) and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group of the ellipse has to be *flat* (i.e., all derivatives have to vanish for $\varepsilon = 0$). Popov–Topalov [39] recently extended these results (see also [52]). Further historical remarks on the inverse spectral

problem can be also found in [20]. In [33], [34], [35] Osgood, Phillips and Sarnak showed that isospectral sets are necessarily compact in the C^∞ topology in the space of domains with C^∞ boundary. In [43] Sarnak conjectures that the set of smooth convex domains isospectral to a given smooth convex domain is finite. (For a partial progress on this question, see [11].)

One of the difficulties in working with the length spectrum is that all of this information on the periodic orbits comes in a non-formatted way. For example, we lose track of the rotation number corresponding to each length. A way to overcome this difficulty is to "organize" this set of information in a more systematic way, for instance, by associating to each length the corresponding rotation number. This new set is called the *Marked length spectrum* of Ω and is denoted by $\mathcal{ML}_\Omega$:

$$\mathcal{ML}(\Omega) := \{(\text{length}(\gamma), \text{rot}(\gamma)) : \ \gamma \text{ periodic orbit of the billiard in } \Omega\},$$

where $\text{rot}(\gamma)$ denotes the rotation number of γ.[6]

One could also refine this set of information by considering not the lengths of all orbits, but selecting some of them. More precisely, for each rotation number p/q in lowest terms, one could consider the maximal length among those having rotation number p/q. We call this map the *Maximal Marked length spectrum* of Ω, namely $\mathcal{ML}^{\max}(\Omega) : \mathbb{Q} \cap [0, 1/2] \ \to \mathbb{R}$, given by

$$\mathcal{ML}^{\max}_\Omega(p/q) = \max\Big\{\text{lengths of periodic orbits with rot. number } p/q\Big\}.$$

Remark 11. The maximal marked length spectrum is closely related to *Mather's minimal average action* (or *Mather's β-function*) of the associated billiard map in the domain, as was pointed out in [44]. Briefly speaking, this function — which can be defined for any exact area preserving twist map, not necessarily a billiard map — associates to any fixed rotation number (not only rational ones) the minimal average action of orbits with that rotation number (whose existence, inside a suitable interval, is ensured by the twist condition). These action-minimizing orbits are of particular interest from a dynamical point of view and play a key-role in what is nowadays called the *Aubry-Mather theory*; we refer the reader to [4], [30], [44], [45] for a presentation of this topic.

In the case of billiard maps, since the *action* coincides (up to a negative sign) with the euclidean length of the segment joining two subsequent rebounds, we have that the minimal average action of periodic orbits can be

[6] In the case of negatively curved surfaces without boundary the marked length spectrum consists of pairs of homotopy classes and length of the shortest geodesic in that homotopy class. Guillemin and Kazhdan [16] proved local rigidity with respect to this marked length spectrum. Global version of this result was obtained by Otal [36] and Croke [9].

expressed in terms of the maximal marked length spectrum; namely,

$$\beta_\Omega(p/q) = -\frac{1}{q}\mathcal{M}L_\Omega^{\max}(p/q) \qquad \forall\, 0 < p/q \le 1/2. \tag{4}$$

In particular, this object encodes many interesting dynamical information on the billiard map. For example, using the result in [29], one can deduce that β is differentiable at p/q if and only if there exists a rational caustic of rotation number p/q. See [44] for a detailed presentation of this and many other properties.

Let us now address the following question.

QUESTION: *Let Ω_1 and Ω_2 be two strictly convex planar domains with smooth boundaries, and assume that they have the same maximal marked Length spectrum, namely, $\mathcal{ML}_{\Omega_1}^{\max} \equiv \mathcal{ML}_{\Omega_2}^{\max}$ (or equivalently, $\beta_{\Omega_1} \equiv \beta_{\Omega_2}$). Is it true that Ω_1 and Ω_2 are isometric*?

Remark 12. It is known that if Ω has the same marked length spectrum of a disc, then it is indeed a disc; for a proof of this result, see for example [44, Cor. 3.2.17]. Another proof can be obtained by looking only at the Taylor coefficients of the β-function at 0 (which are related to the so-called *Marvizi-Melrose invariants*); it turns out that the first and the third order coefficients always satisfy an inequality, which becomes an equality if and only if the domain is a disc (see [26, §8] and [46, Cor. 1]).

It would be interesting to find a similar characterization for elliptic billiards, namely, that the maximal marked length spectrum (respectively, the β-function) univocally determines ellipses amongst all possible Birkhoff billiards.

In [46, Prop. 1], by looking at the Taylor expansion of the β-function at 0 (actually, only at the first and third order coefficients), a much weaker result was pointed out, namely, that the isospectrality condition determines univocally a given ellipse within the family of ellipses (up to rigid motions, i.e., the composition of a translation and a rotation).

From our Main Theorem, we can now deduce the following spectral rigidity results for ellipses.

a much weaker result

COROLLARY 13 (Local length–spectral uniqueness of ellipses). *Let Ω be a smooth strictly convex domain Ω sufficiently close to an ellipse.*

(i) *If Ω has the same maximal marked length spectrum (or Mather's β-function) of an ellipse, then it is an ellipse.*
(ii) *If its Mather's β-function is differentiable at all rationals $1/q$ with $q \ge 3$, then Ω is an ellipse.*

Moreover, the following spectral rigidity result holds.

COROLLARY 14 (Spectral rigidity of ellipses).

(i) *Ellipses are* (maximal) marked-length-spectrally rigid, *meaning that if* Ω_t *is a smooth deformation of an ellipse that keeps fixed the* (*maximal*) *marked length spectrum, then it consists of a rigid motion.*
(ii) *Ellipses are* length-spectrally rigid, *meaning that if* Ω_t *is a smooth deformation of an ellipse that keeps fixed the length spectrum, then it consists of a rigid motion.*

Proof of Corollary 13. Assertion (i) follows from assertion (ii), using (4) and recalling that the β-function of an ellipse is differentiable in $[0, 1/2)$, since the corresponding billiard map is integrable. As for the proof of (ii), it follows from the differentiability assumptions on β and from what was recalled at the end of Remark 11 (see also [29], [44]), that there exist integrable rational caustics for all rotation numbers $1/q$ for any $q \geq 3$. Hence our billiard is rationally integrable (see Definition 6). Applying the Main Theorem, since Ω is close to an ellipse, then it must be an ellipse. □

Proof of Corollary 14. Assertion (i) follows from Corollary 13(ii) and the fact that the β function (equivalently, the maximal marked length spectrum) univocally determines a given ellipse within the family of ellipses (up to rigid motions); see [46, Prop. 1].

To prove assertion (ii), one needs to use [44, Prop. 3.2.2], which shows that a C^0 iso-length spectral deformation is necessarily an iso-marked length spectral deformation. Then, the claim follows by applying (i). □

1.3. *Organization of the article.* For the reader's convenience, a brief description of how the article is organized now follows.

In Section 2 we describe our setting and introduce elliptic coordinates (see Section 2.1), while in Section 3 we recall some definitions and some needed properties of elliptic integrals and elliptic functions (see Section 3.1) and use them to provide a more precise description of the billiard dynamics inside an ellipse (see Section 3.2).

In Section 4 we outline the scheme of the proof of our Main Theorem, both for perturbations of circular billiards (see Section 4.1) and for perturbations of general elliptic ones (see Section 4.2); we refer to this latter subsection for a detailed description of the contents of Sections 3–8.

In order to make the presentation clearer and easier to follow, we deferred several proofs of technical claims and some complementary material to Appendices A–E. Finally, in Appendix F we outline a possible strategy to approach the global Birkhoff conjecture, by means of the affine length shortening flow.

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2. **Notation and setting**

Let us consider the ellipse

$$\mathcal{E}_{e_0,c} = \left\{(x,y) \in \mathbb{R}^2 : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\},$$

centered at the origin and with semi-axes of lengths, respectively, $0 < b \leq a$; in particular, e_0 denotes its eccentricity, given by $e_0 = \sqrt{1 - \frac{b^2}{a^2}} \in [0,1)$ and $c = \sqrt{a^2 - b^2}$ the semi-focal distance. Observe that when $e_0 = 0$, then $c = 0$ and $\mathcal{E}_{0,0}$ degenerates to a 1-parameter family of circles centered at the origin.

The family of confocal elliptic caustics in $\mathcal{E}_{e_0,c}$ is given by (see also Figure 3)

$$C_\lambda = \left\{(x,y) \in \mathbb{R}^2 : \ \frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = 1\right\}, \qquad 0 < \lambda < b. \tag{5}$$

Observe that the boundary itself corresponds to $\lambda = 0$, while the limit case $\lambda = b$ corresponds to the the two foci $\mathcal{F}_\pm = (\pm\sqrt{a^2 - b^2}, 0)$. Clearly, for $e_0 = 0$, we recover the family of concentric circles described in Figure 2.

2.1. *Elliptic polar coordinates.* A more convenient coordinate frame for addressing this question is provided by the so-called *elliptic polar coordinates* (or, simply, elliptic coordinates) $(\mu, \varphi) \in \mathbb{R}_{\geq 0} \times \mathbb{R}/2\pi\mathbb{Z}$, given by

$$\begin{cases} x = c \cosh\mu \, \cos\varphi, \\ y = c \sinh\mu \, \sin\varphi, \end{cases}$$

where $c = \sqrt{a^2 - b^2} > 0$ represents the semi-focal distance (In the case $e_0 = 0$, this parametrization degenerates to the usual polar coordinates.) Observe that for each $\mu_* > 0$, the equation $\mu \equiv \mu_*$ represents a confocal ellipse, while for each $\varphi_* \in [0, 2\pi) \setminus \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, the equation $\varphi \equiv \varphi_*$ corresponds to one of the two branches of a confocal hyperbola; these grid-lines are mutually orthogonal. Moreover, the degenerate cases $\mu_* \equiv 0$ and $\varphi_* \equiv 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ describe, respectively, the (cartesian) segment $\{-c \leq x \leq c\}$, and the (cartesian) half-lines $\{x \geq c\}$, $\{y \geq 0\}$, $\{x \leq -c\}$ and $\{y \leq 0\}$.

Therefore, in these elliptic polar coordinates $\mathcal{E}_{e_0,c}$ becomes

$$\mathcal{E}_{e_0,c} = \{(\mu_0, \varphi),\ \varphi \in \mathbb{R}/2\pi\mathbb{Z}\},$$

where $\mu_0 = \mu_0(e_0) := \operatorname{arcosh}(1/e_0)$ (The dependence on c is in the definition of the coordinate frame.)

Let us denote by $\mathcal{E}\ell\ell$ the set of ellipses in $\mathbb{R}^2$ with circles being degenerate points. This is a 5-dimensional family of strictly convex curves parametrized, for example, by the cartesian coordinates of its center $(x_0, y_0) \in \mathbb{R}^2$, the semi-focal distance $c > 0$, the parameter $\mu_0 > 0$ corresponding to the eccentricity, and the angle $\theta \in [0, \pi)$ between the major semi-axis and the x-axis. (Notice that θ is not well defined for circles.) More specifically, for each $(x_0, y_0, c, \mu_0, \theta) \in \mathbb{R}^2 \times (0, +\infty)^2 \times [0, \pi)$, we associate the (parametrized) ellipse

$$\begin{aligned} &\mathcal{E}(x_0, y_0, c, \mu_0, \theta) \\ &:= \left\{ \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} c\cosh\mu_0 \cos\varphi \\ c\sinh\mu_0 \sin\varphi \end{pmatrix}, \varphi \in [0, 2\pi) \right\}. \end{aligned} \tag{6}$$

In the following we will use the shorthand $\mathcal{E}_{e_0,c}$ for $\mathcal{E}(0, 0, c, \mu_0(e_0), 0)$. In particular, $\mathcal{E}_{0,c}$ consists of a 1-parameter family of circles centered at the origin.

3. **Action-angle coordinate of elliptic billiards**

Here we define and study action-angle coordinates for elliptic billiards.

3.1. *Elliptic integrals and Jacobi elliptic functions.* Let us recall some basic definitions on elliptic integrals and elliptic functions that will be used in the following; we refer the reader, for instance, to [1] for a more comprehensive presentation.

Let $0 \leq k < 1$. We define the following *elliptic integrals*:

- *Incomplete elliptic integral of the first kind*:

$$F(\varphi; k) := \int_0^{\varphi} \frac{1}{\sqrt{1 - k^2 \sin^2\varphi}} d\varphi.$$

  In particular, k is called the *modulus* and φ the *amplitude*. Moreover, the quantity $k' := \sqrt{1 - k^2}$ is often called the *complementary modulus*. Observe that for $k = 0$, we have $F(\varphi; 0) = \varphi$; on the other hand, $F(\varphi; 1)$ has a pole at $\varphi = \frac{\pi}{2}$.

- *Complete elliptic integral of the first kind*:

$$K(k) = F\left(\frac{\pi}{2}; k\right).$$

Let us recall that an *elliptic function* is a doubly-periodic meromorphic function, i.e., it is periodic in two directions, and hence it is determined by its values on a fundamental parallelogram. Of course, a non-constant elliptic function cannot be holomorphic, as it would be a bounded entire function, and by Liouville's theorem it would be constant. In particular, elliptic functions must have at least two poles in a fundamental parallelogram (counting multiplicities); it is easy to check, using the periodicity, that a contour integral around its boundary must vanish, implying that the residues of all simple poles must cancel out.

Jacobi Elliptic functions are obtained by inverting incomplete elliptic integrals of the first kind. More specifically, let

$$u = F(\varphi; k) = \int_0^{\varphi} \frac{d\tau}{\sqrt{1 - k^2 \sin \tau}}; \tag{7}$$

u is often called the *argument*. If u and φ are related as above (we can also write $\varphi = \operatorname{am}(u; k)$, called the *amplitude* of u), then we define the Jacobi elliptic functions as

$$\operatorname{sn}(u; k) := \sin(\operatorname{am}(u; k)),$$
$$\operatorname{cn}(u; k) := \cos(\operatorname{am}(u; k)).$$

Remark 15. These two elliptic functions have periods $4K(k)$ (in the real direction) and $4iK(k')$ (in the imaginary direction). Moreover, they have two simple poles: at $u_1 = iK(k')$ with residue, respectively, $1/k$ and $-i/k$, and at $u_2 = 2K(k) + iK(k')$ with residue, respectively, $-1/k$ and i/k.

3.2. *Elliptic billiard dynamics and caustics.* Now we want to provide a more precise description of the billiard dynamics in $\mathcal{E}_{e_0,c}$.

The following result has been proven in [8] (see also [10, Lemma 2.1]).

PROPOSITION 16. *Let* $\lambda \in (0, b)$, *and let*

$$k_\lambda^2 := \frac{a^2 - b^2}{a^2 - \lambda^2} \quad \text{and} \quad \delta_\lambda := 2\, F(\arcsin(\lambda/b); k_\lambda).$$

Let us denote, in cartesian coordinates, $q_\lambda(t) := (a \operatorname{cn}(t; k_\lambda), b \operatorname{sn}(t; k_\lambda))$. *Then, for every* $t \in [0, 4K(k_\lambda))$, *the segment joining* $q_\lambda(t)$ *and* $q_\lambda(t + \delta_\lambda)$ *is tangent to the caustic* C_λ.

Observe that

- k_λ is a strictly increasing function of $\lambda \in (0, b)$; in particular, $k_\lambda \to e_0$ as $\lambda \to 0^+$, while $k_\lambda \to 1$ as $\lambda \to b^-$. Observe that k_λ represents the eccentricity of the ellipse C_λ.

- δ_λ is also a strictly increasing function of $\lambda \in (0, b)$; in fact, $F(\varphi; k)$ is clearly strictly increasing in both φ and $k \in [0, 1)$. Moreover, $\delta_\lambda \to 0$ as $\lambda \to 0^+$, and $\delta_\lambda \to +\infty$ as $\lambda \to b^-$.

Remark 17. Using elliptic polar coordinates, one can easily check that $\tanh^2 \mu = 1 - \frac{a^2-b^2}{a^2-\lambda^2}$ and therefore

$$k(\mu) = \sqrt{1 - \tanh^2 \mu} = \frac{1}{\cosh \mu},$$

which is exactly the eccentricity of the confocal ellipse of parameter μ.

Let us now consider the parametrization of the boundary induced by the dynamics on the caustic C_λ:

$$\begin{aligned} Q_\lambda : \mathbb{R}/2\pi\mathbb{Z} &\longrightarrow \mathbb{R}^2 \\ \theta &\longmapsto q_\lambda \left(\frac{4K(k_\lambda)}{2\pi}\, \theta \right). \end{aligned}$$

We define the *rotation number* associated to the caustic C_λ to be

$$\omega_\lambda := \frac{\delta_\lambda}{4K(k_\lambda)} = \frac{F(\arcsin(\lambda/b); k_\lambda)}{2K(k_\lambda)}. \tag{8}$$

In particular, ω_λ is strictly increasing as a function of λ and $\omega_\lambda \longrightarrow 0$ as $\lambda \to 0^+$, while $\omega_\lambda \to \frac{1}{2}$ as $\lambda \to b^-$.

It is easy to deduce from the above expressions that, in elliptic coordinates (μ, φ), the boundary parametrization induced by the caustic C_λ is given by

$$S_\lambda(\theta) := (\mu_\lambda(\theta), \varphi_\lambda(\theta)) = \left(\mu_0, \operatorname{am} \left(\frac{4K(k_\lambda)}{2\pi}\, \theta; k_\lambda \right) \right). \tag{9}$$

More precisely, the orbit starting at $S_\lambda(\theta)$ and tangent to C_λ hits the boundary at $S_\lambda(\theta + 2\pi\, \omega_\lambda)$.

4. **Outline of the proof**

In this section we provide a description of the strategy that we will follow to prove our Main Theorem.

4.1. *A scheme for proving the Main Theorem for circular billiards.* For small eccentricities, the Main Theorem was proven in [3], and we now describe the proof therein. Let us start with the simplified setting of integrable infinitesimal deformations of a circle. This provides an insight into the strategy of the proof in the general case.

Let $\mathcal{E}^{\rho_0}_{0,0}$ be a circle centered at the origin and radius $\rho_0 > 0$. Let Ω_ε be a 1-parameter family of deformations given in the polar coordinates (ρ, φ) by

$$\partial\Omega_\varepsilon = \{(\rho, \varphi) = (\rho_0 + \varepsilon\rho(\varphi) + O(\varepsilon^2), \varphi)\}.$$

Consider the Fourier expansion of ρ :

$$\rho(\varphi) = \rho_0' + \sum_{k>0} \rho_k \sin(k\varphi) + \rho_{-k} \cos(k\varphi).$$

THEOREM 18 (Ramírez-Ros [41]). *If Ω_ε has an integrable rational caustic $\Gamma_{1/q}$ of rotation number $1/q$, for any ε sufficiently small, then we have $\rho_{kq} = 0$ for any integer k.*

Let us now assume that the domains Ω_ε are 2-rationally integrable for all sufficiently small ε and ignore for a moment dependence of parametrization: then the above theorem implies that $\rho_k = \rho_{-k} = 0$ for $k > 2$, i.e.,

$$\begin{aligned}\rho(\varphi) &= \rho_0' + \rho_{-1}\cos\varphi + \rho_1 \sin\varphi + \rho_{-2}\cos 2\varphi + \rho_2 \sin 2\varphi \\ &= \rho_0^* + \rho_1^* \cos(\varphi - \varphi_1) + \rho_2^* \cos 2(\varphi - \varphi_2),\end{aligned}$$

where φ_1 and φ_2 are appropriately chosen phases.

Remark 19. Observe that

- ρ_0^* corresponds to an homothety;
- ρ_1^* corresponds to a translation in the direction forming an angle φ_1 with the polar axis $\{\varphi = 0\}$;
- ρ_2^* corresponds to a deformation of the circle into an ellipse of small eccentricity, whose major axis forms an angle φ_2 with the polar axis.

This implies that, infinitesimally (as $\varepsilon \to 0$), rationally integrable deformations of a circle are tangent to the 5-parameter family of ellipses.

Notice that, in the above strategy, one needs to take $\varepsilon \to 0$ as $q \to \infty$. This means that we cannot take $\varepsilon > 0$ small, but only infinitesimal; hence one cannot directly use the above theorem to prove the main result. A more elaborate strategy is needed.

4.2. *Our scheme of the proof of the main theorem for elliptic billiards.* One of the noteworthy contributions of this paper is the analysis of perturbations of ellipses of arbitrary eccentricity $0 \le e_0 < 1$. Let us outline the main steps involved in the proof.

Let $\mathcal{E}_{e_0,c}$ be an ellipse of eccentricity $0 < e_0 < 1$ and semi-focal distance $c > 0$, and let (μ, φ) be the associated elliptic coordinates. Any domain Ω close to $\mathcal{E}_{e_0,c}$ can be written (in the elliptic coordinates associated to $\mathcal{E}_{e_0,c}$) in the form

$$\partial\Omega = \{(\mu_0 + \mu_1(\varphi), \varphi) : \varphi \in [0, 2\pi)\},$$

where μ_1 is a smooth 2π-periodic function (see also (11)). Recall that the ellipse $\mathcal{E}_{e_0,c}$ admits all integrable rational caustics of rotation number $1/q$ for $q > 2$.

By analogy with [3] we proceed as follows.

Step 1 (*Dynamical modes*). In Section 5, we consider the 1-parameter integrable deformation of an ellipse $\mathcal{E}_{e_0,c}$, given by the family of rationally integrable domains Ω_ε, whose boundaries are given, using the elliptic coordinates associated to $\mathcal{E}_{e_0,c}$, by

$$\partial\Omega_\varepsilon := \{(\mu_0 + \varepsilon\mu_1(\varphi) + O(\varepsilon^2), \varphi) : \varphi \in [0, 2\pi)\}.$$

In Lemma 21 we show that if for all ε, Ω_ε has an integrable rational caustic $\Gamma^\varepsilon_{1/q}$ of rotation number $1/q$, with $q > 2$, then

$$\langle \mu_1, c_q\rangle_{L^2} = 0, \quad \langle \mu_1, s_q\rangle_{L^2} = 0, \tag{10}$$

where $\langle\cdot,\cdot\rangle_{L^2}$ denotes the standard inner product in $L^2(\mathbb{R}/2\pi\mathbb{Z})$ and $\{c_q, s_q : q > 2\}$ are suitable *dynamical modes*, which can be explicitly defined using the action-angle coordinates; see (15). Also see Remark 22 for a more quantitative version, which we need since we are interested in perturbations of ellipses and not necessarily deformations.

Step 2 (*Elliptic motions*). In Section 6 we consider infinitesimal deformations of ellipses by homotheties, translations, rotations and hyperbolic rotations (we call them *elliptic motions* since they preserve the class of ellipses) and derive their infinitesimal generators e_h, e_{τ_1}, e_{τ_2}, e_{hr} and e_r; see (16)–(20). Moreover, in Proposition 23 we prove a certain approximation result for ellipses.

Step 3 (*Basis property*). In Section 7 we show that the collection of dynamical modes and elliptic motions form a basis of $L^2(\mathbb{R}/2\pi\mathbb{Z})$. In subsections 7.1 and 7.2 we will consider their complex extensions and study their singularities in detail; this analysis will be important to deduce their linear independence (Proposition 28). Moreover, in Proposition 33 we show that they do generate the whole $L^2(\mathbb{R}/2\pi\mathbb{Z})$, hence they form a (non-orthogonal) basis.

Step 4. (Approximation) In Section 8 we prove an approximation lemma (Lemma 34) and use it to complete the proof of the Main Theorem (see Section 8.1) by means of an approximation procedure similar to the one in [3, §8].

5. **Preservation of rational caustics**

In this section we want to investigate perturbations of ellipses, for which the associated billiard map continues to admit rationally integrable caustics corresponding to some rational rotation numbers.

Let us consider an ellipse $\mathcal{E}_{e_0,c}$ and let $\partial\Omega_\varepsilon$ be an infinitesimal perturbation of the form

$$\begin{cases} x = c\,\cosh(\mu_0 + \varepsilon\mu_1(\varphi) + O(\varepsilon^2))\cos\varphi, \\ y = c\,\sinh(\mu_0 + \varepsilon\mu_1(\varphi) + O(\varepsilon^2))\sin\varphi \end{cases} \tag{11}$$

for $\varepsilon \to 0^+$. To simplify notation we write

$$\partial\Omega_\varepsilon = \mathcal{E}_{e_0,c} + \varepsilon\mu_1 + O(\varepsilon^2),$$

which must be understood in the elliptic coordinates with semi-focal distance c.

Let us denote $\mu_\varepsilon := \varepsilon\mu_1 + O(\varepsilon^2)$ and let h_ε be the generating function of the billiard map inside Ω_ε; in particular,

$$h_\varepsilon(\varphi,\varphi') = h_0(\varphi,\varphi') + \varepsilon h_1(\varphi,\varphi') + O_{e_0,c,\|\mu_\varepsilon\|_{C^1}}(\varepsilon^2), \tag{12}$$

where h_0 denotes the generating function of the billiard map inside $\mathcal{E}_{e_0,c}$ and $O_{e_0,c,\|\mu_\varepsilon\|_{C^1}}(\varepsilon^2)$ denotes a term bounded by ε^2 times a factor depending on e_0, c, and $\|\mu_\varepsilon\|_{C^1}$. Notice that this formula makes sense only for infinitesimal perturbations.

Let us recall the following result (see [37, Cor. 9 and Prop. 11]).

PROPOSITION 20. *Assume that the billiard map associated to $\partial\Omega_\varepsilon$ has a rationally integrable caustic corresponding to rotation number, in lowest term, $p/q \in (0, 1/2)$.*

If we denote by $\{\varphi^k_{p/q}\}_{k=0}^q$ the periodic orbit of the billiard map in $\mathcal{E}_{e_0,c}$ with rotation number p/q and starting at $\varphi^0_{p/q} = \varphi$ (these orbits are all tangent to a caustic $C_{\lambda_{p/q}}$ for some $\lambda_{p/q} \in (0, b)$ — see (5)), then

$$L_1(\varphi) := \sum_{k=0}^{q-1} h_1(\varphi^k_{p/q}, \varphi^{k+1}_{p/q}) = 2\lambda_{p/q} \sum_{k=1}^{q} \mu_1(\varphi^k_{p/q}) \equiv c_{p/q}, \tag{13}$$

where $c_{p/q}$ is a constant depending only on p/q.

Let us consider rotation numbers $1/q$, with $q \geq 3$, and denote by λ_q the value of λ corresponding to the caustic of rotation number $1/q$. Similarly, k_{λ_q} denotes the associated modulus (see Proposition 16).

Therefore, with respect to the action-angle variables (9), we have that for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$,

$$\sum_{k=1}^{q} \mu_1(\varphi_{\lambda_q}(\theta + 2\pi k/q)) \equiv \text{constant}.$$

If $u(x)$ denotes either $\cos x$ and $\sin x$, then the above equality implies that

$$\int_0^{2\pi} \mu_1(\varphi_{\lambda_q}(\theta))\, u(q\,\theta)\, d\theta = 0$$

which, using the expression in (9), is equivalent to

$$\int_0^{2\pi} \mu_1\left(\text{am}\left(\frac{4K(k_{\lambda_q})}{2\pi}\,\theta; k_{\lambda_q}\right)\right) u(q\,\theta)\, d\theta = 0.$$

Now consider Now the change of coordinates

$$\varphi = \text{am}\left(\frac{4K(k_{\lambda_q})}{2\pi}\,\theta; k_{\lambda_q}\right) \quad \Longleftrightarrow \quad \theta = \frac{2\pi}{4K(k_{\lambda_q})} F(\varphi; k_{\lambda_q}).$$

Then

$$d\theta = \frac{2\pi}{4K(k_{\lambda_q})}\frac{d}{d\varphi}\left(F(\varphi;k_{\lambda_q})\right) = \frac{2\pi}{4K(k_{\lambda_q})}\frac{1}{\sqrt{1-k_{\lambda_q}^2\sin^2\varphi}},$$

and the above integral becomes

$$\int_0^{2\pi}\mu_1(\varphi)\,\frac{u\left(\frac{2\pi\,q}{4K(k_{\lambda_q})}F(\varphi;k_{\lambda_q})\right)}{\sqrt{1-k_{\lambda_q}^2\sin^2\varphi}}\,d\varphi = 0. \tag{14}$$

For each $q \geq 3$, define

$$\begin{aligned} c_q(\varphi) &:= \frac{\cos\left(\frac{2\pi\,q}{4K(k_{\lambda_q})}F(\varphi;k_{\lambda_q})\right)}{\sqrt{1-k_{\lambda_q}^2\sin^2\varphi}}, \\ s_q(\varphi) &:= \frac{\sin\left(\frac{2\pi\,q}{4K(k_{\lambda_q})}F(\varphi;k_{\lambda_q})\right)}{\sqrt{1-k_{\lambda_q}^2\sin^2\varphi}} \end{aligned} \tag{15}$$

or equivalently, in the complex form,

$$E_q(\varphi) := \frac{e^{2\pi i\frac{q}{4K(k_{\lambda_q})}F(\varphi;k_{\lambda_q})}}{\sqrt{1-k_{\lambda_q}^2\sin^2\varphi}}.$$

LEMMA 21. *Assume that the billiard map in $\partial\Omega_\varepsilon = \mathcal{E}_{e_0,c} + \varepsilon\mu_1 + O(\varepsilon^2)$ has rationally integrable caustics corresponding to rotation numbers $1/q$ for all $q \geq 3$. Then,*

$$\int_0^{2\pi}\mu_1(\varphi)\,c_q(\varphi)\,d\varphi = \int_0^{2\pi}\mu_1(\varphi)\,s_q(\varphi)\,d\varphi = 0 \qquad \forall\, q \geq 3.$$

Moreover, if we denote $\mu_\varepsilon = \varepsilon\mu_1 + O(\varepsilon^2)$, then

$$\int_0^{2\pi}\mu_\varepsilon(\varphi)\,c_q(\varphi)d\varphi = \int_0^{2\pi}\mu_\varepsilon(\varphi)\,s_q(\varphi)d\varphi = O_{e_0,c,q}(\varepsilon^2),$$

where $O_{e_0,c,q}(\varepsilon^2)$ is a term whose absolute value is bounded by ε^2 times a factor depending on e_0, c, and q.

Remark 22. It follows from [3, Lemma 13] that assuming $q < c(e)\|\mu\|^{-1/8}$, we have

$$\int_0^{2\pi}\mu_\varepsilon(\varphi)\,c_q(\varphi)d\varphi = \int_0^{2\pi}\mu_\varepsilon(\varphi)\,s_q(\varphi)d\varphi = O_{e_0,c}(q^8\|\mu\|_{C^1}^2),$$

where $O_{e_0,c}(q^8\|\mu\|_{C^1}^2)$ is a term whose absolute value is bounded by $q^8\|\mu\|_{C^1}^2$ times a factor depending on e_0, c and C^5-norm of μ.

In order to apply [3, Lemma 13] we need to translate notation: in [3, §4, pp. 7–8], action-angle variables are introduced, and in [3, middle of p. 16], X_q is defined, which coincides with what we denote φ_{λ_q}. (Compare with (9), where

λ_q is such that $\omega_{\lambda_q} = 1/q$, or with Appendix E.) With this notation, the above integral is estimated as in [3, Lemma 13]. Notice also that the Lazutkin density μ in [3, (14) on p. 14] coincides with our (37). Thus, integrating with respect to Lazutkin parametrization with Lazutkin density is the same as integrating with respect to φ.

Proof. The first part follows from (14). As for the second part, observe that

$$\int_0^{2\pi} |c_q(\varphi)|\, d\varphi = \int_0^{2\pi} \frac{\left|\cos\left(\frac{2\pi\, q}{4K(k_{\lambda_q})} F(\varphi; k_{\lambda_q})\right)\right|}{\sqrt{1 - k_{\lambda_q}^2 \sin^2\varphi}} d\varphi$$
$$= \frac{4K(k_{\lambda_q})}{2\pi q} \int_0^{2\pi q} |\cos t|\, dt = \frac{8K(k_{\lambda_q})}{\pi}.$$

In particular, recall that $e_0 < k_{\lambda_q} \le k_{\lambda_3} < 1$ for all $q \ge 3$ and that $k_{\lambda_q} \longrightarrow e_0$ as $q \to +\infty$. Hence, using the first statement of the proposition,

$$\int_0^{2\pi} \mu_\varepsilon(\varphi)\, c_q(\varphi) d\varphi = \int_0^{2\pi} O(\varepsilon^2) c_q(\varphi)\, d\varphi = O_{e_0,c,q}(\varepsilon^2). \qquad \square$$

6. **Elliptic motions**

We call translations, rotations, hyperbolic rotations, and homothety *elliptic motions*; indeed, all of these transformations keep the class of ellipses invariant.

In Appendix B, we show that infinitesimal perturbations of an ellipse $\mathcal{E}_{e_0,c}$ by these motions correspond to these functions (expressed in the elliptic coordinate frame with semi-focal distance c).

- Translations:

$$e_{\tau 1}(\varphi) := \frac{\cos\varphi}{1 - e_0^2 \cos^2\varphi}, \tag{16}$$

$$e_{\tau 2}(\varphi) := \frac{\sin\varphi}{1 - e_0^2 \cos^2\varphi}; \tag{17}$$

- Rotations:

$$e_r(\varphi) := \frac{\sin(2\varphi)}{1 - e_0^2 \cos^2\varphi}; \tag{18}$$

- Homotheties:

$$e_h(\varphi) := \frac{1}{1 - e_0^2 \cos^2\varphi}; \tag{19}$$

- Hyperbolic rotations:

$$e_{hr}(\varphi) := \frac{\cos(2\varphi)}{1 - e_0^2 \cos^2\varphi}. \tag{20}$$

We say that a strictly convex smooth domain Ω is a *deformation of an ellipse* if there exist $\mathcal{E} = \mathcal{E}(x_0, y_0, c, \mu_0, \theta)$ and a function

$$\mu_1 = \mu_1(x_0, y_0, c, \mu_0, \theta) \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$$

such that

$$\partial\Omega = \left\{ \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} c\cosh(\mu_0 + \mu_1(\varphi))\cos\varphi \\ c\sinh(\mu_0 + \mu_1(\varphi))\sin\varphi \end{pmatrix},\ \varphi \in [0, 2\pi) \right\}.$$

By abusing notation, in the following we will write

$$\partial\Omega = \mathcal{E}(x_0, y_0, c, \mu_0 + \mu_1, \theta) = \mathcal{E}(x_0, y_0, c, \mu_0, \theta) + \mu_1. \tag{21}$$

We will need the following approximation result.

PROPOSITION 23. *Let us consider the ellipse $\mathcal{E}_{e_0,c}$ and let*

$$\mu_1(\varphi) := a_0 e_h(\varphi) + a_1 e_{\tau 1}(\varphi) + b_1 e_{\tau 2}(\varphi) + a_2 e_{hr}(\varphi) + b_2 e_r(\varphi),$$

where a_0, a_1, b_1, a_2, b_2 are assumed to be sufficiently small. Then, there exist a constant $C = C(e_0, c)$ and an ellipse $\widetilde{\mathcal{E}} = \mathcal{E}_0 + \mu_{\widetilde{\mathcal{E}}}$ such that

$$\|\mu_1 - \mu_{\widetilde{\mathcal{E}}}\|_{C^1} \le C\|\mu_1\|_{C^1}^2.$$

The proof is presented in Appendix B.

7. An adapted basis for $L^2(\mathbb{T})$

In this section we want to determine a suitable basis of $L^2(\mathbb{T})$, where hereafter $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. This basis will be constructed by means of elliptic motions $\{e_h, e_{\tau 1}, e_{\tau 2}, e_{hr}, e_r\}$ (see (16)–(20)) and the functions $\{c_q, s_q\}_{q\ge 3}$ defined in (15).

In order to prove their linear independence, we need to consider their analytic extension to $\mathbb{C}$ and study their singularities.

7.1. *Analyticity properties of c_q and s_q.* Let us start by considering the complex extensions of the functions $\{c_q, s_q\}_{q\ge 3}$ defined in (15),

$$\begin{aligned} c_q(z) &:= \frac{\cos\left(\frac{2\pi q}{4K(k_q)} F(z; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2 z}}, \\ s_q(z) &:= \frac{\sin\left(\frac{2\pi q}{4K(k_q)} F(z; k_q)\right)}{\sqrt{1 - k_q^2 \sin^2 z}} \end{aligned} \tag{22}$$

where, to simplify the notation, we have denoted $k_q := k_{\lambda_q}$. In particular, k_q represents the eccentricity of the caustic $C_q := C_{\lambda_q}$ with rotation number $1/q$; moreover, $k_q \in (e_0, 1)$ for all $q \ge 3$ (e_0 denotes the eccentricity of the boundary), it is strictly decreasing in q, and $k_q \longrightarrow e_0$ as $q \to +\infty$. Denote $\rho_{k_q} = \operatorname{arcosh}\left(\frac{1}{k_q}\right)$ and $\rho_0 = \operatorname{arcosh}\left(\frac{1}{e_0}\right)$.

We are interested in the complex extensions of these functions and in their singularities.

PROPOSITION 24. *For $q \geq 3$, the functions c_q and s_q have an holomorphic extension to the complex strip $\Sigma_q = \left\{z \in \mathbb{C} : \ |\mathrm{Im}(z)| < \rho_{k_q}\right\}$. This extension is maximal in the sense that these functions have singularities at $\frac{\pi}{2} + \pi n \pm i\rho_{k_q}$ (which are ramification singularities).*

This proposition will be proven in Appendix C.1.

Remark 25. Observe that ρ_{k_q} is a strictly increasing function as a function of q and $\rho_{k_3} \leq \rho_{k_q} \longrightarrow \mathrm{arcosh}\,(1/e_0) = \rho_0$ as $q \to +\infty$. Moreover, since $k_3(e_0)$ is a strictly increasing function of e_0 and $k_3(e_0) \longrightarrow 1$ as $e_0 \to 1^-$, then $\rho_{k_3}(e_0)$ is a strictly decreasing function of e_0 and $\rho_{k_3}(e_0) \longrightarrow 0$ as $e_0 \to 1^-$.

7.2. *Analyticity properties of $e_{\tau 1}$, $e_{\tau 2}$, e_r, e_h and e_{hr}.* Now let us discuss the analyticity properties of the complex extensions of the elliptic motions defined in (16)–(20):

$$
\begin{aligned}
e_h(z) &:= \frac{1}{1 - e_0^2 \cos^2 z}, \\
e_{\tau 1}(z) &:= \frac{\cos z}{1 - e_0^2 \cos^2 z} = e_h(z)\, \cos z, \\
e_{\tau 2}(z) &:= \frac{\sin z}{1 - e_0^2 \cos^2 z} = e_h(z)\, \sin z, \\
e_r(z) &:= \frac{\sin(2z)}{1 - e_0^2 \cos^2 z} = e_h(z)\, \sin(2z), \\
e_{hr}(z) &:= \frac{\cos(2z)}{1 - e_0^2 \cos^2 z} = e_h(z)\, \cos(2z).
\end{aligned}
$$

The analyticity and the singularities of these functions are the same as those of $e_h(z)$. More specifically,

PROPOSITION 26. *The function $e_h(z)$ is analytic except at the following singular points (which are poles):*

$$\zeta_n = n\pi \pm i\,\rho_0 \qquad \text{for } n \in \mathbb{Z}.$$

In particular, its maximal strip of analyticity is given by

$$\Sigma_{\rho_0} = \{z \in \mathbb{C} : \ |\mathrm{Im}(z)| < \rho_0\}.$$

We will prove this proposition in Appendix C.2.

Remark 27. Since $\rho_0 > \rho_{k_q}$ for any $q \geq 3$, we conclude that $e_{\tau 1}$, $e_{\tau 2}$, e_r, e_h, e_{hr} cannot be generated as a finite linear combination of functions $s_q(z)$ and $c_q(z)$ with $q \geq 3$.

7.3. *Linear independence.* It follows from the discussion in Sections 7.1 and 7.2 that looking at singularities of these functions, it is possible to deduce the following proposition.

PROPOSITION 28. *The functions* e_h, $e_{\tau 1}$, $e_{\tau 2}$, e_{hr}, e_r, $\{s_q\}_{q\geq 3}$ *and* $\{c_q\}_{q\geq 3}$ *are linearly independent, namely, none of them can be written as a finite linear combination of the others.*

Proof. Clearly, $e_{\tau 1}$, $e_{\tau 2}$, e_r, e_h, e_{hr} are linear independent. Looking at the singularities of the complex extensions of these functions, it follows that

- e_h, $e_{\tau 1}$, $e_{\tau 2}$, e_{hr}, e_r cannot be generated as a finite linear combination of s_q and c_q with $q \geq 3$;
- for any $q_0 \geq 3$, s_{q_0} and c_{q_0} cannot be generated as a finite linear combination of $e_{\tau 1}$, $e_{\tau 2}$, e_r, e_h, e_{hr}, $\{s_q\}_{q\neq q_0}$ and $\{c_q\}_{q\neq q_0}$. □

Remark 29. A more subtle and delicate issue is to understand whether these function can be obtained as infinite combinations of the others. This matter is related to our discussion in Section 7.5 and in Appendix D.

7.4. *Weighted* $L^2(\mathbb{T})$ *space.* Let us denote by $\|\cdot\|_{L^2_{e_0}}$ the L^2-norm induced by the inner product with weight $w_{e_0}(\varphi) := (1 - e_0^2 \cos^2 \varphi)$, i.e.,

$$\langle f, g\rangle_{L^2_{e_0}} := \langle w_{e_0} f, w_{e_0} g\rangle_{L^2}.$$

For $0 \leq e_0 < 1$, this norm is clearly equivalent to the usual L^2-norm; in fact for each $f \in L^2(\mathbb{T})$, we have

$$(1 - e_0^2)\|f\|_{L^2} \leq \|f\|_{L^2_{e_0}} \leq \|f\|_{L^2}.$$

We denote by $L^2_{e_0}(\mathbb{T})$ the space $L^2(\mathbb{T})$ equipped with $\|\cdot\|_{L^2_{e_0}}$. Clearly, with the choice of this weighted norm, the functions $e_h, e_{\tau 1}, e_{\tau 2}, e_r, e_{hr}$ are mutually orthogonal in $L^2_{e_0}$. (Observe in fact that when multiplied by the weight, they become $\cos(k\varphi)$ for some $k = 0, 1, 2$ or $\sin(k\varphi)$ for some $k = 1, 2$.)

In particular,

$$\|e_h\|_{L^2_{e_0}} = \sqrt{2\pi}, \qquad \|e_{\tau 1}\|_{L^2_{e_0}} = \|e_{\tau 2}\|_{L^2_{e_0}} = \|e_r\|_{L^2_{e_0}} = \|e_{hr}\|_{L^2_{e_0}} = \sqrt{\pi}.$$

On the other hand, for $q \geq 3$,

$$\begin{aligned}
\|c_q\|^2_{L^2_{e_0}} &= \int_0^{2\pi} \frac{\cos^2\left(\frac{2\pi\, q}{4K(k_{\lambda_q})} F(\varphi; k_{\lambda_q})\right)}{1 - k^2_{\lambda_q} \sin^2 \varphi} (1 - e_0^2 \cos^2 \varphi)^2 d\varphi \\
&= 4K(k_{\lambda_q}) \int_0^1 \cos^2 (2\pi\, q\, \xi)\, \frac{(1 - e_0^2 \cos^2 \varphi(\xi))^2}{\sqrt{1 - k^2_{\lambda_q} \sin^2 \varphi(\xi)}}\, d\xi \\
&= \frac{2K(k_{\lambda_q})}{q\pi} \int_0^{2\pi q} \cos^2 t\, \frac{(1 - e_0^2 \cos^2 \varphi(\xi(t)))^2}{\sqrt{1 - k^2_{\lambda_q} \sin^2 \varphi(\xi(t))}}\, dt.
\end{aligned}$$

Hence,

$$(1-e_0^2)^2 2K(k_{\lambda_q}) \le \|c_q\|^2_{L^2_{e_0}} \le \frac{2K(k_{\lambda_q})}{\sqrt{1-k^2_{\lambda_q}}}.$$

In particular, using that $K(\cdot)$ is an increasing function and k_{λ_q} is decreasing with respect to q, we can obtain uniform bounds

$$(1-e_0^2)^2 2K(e_0) \le \|c_q\|^2_{L^2_{e_0}} \le \frac{2K(k_{\lambda_3})}{\sqrt{1-k^2_{\lambda_3}}} \qquad \forall q \ge 3;$$

similarly, for the functions s_q.

In order to simplify our notation, hereafter we will denote

$$\mathbb{e}_0 := \frac{e_h}{\sqrt{2\pi}}, \quad \mathbb{e}_1 := \frac{e_{\tau 2}}{\sqrt{\pi}}, \quad \mathbb{e}_2 := \frac{e_{\tau 1}}{\sqrt{\pi}}, \quad \mathbb{e}_3 := \frac{e_r}{\sqrt{\pi}} \quad \mathbb{e}_4 := \frac{e_{hr}}{\sqrt{\pi}} \tag{23}$$

and

$$\mathbb{e}_{2k} := \frac{c_k}{\|c_k\|_{L^2_{e_0}}}, \quad \mathbb{e}_{2k-1} := \frac{s_k}{\|s_k\|_{L^2_{e_0}}} \qquad \forall\, k \ge 3. \tag{24}$$

The family $\{\mathbb{e}_k\}_{k=0}^{+\infty}$ consists of linearly independent normal vectors in $L^2_{e_0}$. We want to show that they are a basis.

7.5. *Basis property.* In this subsection we want to prove that $\{\mathbb{e}_k\}_{k\ge 0}$ form a basis of $L^2_{e_0}(\mathbb{T})$, or equivalently of $L^2(\mathbb{T})$. We need to show that they form a complete set of generators.

Let us start with the following proposition.

PROPOSITION 30. *Let $q_0 \ge 3$; then*

$$\langle \{\mathbb{e}_k\}_{0\le k\le 2q_0}\rangle \cap \overline{\langle \{\mathbb{e}_k\}_{k>2q_0}\rangle} = \{0\}.$$

The proof of this proposition is postponed to Appendix D.

Let us now introduce the linear map $\mathcal{L}_{q_0} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined by mapping the standard Fourier basis into the following functions:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \longmapsto \frac{1}{\sqrt{2\pi}},\\
&\frac{1}{\sqrt{\pi}}\cos(q\varphi) \longmapsto \frac{1}{\sqrt{\pi}}\cos(q\varphi) && \text{for } 0<q\le q_0,\\
&\frac{1}{\sqrt{\pi}}\sin(q\varphi) \longmapsto \frac{1}{\sqrt{\pi}}\sin(q\varphi) && \text{for } 0<q\le q_0,\\
&\frac{1}{\sqrt{\pi}}\cos(q\varphi) \longmapsto c_q(\varphi) && \text{for } q>q_0,\\
&\frac{1}{\sqrt{\pi}}\sin(q\varphi) \longmapsto s_q(\varphi) && \text{for } q>q_0.
\end{aligned} \tag{25}$$

LEMMA 31. *Suppose there is $q_0 \geq 3$ such that the linear map $\mathcal{L}_{q_0}$ is invertible. Then, $\{\mathbb{e}_k\}_{k\geq 0}$ is a basis of $L^2(\mathbb{T})$.*

Proof. Since the corresponding linear map is invertible, then the collection

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(q\varphi), \frac{1}{\sqrt{\pi}}\sin(q\varphi)\right\}_{0<q\leq q_0} \cup \{c_q(\varphi), s_q(\varphi)\}_{q>q_0}$$

also forms a basis and, in particular, it spans the whole space $L^2(\mathbb{T})$. This implies that the subspace

$$\overline{\langle\{c_q, s_q\}_{q>q_0}\rangle} = \overline{\langle\{\mathbb{e}_k\}_{k>2q_0}\rangle}$$

has codimension $2q_0 + 1$.

It follows from Proposition 28 that the subspace spanned by $\{\mathbb{e}_k\}_{0\leq k\leq 2q_0}$ has dimension $2q_0 + 1$ and from Proposition 30 that

$$\overline{\langle\{\mathbb{e}_k\}_{k>2q_0}\rangle} \cap \langle\{\mathbb{e}_k\}_{0\leq k\leq 2q_0}\rangle = \{0\}.$$

We can conclude from this that

$$\overline{\langle\{\mathbb{e}_k\}_{k>0}\rangle} = L^2(\mathbb{T}).$$

Hence, $\{\mathbb{e}_k\}_{k\geq 0}$ form a set of generators of $L^2(\mathbb{T})$ and therefore a basis. □

The problem now reduces to show that the linear map $\mathcal{L}_{q_0}$, defined by (25), is invertible for some $q_0 \geq 3$.

For $q \in \mathbb{Z}_+$ and $j \geq 3$, let us consider the elements of the (infinite) correlation matrix $\widetilde{A} = (\tilde{a}_{i,h})_{i,h=0}^{\infty}$, whose entries are

$$\begin{aligned}
\tilde{a}_{2q,2j} &:= \langle\cos(q\varphi), c_j\rangle_{L^2},\\
\tilde{a}_{2q,2j+1} &:= \langle\cos(q\varphi), s_j\rangle_{L^2},\\
\tilde{a}_{2q+1,2j} &:= \langle\sin(q\varphi), c_j\rangle_{L^2},\\
\tilde{a}_{2q+1,2j+1} &:= \langle\sin(q\varphi), s_j\rangle_{L^2}.
\end{aligned} \tag{26}$$

LEMMA 32. *There exists $\rho = \rho(e_0, c) > 0$ such that for all $q \in \mathbb{N}$ and $j \geq 6$,*

$$\tilde{a}_{q,j} = 2K(k_{[j/2]})\,\delta_{q,j} + O_{e_0,c}\left(j^{-1}\,e^{-\rho\,|q-j|}\right),$$

where $[\cdot]$ denotes the integer part, $\delta_{q,j}$ the Dirac's delta, and $O_{e_0,c}(*)$ means that the absolute value of the corresponding term is bounded by $*$ times a factor depending only on e_0 and c.

The proof of the above lemma will be given in Appendix E.

PROPOSITION 33. *There exists $q_0 = q_0(e_0, c) \geq 3$ such that $\mathcal{L}_{q_0}$ is invertible as an operator acting on $L^2(\mathbb{T})$. In particular, it follows from Lemma 31 that $\{\mathbb{e}_k\}_{k\geq 0}$ is a basis of $L^2(\mathbb{T})$.*

Proof. Let us show that there exists $q_0 = q_0(e_0, c) \geq 3$ such that the linear map $\mathcal{L}_{q_0}$ is invertible. Consider the infinite dimensional matrix $A = (a_{q,j})_{q,j}$ defined as

$$a_{q,j} = \begin{cases} \delta_{q,j} & \text{if } j < 2q_0,\ q \geq 0, \\ \frac{1}{\sqrt{2\pi}} \tilde{a}_{0,j} & \text{if } j \geq 2q_0, \\ \frac{1}{\sqrt{\pi}} \tilde{a}_{q,j} & \text{if } j \geq 2q_0,\ q \geq 1. \end{cases} \tag{27}$$

Using Lemma 32 and the fact that $K(k_j) \geq K(e_0) > 0$ for all $j \geq 3$, we obtain

$$|a_{q,q}| \geq \min \left\{ 1, \frac{2}{\sqrt{\pi}}\ K(e_0) + O_{e_0,c}\left(\frac{1}{q\, e^{\rho}}\right) \right\}.$$

Observe that since $K(e_0) \geq \frac{\pi}{2}$ for $0 \leq e_0 < 1$, then if one chooses q_0 sufficiently large, the above minimum is achieved by 1.

Denote by $\mathcal{D}_{q_0}$ the diagonal linear operator given by the diagonal elements of $\mathcal{L}_{q_0}$. Notice that $\mathcal{D}_{q_0}$ is invertible and has bounded norm of the inverse; in particular, for q_0 sufficiently large, $\|\mathcal{D}_{q_0}^{-1}\|_2 \leq 1$. Again using Lemma 32 we also have for each $q \geq 0$,

$$\sum_{q \geq q_0} \sum_{j=0, j \neq q}^{\infty} |a_{q,j}|^2 \leq \frac{C}{q_0}$$

for some suitable constant $C = C(e_0, c)$. For any predetermined $\delta = \delta(e_0, c) > 0$, by choosing q_0 large enough we obtain

$$\sum_{q \geq q_0} \sum_{j=0, j \neq q}^{\infty} |a_{q,j}|^2 < \delta(e_0, c). \tag{28}$$

Using Cauchy-Schwarz, (28) implies that with respect to the L^2-norm $\|\cdot\|_2$ we have

$$\|\mathcal{L}_{q_0} - \mathcal{D}_{q_0}\|_2 \leq \delta(e_0, c) \leq \frac{1}{2} \leq \frac{1}{2} \|\mathcal{D}_{q_0}^{-1}\|_2^{-1}.$$

This implies that $\mathcal{L}_{q_0}$ is invertible and concludes the proof. □

8. **Proof of the Main Theorem**

In this section we prove our Main Theorem. Let us first start by stating and proving the following approximation lemma similar to [3, Lemma 24].

Lemma 34 (Approximation Lemma). *Let us consider the ellipse $\mathcal{E}_{e_0,c}$ and let $\partial\Omega$ be a rationally integrable C^{39}-deformation of $\mathcal{E}_{e_0,c}$, identified by a C^{39} function μ, i.e., $\partial\Omega = \mathcal{E}_{e_0,c} + \mu$. For every $L > 0$, there exists a constant $C = C(e_0, c, L)$ such that if $\|\mu\|_{C^{39}} \leq L$, then the following holds. There exist an ellipse $\overline{\mathcal{E}} = \overline{\mathcal{E}}(\bar{x}_0, \bar{y}_0, \bar{c}, \bar{\mu}_0, \bar{\theta})$ and a function $\overline{\mu} = \overline{\mu}(\overline{\varphi})$ (where $\overline{\varphi}$ is the*

angle with respect to the elliptic coordinate frame associated to $\overline{\mathcal{E}}$), *such that* $\partial\Omega = \overline{\mathcal{E}} + \overline{\mu}$ (*see* (21)) *and*

$$\|\overline{\mu}\|_{C^1} \leq C(e_0, c, L)\|\mu\|_{C^1}^{703/702}.$$

Proof. Let us consider the basis $\mathcal{B}_{e_0} := \{\mathtt{e}_j\}_{j\geq 0}$ of $L^2_{e_0}(\mathbb{T})$, introduced in (23) and (24); moreover, we denote by

$$\mathcal{V}_{e_0} := \langle \mathtt{e}_0, \mathtt{e}_1, \mathtt{e}_2, \mathtt{e}_3, \mathtt{e}_4 \rangle$$

the 5-dimensional space generated by elliptic motions. Let us decompose

$$\mu = \mu_{\mathcal{V}_{e_0}} + \mu^{\perp},$$

where $\mu^{\perp}$ is orthogonal in $L^2_{e_0}(\mathbb{T})$ to the subspace $\mathcal{V}_{e_0}$ and $\mu_{\mathcal{V}_{e_0}} := \sum_{j=0}^4 a_j \mathtt{e}_j \in \mathcal{V}_{e_0}$. Using the orthogonality in $L^2_{e_0}(\mathbb{T})$ and the fact that $\mathcal{B}_{e_0}$ is a basis, we obtain

$$\|\mu_{\mathcal{V}_{e_0}}\|^2_{L^2_{e_0}} + \|\mu^{\perp}\|^2_{L^2_{e_0}} = \|\mu\|^2_{L^2_{e_0}} \leq C\|\mu\|^2_{C^1}$$

for some $C = C(e_0, c)$. This implies that $a_j = O_{e_0,c}(\|\mu\|_{C^1})$ for $0 \leq j \leq 4$; since the functions $\mathtt{e}_j$ are analytic, we obtain

$$\|\mu_{\mathcal{V}_{e_0}}\|_{C^k} \leq C(e_0, c, k)\|\mu\|_{C^1}. \tag{29}$$

We claim that

$$\|\mu^{\perp}\|_{C^1} \leq C(e_0, c, \|\mu\|_{C^k})\|\mu\|_{C^1}^{1+\delta}, \tag{30}$$

where the above constant depends monotonically on $\|\mu\|_{C^k}$, and δ will turn out to be equal to $1/702$. This is enough to complete the proof. In fact, applying Proposition 23 with $\mathcal{E}_{e_0,c}$ and $\mu_{\mathcal{V}_{e_0}}$, we obtain an ellipse $\widetilde{\mathcal{E}} = \mathcal{E}_{e_0,c} + \mu_{\widetilde{\mathcal{E}}}$ such that

$$\|\mu_{\mathcal{V}_{e_0}} - \mu_{\widetilde{\mathcal{E}}}\|_{C^1} \leq C\|\mu_{\mathcal{V}_{e_0}}\|^2_{C^1} \leq C\|\mu\|^2_{C^1},$$

where the last inequality follows from (29). We choose $\overline{\mathcal{E}} := \widetilde{\mathcal{E}}$; if we consider $\partial\Omega = \overline{\mathcal{E}} + \overline{\mu}$, then we conclude from Lemma 36 that

$$\begin{aligned}\|\overline{\mu}\|_{C^1} &\leq C(e_0, c)\|\mu - \mu_{\widetilde{\mathcal{E}}}\|_{C^1} = C(e_0, c)\|\mu_{\mathcal{V}_{e_0}} + \mu^{\perp} - \mu_{\widetilde{\mathcal{E}}}\|_{C^1} \\ &\leq C(e_0, c)\left(\|\mu_{\mathcal{V}_{e_0}} - \mu_{\widetilde{\mathcal{E}}}\| + \|\mu^{\perp}\|_{C^1}\right) \\ &\leq C(e_0, c, \|\mu\|_{C^k})\,\|\mu\|_{C^1}^{1+\delta}.\end{aligned}$$

Therefore, let us prove (30). Let us define the Fourier coefficients

$$\widehat{\mu}^{\perp}_j := \langle \mu^{\perp}, \mathtt{e}_j \rangle_{e_0},$$

which are clearly zero for $j = 0, \ldots, 4$ (due to orthogonality). In particular, we have (see, for example, [3, Cor. 23])

$$\|\mu^{\perp}\|^2_{L^2_{e_0}} \leq C(e_0, c)\sum_{j=5}^{\infty}\left|\widehat{\mu}^{\perp}_j\right|^2.$$

It follows from Lemma 21 and Remark 22 that

$$\left|\widehat{\mu}_j^{\perp}\right| = O_{e_0,c}(q^8\|\mu\|_{C^1}^2).$$

Fix some positive $\alpha < 1/8$. Choose $q_0 = [\|\mu\|_{C^1}^{-\alpha}]$, where $[\cdot]$ denotes the integer part and $\alpha > 0$ will be determined in the following. Below, $C(e_0, c)$ denotes a constant depending on e_0 and c. Using the above estimates we get for $5 \le q \le q_0$,

$$\left|\widehat{\mu}_j^{\perp}\right| \le C(e_0,c)q^8\|\mu\|_{C^1}^2 \le C(e_0,c)\|\mu\|_{C^1}^{2-8\alpha}.$$

Then, summing over $5 \le q \le q_0$, we obtain

$$\sum_{q=5}^{q_0} \left|\widehat{\mu}_j^{\perp}\right|^2 \le C(e_0,c)\|\mu\|_{C^1}^{4-17\alpha}.$$

On the other hand, Lemma 49 gives

$$\left|\mu_j^{\perp}\right|^2 \le C(e_0,c)\frac{\|\mu\|_{C^1}^2}{q^2}.$$

Therefore, summing over $q > q_0$ we conclude that

$$\sum_{q=q_0+1}^{+\infty} \left|\widehat{\mu}_j^{\perp}\right|^2 \le C(e_0,c)\|\mu\|_{C^1}^{2+\alpha}.$$

Combining the two above estimates and optimizing for α (i.e., choosing $\alpha = 1/9$), we conclude that $\left|\widehat{\mu}_j^{\perp}\right| \le C(e_0,c)\|\mu\|_{C^1}^{19/18}$.

Now, observe that

$$\|\mu^{\perp}\|_{C^1} \le \|D\mu^{\perp}\|_{L^1} + \|D^2\mu^{\perp}\|_{L^1} \le \|D\mu^{\perp}\|_{L^2} + \|D^2\mu^{\perp}\|_{L^2}.$$

Using standard Sobolev interpolation inequalities (see, for example, [13]), for any $\delta > 0$ and any $1 \le j \le 2$, we have

$$\|D^j\mu^{\perp}\|_{L^2} \le C\left(\Delta\|\mu^{\perp}\|_{C^k} + \Delta^{-j/(k-j)}\|\mu^{\perp}\|_{L^2}\right).$$

Optimizing the above estimate, we choose $\Delta = \|\mu\|_{C^1}^{703/702}$.

Using the above estimates and the fact that $\|\cdot\|_{L^2_{e_0}}$ is equivalent to $\|\cdot\|_{L^2}$, we conclude that (30) holds, by taking $\delta = \frac{1}{702}$. □

8.1. *Proof of the Main Theorem.* First of all, observe that up to applying a rotation and a translation (which do not alter rational integrability, nor the other hypotheses), we can assume that $\mathcal{E}_0 = \mathcal{E}_{e_0,c}$.

Let us denote by $\mathcal{E}\ell\ell_\sigma(\mathcal{E}_0)$ the set of ellipses whose Hausdorff distance from $\mathcal{E}_0$ is not larger than σ:

$$\mathcal{E}\ell\ell_\sigma(\mathcal{E}_0) = \left\{\mathcal{E}' \subset \mathbb{R}^2 : \operatorname{dist}_H(\mathcal{E}', \mathcal{E}_0) \le \sigma\right\},$$

where σ is sufficiently small (to be determined).

Let us denote by $\mathcal{P}_\sigma(\mathcal{E}_0)$ the set of parameters corresponding to ellipses in $\mathcal{E}\ell\ell_\sigma(\mathcal{E}_0)$:

$$\mathcal{P}_\sigma(\mathcal{E}_0) := \left\{(x, y, c, \mu, \theta) \in \mathbb{R}^2 \times (0, +\infty)^2 \times [0, \pi) : \ \mathcal{E}(x, y, c, \mu, \theta) \in \mathcal{E}\ell\ell_\sigma(\mathcal{E}_0)\right\}.$$

Then, $\mathcal{P}_\sigma(\mathcal{E}_0)$ is compact in $\mathbb{R}^2 \times (0, +\infty)^2 \times [0, \pi)$. Notice that the size of this set is independent of ε.

Let μ be a C^k perturbation, with $\|\mu\|_{C^k} < K$ and $\|\mu\|_{C^1} < \varepsilon$, and consider the domain given by

$$\partial\Omega = \mathcal{E}_0 + \mu.$$

Observe that there exists a constant $M = M(e_0, c, K)$ such that if $\mathcal{E} \in \mathcal{E}\ell\ell_\sigma(\mathcal{E}_0)$ and $\partial\Omega = \mathcal{E} + \widetilde{\mu}$, then

$$\operatorname{dist}_H(\mathcal{E}, \partial\Omega) \leq M\|\widetilde{\mu}\|_{C^0}. \tag{31}$$

For any $\nu \in \mathcal{P}_\sigma(\mathcal{E}_0)$, let us denote by $\mathcal{E}_\nu$ the corresponding ellipse and by μ_ν the perturbation such that $\partial\Omega = \mathcal{E}_\nu + \mu_\nu$. Observe that the elliptic coordinate frame corresponding to $\mathcal{E}_\nu$ varies analytically with respect to ν; hence, μ_ν also changes analytically with respect to ν. In particular, we can assume ε sufficiently small so that for any $\nu \in \mathcal{P}_\sigma(\mathcal{E}_0)$, we have $\|\mu_\nu\|_{C^k} < 2K$.

The function $\nu \longmapsto \|\mu_\nu\|_{C^1}$ is, therefore, continuous and, being $\mathcal{P}_\sigma(\mathcal{E}_0)$ compact, it achieves a minimum at some $\nu^* \in \mathcal{P}_\sigma(\mathcal{E}_0)$.

$$0 \leq \|\mu_{\nu^*}\|_{C^1} \leq \|\mu\|_{C^1} < \varepsilon.$$

Let us assume that $\|\mu_{\nu^*}\|_{C^1} \neq 0$ and apply Lemma 34 to $\mathcal{E}_{\nu^*}$ and μ_{ν^*}, thus obtaining $\overline{\mathcal{E}}_{\nu^*}$ and $\overline{\mu}_{\nu^*}$, such that

$$\|\overline{\mu}_{\nu^*}\|_{C^1} \leq C\|\mu_{\nu^*}\|_{C^1}^{1+\delta} < \|\mu_{\nu^*}\|_{C^1}, \tag{32}$$

where we have assumed ε to be sufficiently small. Notice that as $\|\mu_{\nu^*}\|_{C^1}$ decreases, $\|\overline{\mu}_{\nu^*}\|_{C^1}$ decreases. Therefore, ε is small enough, $\bar{\mathcal{E}}$ from Lemma 34 belongs to the set $\mathcal{P}_\sigma(\mathcal{E}_0)$, which has non-emtpy interior and is independent of ε.

Using the triangle inequality, for sufficiently small ε, we have

$$\begin{aligned}\operatorname{dist}_H(\mathcal{E}_0, \overline{\mathcal{E}}_{\nu^*}) &\leq \operatorname{dist}_H(\mathcal{E}_0, \partial\Omega) + \operatorname{dist}_H(\partial\Omega, \overline{\mathcal{E}}_{\nu^*}) \\ &\leq 2M\varepsilon \leq \sigma.\end{aligned}$$

Hence $\overline{\mathcal{E}}_{\nu^*} \in \mathcal{E}\ell\ell_\sigma(\mathcal{E}_0)$, and therefore $\overline{\mathcal{E}}_{\nu^*} = \mathcal{E}_{\overline{\nu}^*}$ for some $\overline{\nu}^* \in \mathcal{P}_\sigma(\mathcal{E}_0)$. This and (32) contradict the minimality of ν^* in $\mathcal{P}_\sigma(\mathcal{E}_0)$. As a consequence, $\mu_{\nu^*} \equiv 0$ and therefore $\partial\Omega \in \mathcal{P}_\sigma(\mathcal{E}_0)$. □

Appendix A. Parametrizing ellipses

Let us consider the ellipse

$$\mathcal{E}_{e_0,c} = \left\{(x,y) \in \mathbb{R}^2 : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$$

centered at the origin and with semi-axes of lengths, respectively, $0 < b \le a$; in particular, as before, $e_0 = \sqrt{1 - \frac{b^2}{a^2}} \in [0,1)$ denotes its eccentricity, while $c = \sqrt{a^2 - b^2}$ denotes the semi-focal distance.

We want to recall various parametrizations of ellipses that have been mentioned and used in the proofs.

- *Polar coordinates*: $(r,\varphi) \in (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}$:

$$\mathcal{E}_{e_0,c} : \begin{cases} x = a\cos\varphi, \\ y = b\sin\varphi. \end{cases}$$

  Observe that this choice parametrizes the ellipse counterclockwise, with $(x(0), y(0)) = (a, 0)$.

  In these coordinates the radius of curvature of the ellipse is given by

$$\begin{aligned} \rho(\varphi) &= \left| \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \right| = \frac{(a^2\sin^2\varphi + b^2\cos^2\varphi)^{\frac{3}{2}}}{ab} \\ &= \frac{a^2}{b}(1 - e_0^2\sin^2\varphi)^{\frac{3}{2}}. \end{aligned} \tag{33}$$

- *Arc-length parametrization*:

$$\mathcal{E}_{e_0,c} : \begin{cases} x = x(s), \\ y = y(s) \end{cases} \qquad \text{for } s \in [0, |\mathcal{E}_{e_0,c}|),$$

  where $|\mathcal{E}_{e_0,c}|$ denotes the perimeter of $\mathcal{E}_{e_0,c}$ and we fix, for example, the starting point at $(x(0), y(0)) = (a, 0)$ and the counterclockwise orientation. In terms of the polar coordinate φ, we have

$$s(\varphi) = a\int_0^\varphi \sqrt{1 - e_0^2\sin^2\varphi}\, d\varphi, \tag{34}$$

  from which

$$\frac{ds(\varphi)}{d\varphi} = a\sqrt{1 - e_0^2\sin^2\varphi}. \tag{35}$$

  In particular, the perimeter of $\mathcal{E}_{e_0,c}$ can be computed quite explicitly:

$$\begin{aligned} |\mathcal{E}_{e_0,c}| &= \int_0^{2\pi} \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}\, d\varphi = a\int_0^{2\pi} \sqrt{1 - e_0^2\sin^2\varphi}\, d\varphi \\ &= 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - e_0^2\sin^2\varphi}\, d\varphi =: 4a\, E(e_0), \end{aligned}$$

where $E(e_0) := \int_0^{\frac{\pi}{2}} \sqrt{1 - e_0^2 \sin^2 \varphi}\, d\varphi$ is called *complete elliptic integral of the second type*; see, for instance, [1].

- *Elliptic polar coordinates*: See Section 2.1.
- *Lazutkin parametrization*: Following an idea by Lazutkin in [25], let us introduce the following reparametrization

$$x_\ell(s) := C_\ell^{-1} \int_0^s \rho^{-\frac{2}{3}}(\tau) d\tau, \tag{36}$$

where s denotes the arc-length parameter, ρ the radius of curvature computed in (33), and $C_\ell := \int_0^{|\mathcal{E}_{e_0,c}|} \rho^{-\frac{2}{3}}(\tau) d\tau$ is a normalizing factor so that $x_\ell(|\mathcal{E}_{e_0,c}|) = 1$ (sometimes it is called the *Lazutkin perimeter*).

Observe that using (33), (35), and (36), we obtain x_ℓ as a function of the polar angular coordinate φ:

$$\begin{aligned} x_\ell(\varphi) &= C_\ell^{-1} \int_0^\varphi \rho^{-\frac{2}{3}}(s(\varphi)) \frac{ds(\varphi)}{d\varphi}\, d\varphi \\ &= C_\ell^{-1} \frac{b^{\frac{2}{3}}}{a^{\frac{1}{3}}} \int_0^\varphi \frac{d\varphi}{\sqrt{1 - e_0^2 \sin^2 \varphi}}. \end{aligned}$$

In particular,

$$\frac{dx_\ell(\varphi)}{d\varphi} = \frac{b^{\frac{2}{3}}}{a^{\frac{1}{3}}} \frac{C_\ell^{-1}}{\sqrt{1 - e_0^2 \sin^2 \varphi}}. \tag{37}$$

Remark 35. For any smooth strictly convex domain Ω, let us denote by $|\partial\Omega|$ the perimeter of Ω. Let us consider the *Lazutkin change of coordinates* $L_\Omega : [0, |\partial\Omega|) \times [0, \pi] \longrightarrow \mathbb{R}/\mathbb{Z} \times [0, \delta]$:

$$(s, \varphi) \longmapsto \left(x = C_\Omega^{-1} \int_0^s \rho^{-2/3}(s) ds,\ y = 4 C_\Omega^{-1} \rho^{1/3}(s)\ \sin \varphi/2 \right),$$

where $C_\Omega := \int_0^{|\partial\Omega|} \rho^{-2/3}(s) ds$ and $\delta > 0$ is sufficiently small.

In these new coordinates the billiard map becomes very simple (see [25]):

$$f_{L_\Omega}(x, y) = \Big(x + y + O(y^3), y + O(y^4) \Big) \tag{38}$$

In particular, near the boundary $\{\varphi = 0\} = \{y = 0\}$, the billiard map f_{L_Ω} reduces to a small perturbation of the integrable map $(x, y) \longmapsto (x + y, y)$. Using this result and the KAM theorem, Lazutkin proved in [25] that if $\partial\Omega$ is sufficiently smooth (smoothness is determined by the KAM theorem), then there exists a positive measure set of caustics (which correspond to KAM invariant curves), which accumulates on the boundary and on which the motion is smoothly conjugate to a rigid rotation with irrational rotation number.

Appendix B. Elliptic motions and a proof of Proposition 23

We start by studying perturbations of ellipses within the family of ellipses. Once enough analytic tools are developed, we prove Proposition 23. Up to suitable translation and rotation, we can assume — using the parametrization introduced in (6) — that the unperturbed ellipse has the form $\mathcal{E}_{e_0,c} = \mathcal{E}(0,0,c,\mu_0,0)$; in particular, its eccentricity is $e_0 = 1/\cosh\mu_0$.

Perturbing by an homothety. Let $\lambda \in \mathbb{R}$, and consider an homothety of factor e^λ. We want to write the dilated/contracted ellipse $\mathcal{E}_\lambda := e^\lambda \mathcal{E}_{e_0,c}$ as

$$\mathcal{E}_\lambda = \mathcal{E}_{e_0,c} + \mu_\lambda,$$

which is equivalent to

$$\mathcal{E}(0,0,e^\lambda c,\mu_0,0) = \mathcal{E}(0,0,c,\mu_0+\mu_\lambda,0).$$

Hence, we have to solve the following system of equations:

$$\begin{cases} c\cosh(\mu_0+\mu_\lambda(\varphi))\cos\varphi = e^\lambda c\cosh\mu_0\cos\varphi_\lambda, \\ c\sinh(\mu_0+\mu_\lambda(\varphi))\sin\varphi = e^\lambda c\sinh\mu_0\sin\varphi_\lambda, \end{cases}$$

where one should observe that the angle φ changes as well. In particular, $\mu_\lambda = o(1)$ and $\Delta\varphi := \varphi_\lambda - \varphi = o(1)$. Applying Taylor formula and simplifying, we obtain

$$\begin{cases} \big[\cosh\mu_0 + \sinh\mu_0\,\mu_\lambda(\varphi) + o(\lambda)\big]\cos\varphi = (1+\lambda)\cosh\mu_0[\cos\varphi - \sin\varphi\,\Delta\varphi] + o(\lambda), \\ \big[\sinh\mu_0 + \cosh\mu_0\,\mu_\lambda(\varphi) + o(\lambda)\big]\sin\varphi = (1+\lambda)\sinh\mu_0[\sin\varphi + \cos\varphi\,\Delta\varphi] + o(\lambda), \end{cases}$$

$$\begin{cases} \sinh\mu_0\,\cos\varphi\,\mu_\lambda + \cosh\mu_0\,\sin\varphi\,\Delta\varphi = \lambda\cosh\mu_0\,\cos\varphi + o(\lambda), \\ \cosh\mu_0\,\sin\varphi\,\mu_\lambda - \sinh\mu_0\,\cos\varphi\,\Delta\varphi = \lambda\sinh\mu_0\,\sin\lambda\varphi + o(\lambda). \end{cases}$$

Therefore (we are interested in μ_λ),

$$\begin{aligned} \mu_\lambda(\varphi) &= \frac{\lambda\,\sinh\mu_0\cosh\mu_0}{(\sinh^2\mu_0\,\cos^2\varphi + \cosh^2\mu_0\,\sin^2\varphi)} + o(\lambda) \\ &= \frac{\lambda\sqrt{1-e_0^2}}{1-e_0^2\cos^2\varphi} + o(\lambda). \end{aligned} \tag{39}$$

Perturbing by a translation. Let $\tau = (\tau_x,\tau_y) \in \mathbb{R}^2$, and consider a translation by τ. We want to write the translated ellipse $\mathcal{E}_\tau$ as

$$\mathcal{E}_\tau = \mathcal{E}_{e_0,c} + \mu_\tau,$$

which is equivalent to

$$\mathcal{E}(\tau_x,\tau_y,c,\mu_0,0) = \mathcal{E}(0,0,c,\mu_0+\mu_\tau,0).$$

Hence, we have to solve the following system of equations:

$$\begin{cases} c\cosh(\mu_0+\mu_\tau(\varphi))\cos\varphi = \tau_x + c\cosh\mu_0\cos\varphi_\tau, \\ c\sinh(\mu_0+\mu_\tau(\varphi))\sin\varphi = \tau_y + c\sinh\mu_0\sin\varphi_\tau, \end{cases}$$

where one should observe that the angle φ changes as well. In particular, $\mu_\tau = o(1)$ and $\Delta\varphi := \varphi_\tau - \varphi = o(1)$. Applying Taylor formula and simplifying, we obtain

$$\begin{cases} \big[\cosh\mu_0 + \sinh\mu_0\,\mu_\tau + o(\|\tau\|)\big]\cos\varphi = \frac{\tau_x}{c} + \cosh\mu_0[\cos\varphi - \sin\varphi\,\Delta\varphi + o(\|\tau\|)], \\ \big[\sinh\mu_0 + \cosh\mu_0\,\mu_\tau + o(\|\tau\|)\big]\sin\varphi = \frac{\tau_y}{c} + \sinh\mu_0[\sin\varphi + \cos\varphi\,\Delta\varphi + o(\|\tau\|)], \end{cases}$$

$$\begin{cases} \sinh\mu_0\,\cos\varphi\,\mu_\tau + \cosh\mu_0\,\sin\varphi\,\Delta\varphi = \frac{\tau_x}{c} + o(\|\tau\|), \\ \cosh\mu_0\,\sin\varphi\,\mu_\tau - \sinh\mu_0\,\cos\varphi\,\Delta\varphi = \frac{\tau_y}{c} + o(\|\tau\|). \end{cases}$$

Therefore,

$$\begin{aligned} \mu_\tau(\varphi) &= \frac{1}{(\sinh^2\mu_0\,\cos^2\varphi + \cosh^2\mu_0\,\sin^2\varphi)} \\ &\quad\cdot\left[\frac{\tau_x}{c}\sinh\mu_0\,\cos\varphi + \frac{\tau_y}{c}\cosh\mu_0\,\sin\varphi\right] + o(\|\tau\|) \\ &= \frac{e_0}{c(1-e_0^2\cos^2\varphi)}\left[\tau_x\sqrt{1-e_0^2}\cos\varphi + \tau_y\sin\varphi\right] + o(\|\tau\|). \end{aligned} \tag{40}$$

Perturbing by a rotation. Let $\theta \in [0, 2\pi)$, and consider a rotation by θ (counterclockwise); we denote

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We are interested in the rotated ellipse $\mathcal{E}_\theta$, and we want to write it (in elliptic coordinates) as

$$\mathcal{E}_\theta = \mathcal{E}_{e_0,c} + \mu_\theta,$$

which is equivalent to

$$\mathcal{E}(0,0,c,\mu_0,\theta) = \mathcal{E}(0,0,c,\mu_0+\mu_\theta,0).$$

Hence, we have to solve the following system of equations:

$$\begin{pmatrix} c\cosh(\mu_0+\mu_\theta(\varphi))\cos\varphi \\ c\sinh(\mu_0+\mu_\theta(\varphi))\sin\varphi \end{pmatrix} = R_\theta \begin{pmatrix} c\cosh\mu_0\cos\varphi_\theta \\ c\sinh\mu_0\sin\varphi_\theta \end{pmatrix},$$

where one should observe that the angle φ changes as well. In particular, $\mu_\theta = o(1)$ and $\Delta\varphi := \varphi_\theta - \varphi = o(1)$. Applying Taylor formula and simplifying, we obtain

$$\begin{pmatrix} \big[\cosh\mu_0 + \sinh\mu_0\,\mu_\theta\big]\cos\varphi \\ \big[\sinh\mu_0 + \cosh\mu_0\,\mu_\theta\big]\sin\varphi \end{pmatrix} = \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix}\begin{pmatrix} \cosh\mu_0[\cos\varphi - \sin\varphi\,\Delta\varphi + o(\theta)] \\ \sinh\mu_0[\sin\varphi + \cos\varphi\,\Delta\varphi + o(\theta)] \end{pmatrix},$$

which implies

$$\begin{pmatrix} \sinh\mu_0\,\cos\varphi \\ \cosh\mu_0\,\sin\varphi \end{pmatrix}\mu_\theta - \begin{pmatrix} \cosh\mu_0\,\sin\varphi \\ \sinh\mu_0\,\cos\varphi \end{pmatrix}\Delta\varphi = \begin{pmatrix} -\sinh\mu_0\,\sin\varphi \\ \cosh\mu_0\,\cos\varphi \end{pmatrix}\theta + o(\theta).$$

Hence, we conclude

$$\begin{aligned} \mu_\theta &= \frac{\theta\,\left[\sin\varphi\cos\varphi(\cosh^2\mu_0 - \sinh^2\mu_0)\right]}{(\sinh^2\mu_0\,\cos^2\varphi + \cosh^2\mu_0\,\sin^2\varphi)} + o(\theta) \\ &= \frac{\theta\,e_0^2}{2(1-e_0^2\cos^2\varphi)}\sin 2\varphi + o(\theta). \end{aligned} \tag{41}$$

Perturbing by an hyperbolic rotation. Let us consider the matrix

$$\Lambda = \Lambda(\lambda) := \begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix} \qquad \text{with} \quad \lambda \in \mathbb{R};$$

we are interested in the ellipse $\mathcal{E}_\Lambda$ obtained by applying this transformation to $\mathcal{E}_0$, and we want to write it (in elliptic coordinates) as

$$\mathcal{E}_\Lambda = \mathcal{E}_{e_0,c} + \mu_\Lambda,$$

which is equivalent to

$$\mathcal{E}(0,0,c,\mu_0,\theta) = \mathcal{E}(0,0,c,\mu_0+\mu_\Lambda,0).$$

Hence, we have to solve the following system of equations:

$$\begin{pmatrix} c\cosh(\mu_0+\mu_\Lambda(\varphi))\cos\varphi \\ c\sinh(\mu_0+\mu_\Lambda(\varphi))\sin\varphi \end{pmatrix} = \Lambda\begin{pmatrix} c\cosh\mu_0\cos\varphi_\Lambda \\ c\sinh\mu_0\sin\varphi_\Lambda \end{pmatrix},$$

where one should observe that the angle φ changes as well. In particular, $\mu_\Lambda = o(1)$ and $\Delta\varphi := \varphi_\Lambda - \varphi = o(1)$. Applying Taylor formula and simplifying, we obtain

$$\begin{aligned} &\begin{pmatrix} \left[\cosh\mu_0 + \sinh\mu_0\,\mu_\Lambda\right]\cos\varphi \\ \left[\sinh\mu_0 + \cosh\mu_0\,\mu_\Lambda\right]\sin\varphi \end{pmatrix} \\ &\qquad = \begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}\begin{pmatrix} \cosh\mu_0[\cos\varphi - \sin\varphi\,\Delta\varphi] \\ \sinh\mu_0[\sin\varphi + \cos\varphi\,\Delta\varphi] \end{pmatrix} + o(\lambda), \end{aligned}$$

which implies

$$\begin{pmatrix} \sinh\mu_0\,\cos\varphi \\ \cosh\mu_0\,\sin\varphi \end{pmatrix}\mu_\Lambda - \begin{pmatrix} \cosh\mu_0\,\sin\varphi \\ \sinh\mu_0\,\cos\varphi \end{pmatrix}\Delta\varphi = \lambda\begin{pmatrix} \cosh\mu_0\,\cos\varphi \\ -\sinh\mu_0\,\sin\varphi \end{pmatrix} + o(\lambda).$$

Hence, we conclude

$$\begin{aligned} \mu_\Lambda &= \frac{\lambda\sinh\mu_0\cosh\mu_0(\cos^2\varphi - \sin^2\varphi)}{(\sinh^2\mu_0\,\cos^2\varphi + \cosh^2\mu_0\,\sin^2\varphi)} \\ &= \frac{\lambda}{1-e_0^2\cos^2\varphi}\cos 2\varphi + o(\lambda). \end{aligned} \tag{42}$$

Perturbation of Ellipses and Proof of Proposition 23. Let us first start with the following lemma, which is similar to [3, Lemma 7].

LEMMA 36. *Let $\mathcal{E}_{e_0,c} = \mathcal{E}(0,0,c,\mu_0,0)$ be an ellipse of eccentricity $e_0 = 1/\cosh\mu_0$ and semi-focal distance c, and suppose that Ω is a perturbation of $\mathcal{E}_{e_0,c}$, which can be written (in the elliptic coordinate frame (μ,φ) associated to $\mathcal{E}_{e_0,c}$) as $\Omega = \mathcal{E}_{e_0,c} + \mu_\Omega(\varphi)$. Consider another ellipse $\overline{\mathcal{E}}$ sufficiently close to $\mathcal{E}_{e_0,c}$, which can be written (in elliptic coordinates frame associated to $\mathcal{E}_{e_0,c}$) as*

$$\overline{\mathcal{E}} = \mathcal{E}_{e_0,c} + \mu_{\overline{\mathcal{E}}}.$$

If $\overline{\mathcal{E}}$ is sufficiently close to $\mathcal{E}_{e_0,c}$, we can write (in the elliptic coordinate frame $(\overline{\mu},\overline{\varphi})$ associated to $\overline{\mathcal{E}}$) $\Omega = \overline{\mathcal{E}} + \overline{\mu}_\Omega(\overline{\varphi})$ for some function $\overline{\mu}_\Omega$. Then, there exists $C = C(e_0,c)$ such that for every $\varphi \in [0,2\pi)$, we have

$$|\mu_\Omega(\varphi) - (\mu_{\overline{\mathcal{E}}}(\varphi) + \overline{\mu}_\Omega(\varphi))| \le C\|\mu_{\overline{\mathcal{E}}}\|_{C^1}\|\|\mu_\Omega - \mu_{\overline{\mathcal{E}}}\|_{C^1}. \tag{43}$$

Moreover, for any $C' > 1$, if $\overline{\mathcal{E}}$ is sufficiently close to $\mathcal{E}_{e_0,c}$, then we have

$$\frac{1}{C'}\|\mu_\Omega - \mu_{\overline{\mathcal{E}}}\|_{C^1} \le \|\overline{\mu}_\Omega\|_{C^1} \le C'\|\mu_\Omega - \mu_{\overline{\mathcal{E}}}\|_{C^1}. \tag{44}$$

Proof. Let

$$\overline{\mathcal{E}} = \mathcal{E}(\overline{x}_0,\overline{y}_0,\overline{c},\overline{\mu}_0,\overline{\theta}) = \mathcal{E}_{e_0,c} + \mu_{\overline{\mathcal{E}}}(\overline{\varphi}).$$

Consider the analytic change of coordinates between the coordinate frame (μ,φ) associated to $\mathcal{E}_{e_0,c}$ and the coordinate frame $(\overline{\mu},\overline{\varphi})$ associated to $\mathcal{E}$; we have

$$\begin{cases} \overline{\mu}(\mu,\varphi) = \overline{\mu}_0 + \big[\mu - \mu_0 - \mu_{\overline{\mathcal{E}}}(\varphi)\big](1 + \rho_\mu(\mu-\mu_0,\varphi)), \\ \overline{\varphi}(\mu,\varphi) = \varphi + \rho_\varphi(\mu-\mu_0,\varphi), \end{cases} \tag{45}$$

where ρ_μ and ρ_φ are analytic functions that are $C_1\|\mu_{\overline{\mathcal{E}}}\|_{C^r}$-small in any C^r-norm, where $C_1 = C_1(e_0,c,r)$. Observe that $\overline{\mu}(\mu_0 + \mu_{\overline{\mathcal{E}}}(\varphi)) \equiv \overline{\mu}_0$.

Let us observe the following facts:

- It follows from (45) that

$$\overline{\mu}_0 + \overline{\mu}_\Omega\left(\overline{\varphi}(\mu_0 + \mu_\Omega(\varphi),\varphi)\right) = \overline{\mu}\left(\mu_0 + \mu_\Omega(\varphi),\varphi\right).$$

Taking the derivatives on both sides and using (45), we obtain

$$\begin{aligned} &\overline{\mu}'_\Omega(\overline{\varphi}(\mu_0 + \mu_\Omega(\varphi),\varphi))\left[1 + \frac{\partial\rho_\varphi}{\partial\mu}(\mu_\Omega(\varphi),\varphi)\,\mu'_\Omega(\varphi) + \frac{\partial\rho_\varphi}{\partial\varphi}(\mu_\Omega(\varphi),\varphi)\right] \\ &\quad = \mu'_\Omega(\varphi)\left[1 + \rho_\mu(\mu_\Omega(\varphi),\varphi) + (\mu_\Omega(\varphi) - \mu_{\overline{\mathcal{E}}}(\varphi))\,\frac{\partial\rho_\mu}{\partial\mu}(\mu_\Omega(\varphi),\varphi)\right] \\ &\qquad - \mu'_{\overline{\mathcal{E}}}(\varphi)\,[1 + \rho_\mu(\mu_\Omega(\varphi),\varphi)] + [\mu_\Omega(\varphi) - \mu_\mathcal{E}(\varphi)]\,\frac{\partial\rho_\mu}{\partial\varphi}(\mu_\Omega(\varphi),\varphi) \end{aligned}$$

$$= \left(\mu'_\Omega(\varphi) - \mu'_{\overline{\mathcal{E}}}(\varphi)\right)\left[1 + \rho_\mu(\mu_\Omega(\varphi), \varphi)\right]$$
$$+ \left(\mu_\Omega(\varphi) - \mu_{\overline{\mathcal{E}}}(\varphi)\right)\left[\frac{\partial \rho_\mu}{\partial \mu}(\mu_\Omega(\varphi), \varphi)\mu'_\Omega(\varphi) + \frac{\partial \rho_\mu}{\partial \varphi}(\mu_\Omega(\varphi), \varphi)\right].$$

Hence,

$$\overline{\mu}'_\Omega(\overline{\varphi}(\mu_0 + \mu_\Omega(\varphi), \varphi)) = \frac{O_{e_0,c}(\|\mu_\Omega - \mu_{\overline{\mathcal{E}}}\|_{C^1})\left[1 + O_{e_0,c}(\|\mu_{\overline{\mathcal{E}}}\|_{C^1})\right]}{1 + O_{e_0,c}(\|\mu_{\overline{\mathcal{E}}}\|_{C^1})}, \tag{46}$$

where $O_{e_0,c}(\cdot)$ means that its absolute value is bounded by the absolute value of $(\cdot)$ and a constant that depends on e_0 and c.

- Let us write $\overline{\varphi}_\Omega(\varphi) := \overline{\varphi}(\mu_0 + \mu_\Omega(\varphi), \varphi)$; it follows from (45) that it is a diffeomorphism and
$$\overline{\varphi}'_\Omega(\varphi) = 1 + O_{e_0,c}(\|\mu_{\overline{\mathcal{E}}}\|_{C^1}).$$
In particular,
$$\overline{\mu}'_\Omega(\overline{\varphi}) = \left(\overline{\mu}_\Omega \circ \overline{\varphi}_\Omega \circ \overline{\varphi}_\Omega^{-1}\right)'(\overline{\varphi})$$
$$= \overline{\mu}'_\Omega(\overline{\varphi}_\Omega(\varphi)) \cdot \overline{\varphi}'_\Omega(\varphi) \cdot \left(\overline{\varphi}_\Omega^{-1}\right)'(\overline{\varphi}).$$
Along with (46), this implies (44).
- Moreover, using that $\overline{\varphi}(\mu_0 + \mu_\Omega(\varphi), \varphi) - \varphi = O_{e_0,c}(\|\mu_{\overline{\mathcal{E}}}\|_{C^0})$, we obtain
$$\overline{\mu}_\Omega\left(\overline{\varphi}(\mu_0 + \mu_\Omega(\varphi), \varphi)\right) - \overline{\mu}_\Omega(\varphi) = \int_\varphi^{\overline{\varphi}(\mu_0 + \mu_\Omega(\varphi), \varphi)} \overline{\mu}'_\Omega(t)dt$$
$$= O_{e_0,c}(\|\mu_\Omega - \mu_{\overline{\mathcal{E}}}\|_{C^1}\|\mu_{\overline{\mathcal{E}}}\|_{C^1}).$$
- Since
$$\Omega = \mathcal{E}_{e_0,c} + \mu_\Omega(\varphi) = \overline{\mathcal{E}} + \overline{\mu}_\Omega(\overline{\varphi}),$$
then we have
$$\overline{\mu}_0 + \overline{\mu}_\Omega(\overline{\varphi}_\Omega(\varphi)) = \overline{\mu}\left(\mu_0 + \mu_\Omega(\varphi), \varphi\right)$$
$$= \overline{\mu}_0 + \left[\mu_\Omega(\varphi) - \mu_{\overline{\mathcal{E}}}(\varphi)\right](1 + \rho_\mu(\mu_\Omega(\varphi), \varphi));$$
therefore,
$$\overline{\mu}_\Omega(\overline{\varphi}_\Omega(\varphi)) - \left(\mu_\Omega(\varphi) - \mu_{\overline{\mathcal{E}}}(\varphi)\right) = \left(\mu_\Omega(\varphi) - \mu_{\overline{\mathcal{E}}}(\varphi)\right)\rho_\mu(\mu_\Omega(\varphi), \varphi)$$
$$= O_{e_0,c}(\|\mu_\Omega - \mu_{\overline{\mathcal{E}}}\|_{C^0}\|\mu_{\overline{\mathcal{E}}}\|_{C^0}).$$

Summarizing all of the above information, we get

$$\overline{\mu}_\Omega(\varphi) - \mu_\Omega(\varphi) + \mu_{\overline{\mathcal{E}}}(\varphi)$$
$$= \left[\overline{\mu}_\Omega(\overline{\varphi}_\Omega(\varphi)) - \mu_\Omega(\varphi) + \mu_{\overline{\mathcal{E}}}(\varphi)\right] + \left[\overline{\mu}_\Omega(\overline{\varphi}_\Omega(\varphi)) - \overline{\mu}_\Omega(\varphi)\right]$$
$$= O_{e_0,c}(\|\mu_\Omega - \mu_{\overline{\mathcal{E}}}\|_{C^1}\|\mu_{\overline{\mathcal{E}}}\|_{C^1}),$$

and this concludes the proof of (43). □

Now we are ready to prove Proposition 23.

Proof of Proposition 23. We use the notation introduced in (16)–(20) and proven in the first part of this section. Moreover, since we will be working with elliptic coordinate frames associated to different ellipses $\mathcal{E}_k$, we will adopt the convention to denote functions with a superscript $^{(k)}$ when we consider them with respect to the angle associated to the ellipse $\mathcal{E}_k$.

Let us denote $\Omega = \mathcal{E}_{e_0,c} + \mu^{(0)}$. We consider different steps of approximation.

(1) Let us now consider the ellipse $\mathcal{E}_1$ obtained by translating $\mathcal{E}_{e_0,c}$ by a vector

$$\tau = \left(\frac{a_1\, c}{e_0\sqrt{1-e_0^2}}, \frac{b_1\, c}{e_0} \right).$$

Let $\mu^{(0)}_{\mathcal{E}_1}$ be such that $\mathcal{E}_1 = \mathcal{E}_{e_0,c} + \mu^{(0)}_{\mathcal{E}_1}$, and let $\mu^{(1)}_1$ be such that $\Omega = \mathcal{E}_1 + \mu^{(1)}_1$. It follows from (40) that

$$\left\| \mu^{(0)}_{\mathcal{E}_1} - (a_1 e^{(0)}_{\tau 1} + b_1 e^{(0)}_{\tau 2}) \right\|_{C^1} = O_{e_0,c}(a_1^2 + b_1^2). \tag{47}$$

Then, using Lemma 36 and (47), we obtain

$$\begin{aligned} &\left\| \mu^{(0)}_1 - (a_0 e^{(0)}_h + a_2 e^{(0)}_{hr} + b_2 e^{(0)}_r) \right\|_{C^1} \\ &\qquad \le \|\mu^{(0)}_1 - (\mu^{(0)} - \mu^{(0)}_{\mathcal{E}_1})\|_{C^1} + \left\| \mu^{(0)}_{\mathcal{E}_1} - (a_1 e^{(0)}_{\tau 1} + b_1 e^{(0)}_{\tau 2}) \right\|_{C^1} \\ &\qquad = O_{e_0,c}\left(\|\mu^{(0)}\|^2_{C^1} \right); \end{aligned}$$

in particular, we have used that

$$\|\mu^{(0)}_{\mathcal{E}_1}\|_{C^1} = O_{e_0,c}(\sqrt{a_1^2 + b_1^2}) = O_{e_0,c}(\|\mu^{(0)}\|_{C^1}).$$

Let us denote by $\overline{\varphi}_1 = \overline{\varphi}_1(\varphi)$ the angle associated to $\mathcal{E}_1$; it follows from computations similar to (40) that

$$\|\overline{\varphi}_1 - \varphi\|_{C^1} = O_{e_0,c}\left(\sqrt{a_1^2 + b_1^2} \right).$$

Then, we conclude that

$$\left\| \mu^{(1)}_1 - (a_0 e^{(1)}_h + a_2 e^{(1)}_{hr} + b_2 e^{(1)}_r) \right\|_{C^1} = O_{e_0,c}\left(\|\mu^{(0)}\|^2_{C^1} \right). \tag{48}$$

(2) Let us consider the dilated/contracted ellipse

$$\mathcal{E}_2 = e^{\frac{a_0}{\sqrt{1-e_0^2}}} \mathcal{E}_1;$$

let $\mu^{(1)}_{\mathcal{E}_2}$ be such that $\mathcal{E}_2 = \mathcal{E}_1 + \mu^{(1)}_{\mathcal{E}_2}$, and let $\mu^{(2)}_2$ be such that $\Omega = \mathcal{E}_2 + \mu^{(2)}_2$. It follows from (39) that

$$\left\| \mu^{(1)}_{\mathcal{E}_2} - a_0 e^{(1)}_h \right\|_{C^1} = O_{e_0,c}(a_0^2). \tag{49}$$

Then, proceeding as above and using Lemma 36, (48) and (49), we obtain

$$\left\| \mu^{(1)}_2 - (a_2 e^{(1)}_{hr} + b_2 e^{(1)}_r) \right\|_{C^1} \le \|\mu^{(1)}_2 - (\mu^{(1)}_1 - \mu^{(1)}_{\mathcal{E}_2})\|_{C^1}$$

$$
\begin{aligned}
&+ \left\| \mu_1^{(1)} - (a_0 e_h^{(1)} + a_2 e_{hr}^{(1)} + b_2 e_r^{(1)}) \right\|_{C^1} \\
&+ \left\| \mu_{\mathcal{E}_2}^{(1)} - a_0 e_h^{(1)} \right\|_{C^1} \\
&= O_{e_0,c}\left(\|\mu^{(0)}\|_{C^1}^2 \right);
\end{aligned}
$$

Let us denote by $\overline{\varphi}_2 = \overline{\varphi}_2(\overline{\varphi}_1)$ the angle associated to $\mathcal{E}_2$; it follows from computations similar to (39) that

$$
\|\overline{\varphi}_2 - \overline{\varphi}_1\|_{C^1} = O_{e_0,c}\left(a_0\right).
$$

Then, we conclude that

$$
\left\| \mu_2^{(2)} - (a_2 e_{hr}^{(2)} + b_2 e_r^{(2)}) \right\|_{C^1} = O_{e_0,c}\left(\|\mu^{(0)}\|_{C^1}^2 \right).
$$

(3) Let us consider the rotated ellipse

$$
\mathcal{E}_3 = R_{\frac{2a_2}{e_0^2}} \mathcal{E}_2;
$$

let $\mu_{\mathcal{E}_3}^{(2)}$ be such that $\mathcal{E}_3 = \mathcal{E}_2 + \mu_{\mathcal{E}_3}^{(2)}$, and let $\mu_3^{(3)}$ be such that $\Omega = \mathcal{E}_3 + \mu_3^{(3)}$. It follows from (41) that

$$
\left\| \mu_{\mathcal{E}_3}^{(2)} - b_2 e_r^{(2)} \right\|_{C^1} = O_{e_0,c}(b_2^2). \tag{50}
$$

Proceeding as above (Lemma 36 and similar estimates), we get

$$
\left\| \mu_3^{(3)} - a_2 e_{hr}^{(3)} \right\|_{C^1} = O_{e_0,c}\left(\|\mu^{(0)}\|_{C^1}^2 \right).
$$

(4) Finally, let us consider the ellipse obtained by means of an hyperbolic rotation $\Lambda(a_2)$:

$$
\mathcal{E}_4 = \Lambda(a_2)\, \mathcal{E}_3.
$$

Let be $\mu_{\mathcal{E}_4}^{(3)}$ such that $\mathcal{E}_4 = \mathcal{E}_3 + \mu_{\mathcal{E}_4}^{(3)}$, and let $\mu_4^{(4)}$ be such that $\Omega = \mathcal{E}_4 + \mu_4^{(4)}$. It follows from (42) that

$$
\left\| \mu_{\mathcal{E}_4}^{(3)} - a_2 e_{hr}^{(3)} \right\|_{C^1} = O_{e_0,c}(a_2^2).
$$

In particular, proceeding as above, we conclude also in this case that

$$
\|\mu_4^{(4)}\|_{C^1} = O_{e_0,c}\left(\|\mu^{(0)}\|_{C^1}^2 \right).
$$

To conclude the proof, we denote $\widetilde{\mathcal{E}} := \mathcal{E}_4$ and we consider $\mu_{\widetilde{\mathcal{E}}}$ such that $\widetilde{\mathcal{E}} = \mathcal{E}_{e_0,c} + \mu_{\widetilde{\mathcal{E}}}$. It follows from Lemma 36 (second part of the statement) that

$$
\|\mu^{(0)} - \mu_{\widetilde{\mathcal{E}}}^{(0)}\|_{C^1} = O_{e_0,c}\left(\|\mu_4^{(4)}\|_{C^1} \right) = O_{e_0,c}\left(\|\mu^{(0)}\|_{C^1}^2 \right),
$$

and this concludes the proof of the proposition. □

Appendix C. Analytic extensions and their singularities

C.1. *Proof of Proposition* 24. Let us start by studying the zeros of

$$h_k(z) = 1 - k^2 \sin^2 z \tag{51}$$

for $0 < k < 1$.

Remark 37. Observe that $k_q > 0$ unless $e_0 = 0$; i.e., the boundary of the billiard is a circle and $k_q \equiv 0$ for any $q \geq 3$. In this latter case, $h_0(z) \equiv 1$ and there are no zeros; in fact, c_q and s_q correspond to $\cos(q\,z)$ and $\sin(q\,z)$, which are entire functions. Hence, we consider only the case $0 < k_q < 1$.

LEMMA 38. *Let k satisfy $0 < k < 1$:*

$$h_k(z) = 0 \iff z_n = \left(\frac{\pi}{2} + n\pi\right) \pm i\,\rho_k \qquad \text{for } n \in \mathbb{Z}.$$

Proof. Recall that

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

therefore,

$$\begin{aligned} \sin^2(x + iy) &= (\sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y) + 2i \sin x \cos x \sinh y \cosh y \\ &= [\sin^2 x \cosh^2 y - \cos^2 x(\cosh^2 y - 1)] + i \sin(2x) \sinh y \cosh y \\ &= [\cosh^2 y(\sin^2 x - \cos^2 x) + \cos^2 x] + i \sin(2x) \sinh y \cosh y \\ &= [-\cosh^2 y \cos(2x) + \cos^2 x] + i \sin(2x) \sinh y \cosh y. \end{aligned} \tag{52}$$

In particular, denoting $z = x + iy$, we have

$$\begin{aligned} h_k(z) &= 1 - k^2 \sin^2 z \\ &= [1 - k^2 \cos^2 x + k^2 \cosh^2 y \cos(2x)] - ik^2 \sin(2x) \sinh y \cosh y \end{aligned} \tag{53}$$

and hence for $0 < k < 1$,

$$h_k(z) = 0 \iff \begin{cases} 1 - k^2 \cos^2 x + k^2 \cosh^2 y \cos(2x) = 0, \\ \sin(2x) \sinh y \cosh y = 0. \end{cases}$$

The second equation has the following solutions:

$$\text{(i)}\ \ x = \frac{m\pi}{2} \quad (\text{with } m \in \mathbb{Z}) \qquad \text{or} \qquad \text{(ii)}\ \ y = 0.$$

If we plug those solutions in the first equation we obtain

(i) Let $x = \frac{m\pi}{2}$, and let us distinguish two cases:
  (a) if $x = n\pi$, then the first equation becomes

$$1 - k^2 + k^2 \cosh^2 y > 0 \qquad \text{for} \quad 0 < k < 1;$$

(b) if $x = \frac{(2n+1)\pi}{2}$, then the first equation becomes

$$1 - k^2 \cosh^2 y = 0,$$

hence

$$\cosh^2 y = 1/k^2 \iff y_\pm = \pm\rho_k := \pm \operatorname{arcosh}(1/k),$$

which is well defined since $0 < k < 1$.

(ii) If $y = 0$, then the first equation becomes

$$\begin{aligned} 0 &= 1 - k^2 \cos^2 x + k^2 \cos(2x) \\ &= 1 - k^2 \sin^2 x, \end{aligned}$$

which does not admit solutions for $0 < k < 1$.

Summarizing, for $0 < k < 1$,

$$h_k(z) = 0 \iff z_n = \left(\frac{\pi}{2} + n\pi\right) \pm i\,\rho_k \qquad \text{for } n \in \mathbb{Z}. \qquad \square$$

If we denote by Σ_ρ the open complex strip of (half) width $\rho > 0$ around the real axis, i.e.,

$$\Sigma_\rho := \{z \in \mathbb{C} : \ |\mathrm{Im}(z)| < \rho\},$$

then we conclude that h_k is an entire function that, for $0 < k < 1$, does not vanish in the strip Σ_{ρ_k}.

Now we want to consider the complex function $\sqrt{h_k(z)}$ and understand its domain of analyticity. Recall the following elementary result from complex analysis:

> *Let f be a nowhere vanishing holomorphic function in a simply connected region Ω. Then f has a holomorphic logarithm, and hence, a holomorphic square-root in Ω.*

Therefore, we can conclude that the functions $\sqrt{h_k(z)}$ and $1/\sqrt{h_k(z)}$ are analytic in Σ_{ρ_k}.

If we consider, for $0 < k < 1$, the function $F(\varphi; k) := \displaystyle\int_0^\varphi \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$, then its complex extension is given by

$$F(z; k) := \int_0^z \frac{d\zeta}{\sqrt{h_k(\zeta)}}.$$

It follows from Cauchy's theorem that this function is well defined and analytic in Σ_{ρ_k}. This completes the proof of Proposition 24. $\square$

C.2. *Proof of Proposition* 26. Observe that, using the notation introduced in (51),

$$e_h(z) = \frac{1}{h_{e_0}\left(z + \frac{\pi}{2}\right)}.$$

It follows from the discussion in Section C.1 that this function has singularities (which are poles) at

$$\zeta_n = n\pi \pm i\,\rho_0 \qquad \text{for } n \in \mathbb{Z},$$

where $\rho_0 = \operatorname{arcosh}(1/e_0) = \mu_0$. In particular, its maximal strip of analyticity is given by

$$\Sigma_{\rho_0} = \{z \in \mathbb{C} : \ |\mathrm{Im}(z)| < \rho_0\}.$$

This concludes the proof of Proposition 26. □

Appendix D. Proof of Proposition 30

Consider the following variational problem.

Given $0 < j \leq 2q_0$, we would like to see how much $\mathbb{e}_j(\varphi)$ is linearly independent of the vector subspace

$$\Lambda_{q_0} := \overline{\langle \{\mathbb{e}_k\}_{k>2q_0} \rangle}.$$

Observe that it suffices to consider an arbitrary q_0, since we already have linear independence for every finite subcollection.

We start by considering the case $j \geq 5$; for the other case, see Remark 46. Let us define v_j as the vector realizing the minimal $L^2_{e_0}$-distance from the unit vector $\mathbb{e}_j$ to the subspace Λ_{q_0}; namely, if

$$v_j := \mathbb{e}_j - \sum_{k>2q_0} d_{jk}\mathbb{e}_k, \tag{54}$$

then we require that v_j be orthogonal to all $\mathbb{e}_k$ for $k > 2q_0$. Hence, we consider the $L^2_{e_0}$-scalar product of v_j with $\mathbb{e}_m$, for $m > 2q_0$, and we impose that it be equal to zero:

$$v_j \cdot \mathbb{e}_m = \mathbb{e}_j \cdot \mathbb{e}_m - \sum_{k>2q_0} d_{jk}\,(\mathbb{e}_k \cdot \mathbb{e}_m) = 0. \tag{55}$$

Strategy of proof. Notice that by definition each vector $v_j,\ 0 < j \leq 2q_0$ is the projection of $\mathbb{e}_j$ onto the orthogonal complement to Λ_{q_0}. If the vectors $\{v_j,\ 0 < j \leq 2q_0\}$ are linearly independent (see Corollary 47), then the subspaces $\langle \{\mathbb{e}_k\}_{0\leq k\leq 2q_0} \rangle$ and $\overline{\langle \{\mathbb{e}_k\}_{k>2q_0} \rangle}$ have zero dimensional intersection (see Proposition 30). This, in turn, implies that $\{\mathbb{e}_j,\ j > 0\}$ form a basis of $L^2(\mathbb{T})$ (see Lemma 31).

The key idea to check linear independence of vectors $\{v_j,\ 0 < j \leq 2q_0\}$ is the same as in the case of finite linear combinations (see Proposition 28). In the case $\{\mathbb{e}_j,\ 0 < j \leq 2q_0\}$, singularities of the complex extensions are explicit and pairwise disjoint for $\mathbb{e}_i$ and $\mathbb{e}_j$ with $i \neq j$. We modify this idea for

$\{v_j,\ 0 < j \le 2q_0\}$ as follows: for each $5 \le j \le 2q_0$, we would like to compare the maximal strips of analyticity of $\mathbb{e}_j$ and v_j related by (54). Notice that the width of the maximal strip of analyticity of $\mathbb{e}_j$ equals $\rho_{k_{\widehat{j}}}$, while the width of the maximal strip of analyticity of $v_j - \mathbb{e}_j$ equals to the strip of analyticity of $\mathbb{e}_j - \sum_{m>2q_0} d_{jm}\,\mathbb{e}_m$, which turns out to equal $\rho_{k_{\widehat{j}}}$ (Corollary 45). Infinite linear independence will then follow; see Corollary 47.

Let us introduce some notation. For $j \le 2q_0 < k, m$, we define

$$a_{km} := \mathbb{e}_k \cdot \mathbb{e}_m = \int_0^{2\pi} \mathbb{e}_k(\varphi)\,\mathbb{e}_m(\varphi)\,(1 - e_0^2\cos^2\varphi)^2 d\varphi \tag{56}$$

and

$$b_{jm} := \mathbb{e}_j \cdot \mathbb{e}_m = \int_0^{2\pi} \mathbb{e}_j(\varphi)\,\mathbb{e}_m(\varphi)\,(1 - e_0^2\cos^2\varphi)^2 d\varphi, \tag{57}$$

where the scalar product is meant in the weighted space $L^2_{e_0}$.

Hence, we obtain the (infinite) row vector

$$\vec{B}_{q_0} := (b_{jm})_{m>2q_0}$$

and the (infinite) square matrix

$$A_{q_0} := (a_{km})_{k,m>2q_0}.$$

In particular, if we denote by $\vec{D}_{q_0}$ the infinite row vector

$$\vec{D}_{q_0} := (d_{jk})_{k>2q_0},$$

then equation (55) becomes

$$\vec{D}_{q_0}\,A_{q_0} = \vec{B}_{q_0}. \tag{58}$$

In particular, if A_{q_0} is invertible, then

$$\vec{D}_{q_0} = \vec{B}_{q_0}\,A_{q_0}^{-1}.$$

Now we need to study A_{q_0} and $\vec{B}_{q_0}$ for large q_0. Notice that the matrix A_{q_0} is a small perturbation of the identity, because by Lemma 39, for $k \ne m \to +\infty$, its elements a_{km} decay exponentially. (We will make this more quantitative in the following.) The vector $\vec{B}_{q_0}$ also has components exponentially decaying in m. (It follows from the estimates in Lemma 39 too.) To compare maximal strips of analyticity of v_j and $\mathbb{e}_j$ for each $j \le 2q_0$, we need to estimate the exponent of the speed of decay of elements of $\vec{D}_{q_0}$. Our analysis starts with the following lemma.

Notation. Hereafter, given an integer $q \in \mathbb{N}$, we will denote $\widehat{q} := \left[\frac{q+1}{2}\right]$, where $[\,\cdot\,]$ denotes the integer part. This cumbersome notation is needed since for every integer q, we have couples $\mathbb{e}_{2q}$ and $\mathbb{e}_{2q-1}$ corresponding to the same rotation number $1/q$. Whenever it is possible, in the forthcoming statements and proofs, we will try to ease notation as much as possible.

THEOREM 39. *For every $e_0 > 0$, there exists $q_0 = q_0(e_0)$ such that the following holds. For each $j \geq 3$, there exists $\lambda_j \in (0,1)$ such that for any $\delta > 0$, there is $C_j := C_j(e_0, \delta) > 0$ such that for each $3 \leq j < m$,*

$$|a_{jm} - \delta_{jm}| \leq C_j(\lambda_j + \delta)^{\widehat{m}}, \tag{59}$$

where $\widehat{m} := \left[\frac{m+1}{2}\right]$. Moreover, for $2q_0 < j \leq m$, we have

$$|a_{jm} - \delta_{jm}| \leq C^* (\lambda^* + \delta)^{\widehat{m}} \tag{60}$$

for some $C^* = C^*(e_0, \delta)$ and $\lambda^* = \lambda^*(e_0) < 1$.

Remark 40. We will see that we can choose $\lambda_j = \exp[-\rho_{k_j}(1 + \kappa^*))]$ for some suitable $\kappa^* = \kappa^*(e_0) > 0$. Moreover, by studying the growth of the constants C_j, we show that we can choose $\lambda^* = \exp[-(\sigma_\infty(\rho_{k_{q_0}}) - \rho_{k_{q_0}})]$, where $\sigma_\infty(\rho_{k_{q_0}}) - \rho_{k_{q_0}} > 0$. (See (63) for a definition of $\sigma_\infty(\cdot)$.)

Proof. Recall from (24) that

$$\mathbb{e}_{2j} := \frac{c_j}{\|c_j\|_{L^2_{e_0}}}, \quad \mathbb{e}_{2j-1} := \frac{s_j}{\|s_j\|_{L^2_{e_0}}} \qquad \forall\, j \geq 3.$$

In particular, up to multiplication by constants, we have

$$\mathbb{e}_{2j}(\varphi) \asymp \frac{\cos(j\frac{2\pi}{4K(k_j)}F(\varphi, k_j))}{\sqrt{1 - k_j^2 \sin^2\varphi}}, \qquad \mathbb{e}_{2j-1}(\varphi) \asymp \frac{\sin(j\frac{2\pi}{4K(k_j)}F(\varphi, k_j))}{\sqrt{1 - k_j^2 \sin^2\varphi}}.$$

Let us now denote

$$t_{2j}(\varphi) \,=\, t_{2j-1}(\varphi) \,:=\, \frac{2\pi}{4K(k_j)} F(\varphi, k_j)$$

and their inverses

$$\varphi_{2j}(t) \,=\, \varphi_{2j-1}(t) \,:=\, \mathrm{am}\left(\frac{4K(k_j)}{2\pi} t, k_j\right);$$

then

$$\mathbb{e}_{2j}(\varphi) \asymp \cos(jt_j(\varphi))\, \frac{dt_j}{d\varphi}(\varphi) \quad \text{and} \quad \mathbb{e}_{2j-1}(\varphi) \asymp \sin(jt_j(\varphi))\, \frac{dt_j}{d\varphi}(\varphi).$$

We need to compute $\mathbb{e}_j \cdot \mathbb{e}_m$. Observe that if $j = m$, then it is 1, since they are unit vectors with respect to the $L^2_{e_0}$-scalar product. Let us assume that $j < m$. Doing a change of coordinate in the corresponding integral, we get (we consider the case in which both indices are even, since the other cases are analogous)

$$\begin{aligned}
\mathbb{e}_{2j} \cdot \mathbb{e}_{2m} &= \int_0^{2\pi} \mathbb{e}_{2j}(\varphi)\, \mathbb{e}_{2m}(\varphi)\, (1 - e_0^2 \cos^2\varphi)^2\, d\varphi \\
&= \int_0^{2\pi} \mathbb{e}_{2j}(\varphi_m(t))\, \cos(mt)\, \frac{dt_m}{d\varphi_m} \frac{d\varphi_m}{dt}\, (1 - e_0^2 \cos^2\varphi_m(t))^2\, dt \\
&= \int_0^{2\pi} \mathbb{e}_{2j}(\varphi_m(t))\, (1 - e_0^2 \cos^2\varphi_m(t))^2\, \cos(mt)\, dt.
\end{aligned}$$

Hence, we are computing the m-th Fourier coefficients of the function

(61)
$$\begin{aligned} E_{jm}(t) &:= \mathfrak{e}_{2j}(\varphi_m(t))\,(1-e_0^2\cos^2\varphi_m(t))^2 \\ &= \frac{\cos\left(j\frac{2\pi}{4K(k_j)}F\left(\varphi_m(t),k_j\right)\right)}{\sqrt{1-k_j^2\sin^2\left(\varphi_m(t)\right)}}\,(1-e_0^2\cos^2\varphi_m(t))^2 \\ &= \frac{\cos\left(jF\left(\mathrm{am}\,(\frac{4K(k_m)}{2\pi}t,k_m),k_j\right)\right)}{\sqrt{1-k_j^2\,\mathrm{sn}^{\,2}\left(\frac{4K(k_m)}{2\pi}t,k_m\right)}}\,\left(1-e_0^2\,\mathrm{cn}^{\,2}\left(\frac{4K(k_m)}{2\pi}t,k_m\right)\right)^2. \end{aligned}$$

In order to compute the decay rate of its Fourier coefficients, we need to analyze its maximal strip of analyticity.

Recall that k_j represents the eccentricity of the caustic of rotation number $1/j$. In particular, it is strictly decreasing with respect to j and

$$k_j > k_m > e_0 \qquad \forall\, 2 < j < m.$$

First of all, observe (see Remark 15) that $\mathrm{sn}\,(z,k)$ and $\mathrm{cn}\,(z,k)$ have simple poles with imaginary parts $iK(k')$, where k' denotes the complementary modulus $k' := \sqrt{1-k^2}$. Hence, $\mathrm{sn}\,(\frac{4K(k_m)}{2\pi}t,k_m)$ has maximal strip of analyticity of width equal to $2\pi\frac{K(k'_m)}{4K(k_m)}$.

On the other hand, $\cos(\cdot)$ is an entire function. Thus, the singularities of E_{jm} can be of two types: singularities of the last bracket and vanishing of the denominator. The first type singularity occurs at $i2\pi\frac{K(k'_m)}{4K(k_m)}$.

Hence, it remains only to study when the denominator of E_{jm} vanishes:

$$1-k_j^2\sin^2\left(\mathrm{am}\,\left(\frac{4K(k_m)}{2\pi}\zeta,k_m\right)\right)=0.$$

Proceeding as in Lemma 38, if follows that the above equality is achieved when

$$\mathrm{am}\,\left(\frac{4K(k_m)}{2\pi}\zeta,k_m\right)=\frac{\pi}{2}+\pi n\pm i\rho_{k_j},$$

where $\rho_{k_j}=\mathrm{arcosh}\,(1/k_j)$. In particular, the solutions of this equation are

$$\begin{aligned} \zeta_n &:= \frac{2\pi}{4K(k_m)}F\left(\frac{\pi}{2}+\pi n\pm i\rho_{k_j},k_m\right) \\ &= \frac{2\pi}{4K(k_m)}\left(F\left(\frac{\pi}{2}\pm i\rho_{k_j},k_m\right)+2nK(k_m)\right) \\ &= \frac{2\pi}{4K(k_m)}F\left(\frac{\pi}{2}\pm i\rho_{k_j},k_m\right)+\pi n. \end{aligned}$$

Observe that $\rho_{k_j}<\rho_{k_m}$, so the points $\frac{\pi}{2}\pm i\rho_{k_j}$ are inside the strip of analyticity of $F(\cdot;k_m)$.

The above expression can be expanded further. In fact, observe that

$$F\left(\frac{\pi}{2} \pm i\rho_{k_j}, k_m\right) = K(k_m) + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}\pm i\rho_{k_j}} \frac{1}{\sqrt{1-k_m^2\sin^2 z}}\, dz$$
$$= K(k_m) \pm i\int_0^{\rho_{k_j}} \frac{1}{\sqrt{1-k_m^2\cosh^2 t}}\, dt,$$

where in the last equality we have used that $\sin^2\left(\frac{\pi}{2}+it\right) = \cosh^2 t$. Hence, the singularities are at

$$\zeta_n := \frac{2\pi}{4K(k_m)} F\left(\frac{\pi}{2} \pm i\rho_{k_j}, k_m\right) + \pi n$$
$$= \frac{\pi}{2} + \pi n \pm i\frac{2\pi}{4K(k_m)} \int_0^{\rho_{kj}} \frac{1}{\sqrt{1-k_m^2\cosh^2 t}}\, dt.$$

The quantity

$$\sigma_m(\rho_{k_j}) := \frac{2\pi}{4K(k_m)} \min\left\{ \int_0^{\rho_{k_j}} \frac{1}{\sqrt{1-k_m^2\cosh^2 t}}\, dt\ ,\ K(\sqrt{1-k_m^2}) \right\}$$
$$= \frac{2\pi}{4K(k_m)} \int_0^{\rho_{k_j}} \frac{1}{\sqrt{1-k_m^2\cosh^2 t}}\, dt \tag{62}$$

provides the width of the strip of analyticity of E_{jm}; the proof of the last equality follows from Lemma 41 with $x = k_m$ and $y = k_j$, observing that $0 < k_m < k_j$ for $j < m$.

Notice that the entries $a_{2j,2m}$, defined by (56), can be viewed as Fourier coefficients of the functions E_{jm}. The latter ones has the strip of analyticity, given by $\sigma_m(\rho_{k_j})$. For fixed j, these widths are strictly decreasing in m and, in the limit as $m \to +\infty$, they tend to

$$\sigma_\infty(\rho_{k_j}) := \frac{2\pi}{4K(e_0)} \int_0^{\rho_{k_j}} \frac{1}{\sqrt{1-e_0^2\cosh^2 t}}\, dt, \tag{63}$$

which is strictly increasing in j. In fact, consider the function

$$W(x,y) := \frac{2\pi}{4K(x)} \int_0^{\operatorname{arcosh} 1/y} \frac{1}{\sqrt{1-x^2\cosh^2 t}}\, dt$$

defined for $0 < x < y < 1$. It suffices to show that it is increasing with respect to x. Since $x = k_m$ is decreasing with respect to m, it will follow that it is decreasing. This can be shown using lengthy, but elementary, manipulation or using Mathematica. In Figure 4 we present two plots: the first one is the graph of W, and the second one is the graph of the partial derivative of W with respect to x, which turns out to be positive.

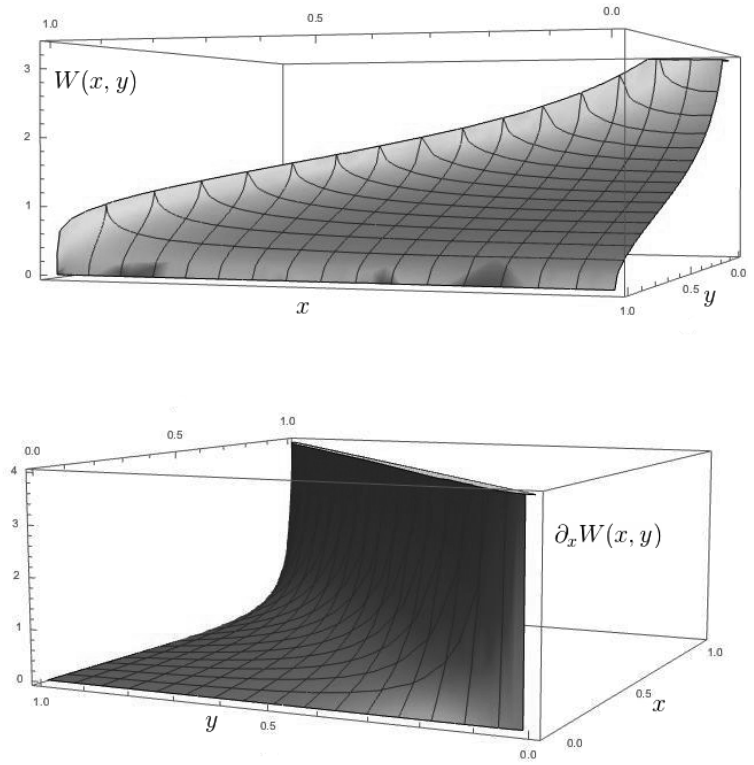

Figure 4. Plots of $W(x,y)$ and $\partial_x W(x,y)$.

We can now deduce (59) by applying Paley-Wiener theorem.899[7] Observe that we can choose $\lambda_m = \exp[-\sigma_\infty(\rho_{k_j})]$. (We will show in Proposition 42 that $\sigma_\infty(\rho_{k_j}) > \rho_{k_m}(1+\kappa^*)$) for some suitable $\kappa^* = \kappa^*(e_0) > 0$ that will be explicitly determined.)

Now, we want to prove (60). In order to do this, we need to get a better control on the constants C_j. In particular, we need to estimate

$$\left|\cos\left(j\frac{2\pi}{4K(k_j)}F\left(\varphi_m(z),k_j\right)\right)\right| = \left|\cos\left(j\frac{2\pi}{4K(k_j)}F\left(\mathrm{am}\left(\frac{4K(k_m)}{2\pi}z,k_m\right),k_j\right)\right)\right|$$

[7] Let us briefly recall the statement of this theorem; see, for example, http://www.math.lsa.umich.edu/~rauch/555/fouriercomplexbw.pdf:

THEOREM (Paley-Wiener). *If f is an analytic periodic function in the strip $\{|\mathrm{Im}\, z| < a\}$ for some $a > 0$, then its Fourier coefficients c_n satisfy the following property: for any $\epsilon > 0$,t there exists $C(\epsilon) > 0$ such that $|c_n| \le C(\epsilon)e^{(-a+\epsilon)|n|}$ for every $n \in \mathbb{Z}$. In particular, $C(\epsilon)$ is bounded from above by the supremum of $|f|$ on the strip $\{|\mathrm{Im}\, z| \le a - \epsilon\}$.*

Conversely, if $\{c_n\}_n$ satisfy the above property, then $f := \sum_{n\in\mathbb{Z}} c_n e^{inz}$ has an analytic continuation to the strip $\{|\mathrm{Im}\, z| < a\}$.

on the complex strip of width $(\rho_{k_j}-\delta)$. Since $|\cos(x+iy)|$ grows like $e^{|y|}$, then we need to estimate

$$\left|\mathrm{Im}\left(\frac{2\pi}{4K(k_j)}F\left(\mathrm{am}\,(\frac{4K(k_m)}{2\pi}t,k_m),k_j\right)\right)\right| = \frac{2\pi}{4K(k_j)}\left|\mathrm{Im}\left(F\left(\mathrm{am}\,(\frac{4K(k_m)}{2\pi}t,k_m),k_j\right)\right)\right|$$

on the strip of width $(\rho_{k_j}-\delta)$.

Since $F(\cdot,k_m)$ and φ_m are one the inverse of the other, then it follows that for $2q_0<j\le m$, and then there exists $\widetilde{C}(e_0,\delta)>0$ such that

$$\left|\mathrm{Im}\left(F\left(\mathrm{am}\,(\frac{4K(k_m)}{2\pi}z,k_m),k_j\right)\right)\right| \le \widetilde{C}\,\frac{4K(k_m)}{2\pi}|\mathrm{Im}\,(z)|$$

for every z in the complex strip of width $\rho_{k_j}-\delta$.

Hence,

$$|E_{jm}(z)| \le C\,\exp(j(\rho_{k_j}-\delta))$$

in the $(\rho_{k_j}-\delta)$-strip for some $C=C(e_0,\delta)$.

Now with this bound at hand we can deduce from the Paley-Wiener theorem (see footnote 7) that

$$\begin{aligned}|\mathfrak{e}_{2j}\cdot\mathfrak{e}_{2m}| &\le C^*(e_0,\delta)\,\exp(j(\rho_{k_j}-\delta))\,\exp(-m\,\sigma_\infty(\rho_{k_j}))\\ &\le C^*(e_0,\delta)\,\exp(-m(\sigma_\infty(\rho_{k_0})-\rho_{k_0}+\delta)).\end{aligned}$$

Since $\rho_{k_j}>\rho_{k_{q_0}}$ for every $j>q_0$, we can choose $\lambda^*=\exp(-(\sigma_\infty(\rho_{k_{q_0}})-\rho_{k_{q_0}}))$. We point out that $\sigma_\infty(\rho_{k_{q_0}})-\rho_{k_{q_0}}>\kappa^*\rho_{k_0}>0$, as it follows from Proposition 42. □

Let us prove this lemma, which was used in the proof of Theorem 39.

LEMMA 41. *For $0<x\le y<1$, we have*

$$I(x,y):=\int_0^{\mathrm{arcosh}(1/y)}\frac{1}{\sqrt{1-x^2\cosh^2 t}}\,dt \le K(\sqrt{1-x^2}),$$

with equality only for $x=y$.

Proof. [8] Clearly, $I(x,y)$ is strictly increasing with respect to x, while $K(\sqrt{1-x^2})$ is strictly decreasing with respect to x.

The claim follows from the fact that for any $0<y<1$, we have

$$I(y,y)=K(\sqrt{1-y^2}).$$

[8] There is an alternative proof of this lemma using the Reduction Theorem for General Elliptic Integrals (see, e.g., https://dlmf.nist.gov/19.29). One can represent both integrals using the canonical form R_F and then relate them using the representation formula for R_F and in terms of R_C (see https://dlmf.nist.gov/19.23).

In fact, consider the following change of variable in the integral defining $I(y,y)$:

$$\cosh^2 t - 1 = (1/y^2 - 1)\sin^2\theta,$$

which implies

$$\sinh t = \sqrt{1 + (1/y^2 - 1)}\sin\theta$$

and

$$\cosh t = \sqrt{1 + (1/y^2 - 1)\sin^2\theta}$$
$$dt = \frac{\sqrt{1/y^2 - 1}\cos\theta\, d\theta}{\sqrt{1 + (1/y^2 - 1)\sin^2\theta}}.$$

Then

$$\begin{aligned} I(y,y) &= \int_0^{\pi/2} \frac{1}{\sqrt{1 - y^2\cos\theta}} \frac{\sqrt{1/y^2 - 1}\cos\theta\, d\theta}{\sqrt{1 + (1/y^2 - 1)\sin^2\theta}} \\ &= \frac{1}{y}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + (1/y^2 - 1)\sin^2\theta}} \\ &= \frac{1}{y}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2\theta + 1/y^2\sin^2\theta}} \\ &= \frac{1}{y}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1/y^2 - (1/y^2 - 1)\cos^2\theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - (1 - y^2)\cos^2\theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - (1 - y^2)\sin^2\theta}} = K(\sqrt{1 - y^2}). \end{aligned}$$

□

The width of the strip of analyticity of $v_j - \mathbb{e}_j$ depends on the exponent of the speed of decay of elements of $\vec{D}_{q_0}$. We will compare now the width of strips of analyticity of v_j and $v_j - \mathbb{e}_j$ for each $j < 2q_0$.

We need the following estimate to compare $\sigma_m(\rho_{k_j})$ with ρ_{k_j}.

PROPOSITION 42. *There is a decreasing sequence $\kappa_m \geq \kappa^* := \kappa^*(e_0) > 0$ such that for any $m > j \geq 3$, we have*

$$\rho_{k_j} < \sigma_m(\rho_{k_j}) - \rho_{k_j}\kappa_m.$$

In particular,

$$\rho_{k_j} < \sigma_\infty(\rho_{k_j}) - \rho_{k_j}\kappa^*.$$

Proof. Recall the definition of $\sigma_m(\rho_{k_j})$ in (62). There is $\kappa'_m = \kappa'_m(k_m) > 0$ such that

$$\begin{aligned} 4\rho_{k_j}K(k_m) &= \rho_{k_j}\int_0^{2\pi}\frac{dt}{\sqrt{1-k_m^2\sin^2 t}} \\ &< \rho_{k_j}\left(\frac{\pi}{\sqrt{1-k_m^2}}+\frac{\pi}{\sqrt{1-k_m^2/2}}\right) \\ &=:\rho_{k_j}\left(\frac{2\pi}{\sqrt{1-k_m^2}}-\kappa'_m\right) \\ &= 2\pi\int_0^{\rho_{k_j}}\frac{dt}{\sqrt{1-k_m^2}} - \rho_{k_j}\kappa'_m \\ &< 2\pi\int_0^{\rho_{k_j}}\frac{dt}{\sqrt{1-k_m^2\cosh^2 t}} - \rho_{k_j}\kappa'_m, \end{aligned} \tag{64}$$

where

$$\kappa'_m := \pi\left(\frac{1}{\sqrt{1-k_m^2}}-\frac{1}{\sqrt{1-k_m^2/2}}\right),$$

which is strictly decreasing[9] as a function of m and, as $m\to+\infty$, tends to

$$\kappa'_\infty := \pi\left(\frac{1}{\sqrt{1-e_0^2}}-\frac{1}{\sqrt{1-e_0^2/2}}\right) > 0.$$

Dividing on both sides of (64) by $4K(k_m)$ we get

$$\rho_{k_j} < \sigma_m(\rho_{k_j}) - \frac{\rho_{k_j}\kappa'_m}{4K(k_m)} \qquad \text{for any } m > j \geq 3.$$

Denote $\kappa_m = \frac{\kappa'_m}{4K(k_m)}$; this function is also strictly decreasing[10] as a function of m and, as $m\to+\infty$, tends to

$$\kappa^* := \frac{\pi}{4K(e_0)}\left(\frac{1}{\sqrt{1-e_0^2}}-\frac{1}{\sqrt{1-e_0^2/2}}\right) > 0. \qquad \square$$

Let $\mathbb{I}$ denote the Identity (infinite) matrix, and let us denote

$$A_{q_0} = \mathbb{I} + \Delta A_{q_0},$$

[9]This follows from the fact that the function $\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2/2}}$ is strictly increasing in $[0,1)$.

[10]This follows from the fact that the function $\frac{1}{K(x)}\left(\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2/2}}\right)$ is strictly increasing in $[0,1)$.

where $\Delta A_{q_0} := (a_{km} - \delta_{km})_{k,m>2q_0}$ and

$$|a_{km} - \delta_{km}| \le C^*(\lambda^* + \delta)^{q_0}.$$

LEMMA 43. *Using the same notation as in Lemma* 39, *assume that* q_0 *is chosen so that*

$$\sum_{m>2q_0} C^*(\lambda^* + \delta)^{\widehat{m}} \le \frac{1}{4}. \tag{65}$$

Then, for any $h, k > 2q_0$, *we have*

$$\left| \sum_{m>2q_0} (a_{hm} - \delta_{hm})(a_{mk} - \delta_{mk}) \right| \le \frac{C^*}{4}(\lambda^* + \delta)^{\max\{\widehat{k},\widehat{h}\}}.$$

In particular, this implies that

$$|(\Delta A_{q_0})^2_{h,k}| \le \frac{C^*}{4}(\lambda^* + \delta)^{\max\{\widehat{k},\widehat{h}\}}. \tag{66}$$

Inductively, one can show that for every $N \ge 2$,

$$|(\Delta A_{q_0})^N_{h,k}| \le \frac{C^*}{4^{N-1}}(\lambda^* + \delta)^{\max\{\widehat{k},\widehat{h}\}}. \tag{67}$$

Proof. Without loss of generality we assume $2q_0 < h \le k$. (Indeed, estimates are symmetric with respect to switching indices h and k.) Using (60) and (65),

$$\begin{aligned}
\left| \sum_{m>2q_0} (a_{hm} - \delta_{hm})(a_{mk} - \delta_{mk}) \right| &\le C^*(\lambda^* + \delta)^{\widehat{k}} \sum_{m>2q_0} |a_{hm} - \delta_{hm}| \\
&\le C^*(\lambda^* + \delta)^{\widehat{k}} \left(\sum_{m>2q_0} C^*(\lambda^* + \delta)^{\widehat{m}} \right) \\
&= \frac{C^*}{4}\ (\lambda^* + \delta)^{\widehat{k}},
\end{aligned}$$

which implies (66). As for (67), it suffices to proceed by induction on N: assume that the estimate holds for $N \ge 2$, then

$$\begin{aligned}
|(\Delta A_{q_0})^{N+1}_{h,k}\| &\le \left| \sum_{m>2q_0} (a_{hm} - \delta_{hm})(\Delta A_{q_0})^N_{m,k} \right| \\
&\le \frac{C^*}{4^{N-1}}(\lambda^* + \delta)^{\widehat{k}} \left(\sum_{m>2q_0} C^*(\lambda^* + \delta)^{\widehat{m}} \right) \\
&= \frac{C^*}{4^N}\ (\lambda^* + \delta)^{\widehat{k}}.
\end{aligned}$$

□

Let us now consider

$$A_{q_0}^{-1} = (\mathbb{I} + \Delta A_{q_0})^{-1} = \mathbb{I} + \sum_{N\geq 1} (-\Delta A_{q_0})^N.$$

Applying Lemma 43, we deduce that the (k, m) entry of the matrix

$$A_{q_0}^{-1} - \mathbb{I},$$

which we denote by a_{km}^-, is bounded by

$$|a_{km}^-| \leq 2C^*(\lambda^* + \delta)^{\widehat{m}}.$$

Then, combining this with the estimates on the decays of the elements of $\vec{B}_{q_0}$ proved in Lemma 39, we obtain the following lemma. (In particular, it uses the that fact that $\sum_{m>2q_0} |b_{j,m}| < +\infty$.)

LEMMA 44. *Let d_{jk} be the (j, k)-entry of*

$$\vec{D}_{q_0} = \vec{B}_{q_0} \cdot A_{q_0}^{-1},$$

with $j \leq 2q_0 < k$. Then there exists $C^* > 0$ such that for all $k > 2q_0$, we have

$$|d_{jk}| \leq C^*(\lambda^* + \delta)^{\widehat{k}}.$$

For each $5 \leq j \leq 2q_0$, we need to compare the maximal strips of analyticity of $\mathbb{e}_j$ and v_j related by (54). Notice that the width of the maximal strip of analyticity of $\mathbb{e}_j$ equals $\rho_{k_{\widehat{j}}}$. On the other hand, using the estimates in Lemma 44 and the analyticity properties of $\mathbb{e}_k$, we conclude that $\sum_{k>2q_0} d_{jk}\mathbb{e}_k$ has strip of analyticity not smaller than $\sigma_\infty(\rho_{k_{q_0}}) - \rho_{k_{q_0}} + \rho_{k_{\widehat{k}}} > \rho_{k_{\widehat{j}}}$, for $j \leq 2q_0 < k$. Hence v_j has width of analyticity $\rho_{k_{\widehat{j}}}$.

COROLLARY 45. *For each $5 \leq j \leq 2q_0$, the functions v_j and $\mathbb{e}_j$ related by (54) are real analytic and have maximal strips of analyticity $\rho_{k_{\widehat{j}}}$.*

Remark 46. The case corresponding to $0 < j \leq 4$ can be treated similarly. Recalling the definitions of these $\mathbb{e}_j$ in Section 7.2 (see also (16)–(20)), it follows that the main modifications correspond to a simpler expression for E_{jm} in (61), in which the denominator disappears and the singularities are given by the ones of $\varphi_m(t)$:

$$E_{jm}(t) = u(\hat{j}\,\varphi_m(t))\left(1 - e_0^2\cos^2\varphi_m(t)\right),$$

where $u(\cdot)$ denotes either sine or cosine. Hence, the corresponding strip of analyticity is independent of j:

$$\sigma_{\widehat{m}} := \frac{2\pi}{4K(k_{\widehat{m}})}\; K(\sqrt{1 - k_{\widehat{m}}^2}).$$

One can prove similarly that the corresponding functions $\{v_j\}_{j=1}^4$ have different strips of analyticity from the ones corresponding to the case $j \geq 5$.

COROLLARY 47. *Any non-trivial linear combination of* $\{v_j\}_{j\le 2q_0}$ *is non-zero*; *i.e.*, *they are linearly independent.*

Proof. The claim easily follows from the fact that we are considering finite linear combinations of analytic functions, with different maximal strips of analyticity. □

Finally, we can conclude the proof of Proposition 30.

Proof of Proposition 30. If we had

$$\sum_{j=1}^{2q_0} \alpha_j \mathbb{e}_j \in \overline{\langle \{\mathbb{e}_j\}_{j>2q_0} \rangle},$$

then

$$\sum_{j=1}^{2q_0} \alpha_j v_j = 0.$$

It follows from Corollary 47 that $\alpha_1 = \cdots = \alpha_{2q_0} = 0$, which completes the proof. □

Appendix E. Some technical lemmata

Let us recall the expression of the angles of the action-angle coordinates; see (9). For the sake of simplicity, as before, we denote by k_q the eccentricity of the caustic of rotation number $1/q$ (with $q \ge 3$):

$$\varphi_q(\xi) := \mathrm{am}\left(\frac{4K(k_q)}{2\pi}\,\xi; k_q\right)$$

and its inverse

$$\xi_q(\varphi) := \frac{2\pi}{4K(k_q)} F\left(\varphi; k_q\right).$$

Similarly, we denote the corresponding functions corresponding to boundary and rotation number 0 (i.e., in the limit as $q \to +\infty$):

$$\varphi_\infty(\xi) := \mathrm{am}\left(\frac{4K(e_0)}{2\pi}\,\xi; e_0\right)$$

and its inverse

$$\xi_\infty(\varphi) := \frac{2\pi}{4K(e_0)} F\left(\varphi; e_0\right),$$

where we have used that $k_q \to e_0^+$ in the limit as $q \to +\infty$.

LEMMA 48. *For each* $q \ge 1$,

$$\xi_q(\xi_\infty) - \xi_\infty = O_{e_0,c}(1/q^2)$$

and

$$k_q - e_0 = O_{e_0,c}(1/q^2).$$

Proof. Observe that

$$\begin{aligned}\xi_q(\xi_\infty) &= \frac{2\pi}{4K(k_q)}\, F\left(\operatorname{am}\left(\frac{4K(e_0)}{2\pi}\,\xi_\infty; e_0\right); k_q\right)\\ &= \xi_\infty + \left[\frac{2\pi}{4K(k_q)}\, F\left(\operatorname{am}\left(\frac{4K(e_0)}{2\pi}\,\xi_\infty; e_0\right); k_q\right) - \xi_\infty\right]\\ &= \xi_\infty + \frac{\pi}{2}\int_{e_0}^{k_q} \underbrace{\frac{\partial}{\partial k}\left(\frac{F\left(\operatorname{am}\left(\frac{4K(e_0)}{2\pi}\,\xi_\infty; e_0\right); k\right)}{K(k)}\right)}_{=:\,\alpha(\xi_\infty,k)}\, dk.\end{aligned} \tag{68}$$

Hence,

$$\begin{aligned}|\xi_q(\xi_\infty) - \xi_\infty| &\le \frac{\pi}{2}\left(\max_{e_0\le k\le k_3(e_0)}\ \max_{\xi_\infty\in[0,2\pi)} |\alpha(\xi_\infty, k)|\right)(k_q - e_0)\\ &\le C(e_0, a)(k_q - e_0).\end{aligned} \tag{69}$$

In order to conclude the proof, we need to estimate $k_q - e_0$. By definition of $k_q = k_{\lambda_q}$ (see Proposition 16), we have

$$k_q^2 = \frac{a^2 - b^2}{a^2 - \lambda_q^2} = \frac{a^2 e_0^2}{a^2 - \lambda_q^2},$$

from which we deduce that

$$\begin{aligned}k_q - e_0 &= \frac{a e_0}{\sqrt{a^2 - \lambda_q^2}} - e_0\\ &= \frac{e_0 \lambda_q^2}{\sqrt{a^2 - \lambda_q^2}\left(a + \sqrt{a^2 - \lambda_q^2}\right)}.\end{aligned} \tag{70}$$

Using definition (8) we obtain

$$\frac{2}{q} = \frac{F(\arcsin\frac{\lambda_q}{b}; k_q)}{K(k_q)} \quad\Longleftrightarrow\quad \frac{2}{q}\,K(k_q) = F(\arcsin\frac{\lambda_q}{b}; k_q). \tag{71}$$

Rewriting, using the definition of both F and K and the fact that $b = a\sqrt{1 - e_0^2}$, we obtain an implicit equation for λ_q (observe that $k_q = k_q(\lambda_q)$):

$$\frac{2}{q}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k_q^2\,\sin^2\varphi}} = \int_0^{\arcsin\lambda_q/(a\sqrt{1-e_0^2})} \frac{d\varphi}{\sqrt{1 - k_q^2\,\sin^2\varphi}}. \tag{72}$$

Since $k_q \in [e_0, k_3]$ for all $q \ge 3$, then

$$1 \le \frac{1}{\sqrt{1 - k_q^2\,\sin^2\varphi}} \le \frac{1}{\sqrt{1 - k_3^2}},$$

and hence if we substitute in (72), we deduce

$$\frac{\pi}{q}\frac{1}{\sqrt{1-k_3^2}} \geq \arcsin\left(\frac{\lambda_q}{a\sqrt{1-e_0^2}}\right).$$

In particular, if $q \geq 2/\sqrt{1-k_3^2} =: q_0(e_0)$, then we have

$$\lambda_q \leq a\sqrt{1-e_0^2}\,\sin\left(\frac{\pi}{q}\frac{1}{\sqrt{1-k_3^2}}\right); \tag{73}$$

namely, $\lambda_q = O_{e_0,a}(1/q)$.

Substituting in (70) and (69), and observing that $c = a\sqrt{1-e_0^2}$, we conclude that

$$\xi_q(\xi_\infty) - \xi_\infty = O_{e_0,c}(1/q^2), \quad \text{and} \quad k_q - e_0 = O_{e_0,c}(1/q^2). \qquad \square$$

LEMMA 49. *Let $f : [0, 2\pi) \longrightarrow \mathbb{R}$ be a C^1 function. Then, there exists $C = C(e_0, c)$ such that for each $q \geq 3$,*

$$\left|\int_0^{2\pi} f(\varphi)c_q(\varphi)\,d\varphi\right| \leq \frac{C\,\|f\|_{C^1}}{q} \qquad \textit{and} \qquad \left|\int_0^{2\pi} f(\varphi)s_q(\varphi)\,d\varphi\right| \leq \frac{C\,\|f\|_{C^1}}{q}.$$

Proof. If follows from the definition of c_q (see (15)), ξ_q, φ_∞ and ξ_∞, that

$$\begin{aligned}\int_0^{2\pi} f(\varphi)c_q(\varphi)\,d\varphi &= \frac{4K(k_q)}{2\pi}\int_0^{2\pi} f(\varphi)\cos(q\,\xi_q(\varphi))\,\xi_q'(\varphi)\,d\varphi\\ &= \frac{4K(k_q)}{2\pi}\int_0^{2\pi} f(\xi_\infty)\cos(q\,\xi_q(\xi_\infty))\,\xi_q'(\xi_\infty)\,\varphi_\infty'(\xi_\infty)d\xi_\infty\\ &= \frac{4K(k_q)}{2\pi}\int_0^{2\pi} f(\xi_\infty)\cos(q\,\xi_q(\xi_\infty))\,\frac{d}{d\xi_\infty}\left(\xi_q(\xi_\infty)\right)\,d\xi_\infty.\end{aligned}$$

Using Lemma 48:

$$\begin{aligned}\int_0^{2\pi} f(\varphi)c_q(\varphi)\,d\varphi &= \frac{4K(k_q)}{2\pi}\int_0^{2\pi}\left(f(\xi_\infty)\cos(q\xi_\infty) + O_{e_0,c}(1/q)\right)\,d\xi_\infty\\ &= \frac{4K(k_q)}{2\pi}\int_0^{2\pi} f(\xi_\infty)\cos(q\xi_\infty)\,d\xi_\infty + O_{e_0,c}\left(\frac{\|f\|_{C^0}}{q}\right).\end{aligned}$$

Observe that $\varphi_\infty = \varphi_\infty(\xi_\infty)$ is an analytic function, so $f(\xi_\infty)$ is C^1 and its q-th Fourier coefficient are $O_{e_0,c}(\|f\|_{C^1}/q)$; hence we conclude

$$\int_0^{2\pi} f(\varphi)c_q(\varphi)\,d\varphi = O_{e_0,c}\left(\frac{\|f\|_{C^1}}{q}\right),$$

which proves the first relation. In the same way, one proves the other one involving s_q. $\square$

For $q \in \mathbb{N}$ and $j \geq 3$, let us consider the elements of the (infinite) correlation matrix $\widetilde{A} = (\tilde{a}_{i,h})_{i,h=0}^{\infty}$, introduced in (26), Section 7.5.

LEMMA 50. *There exists $\rho = \rho(e_0, c) > 0$ such that for all $q \in \mathbb{N}$ and $j \geq 6$,*

$$\tilde{a}_{q,j} = 2K(k_{[j/2]})\,\delta_{q,j} + O_{e_0,c}\left(j^{-1}\, e^{-\rho\,|q-j|}\right),$$

where $[\cdot]$ denotes the integer part and $\delta_{q,j}$ the Dirac's delta.

Proof. We proceed similarly to what done in Lemma 49. In particular, recall formula (68)

$$\xi_q(\xi_\infty) = \xi_\infty + \frac{\pi}{2}\int_{e_0}^{k_q} \alpha(\xi_\infty, k)dk =: \xi_\infty + \Delta_q(\xi_\infty).$$

Observe that Δ_q is analytic in a complex strip of width at least $\rho = \rho(e_0, c) > 0$ (independent of q) and that there exists $C = C(e_0, c)$ such that $q^2\|\Delta_q\|_\rho \leq C$ for all $q \geq 3$, where $\|\cdot\|_\rho$ denotes the analytic norm of the function in the strip $\{|\mathrm{Im} z| \leq \rho\}$ (namely, the sup-norm on this closed strip of the modulus of its complex extension). This follows from the second part of Lemma 48, namely, the fact that $q^2(k_q - e_0)$ is uniformly bounded.

Recalling the definition of c_q, s_q, ξ_q, φ_∞ and ξ_∞, we obtain the following (we prove it only in one case, the proofs of the others are identical):

$$\begin{aligned}
\tilde{a}_{2q,2j} &= \int_0^{2\pi} \cos(q\varphi)\, c_j(\varphi)\, d\varphi \\
&= \frac{4K(k_j)}{2\pi}\int_0^{2\pi} \cos(q\varphi)\cos(j\,\xi_j(\varphi))\,\xi_j'(\varphi)\,d\varphi \\
&= \frac{4K(k_j)}{2\pi}\int_0^{2\pi} \cos(q\xi_\infty)\cos(j\,\xi_j(\xi_\infty))\,\xi_j'(\xi_\infty)\,\varphi_\infty'(\xi_\infty)d\xi_\infty \\
&= \frac{4K(k_j)}{2\pi}\int_0^{2\pi} \cos(q\xi_\infty)\cos(j\,\xi_j(\xi_\infty))\,\frac{d}{d\xi_\infty}\left(\xi_j(\xi_\infty)\right)\,d\xi_\infty \\
&= \frac{4K(k_j)}{2\pi}\int_0^{2\pi} \cos(q\xi_\infty)\cos(j\,\xi_\infty + j\Delta_j(\xi_\infty))\,\left(1 + \frac{d}{d\xi_\infty}\Delta_j(\xi_\infty)\right)\,d\xi_\infty \\
&= \frac{4K(k_j)}{2\pi}\int_0^{2\pi} \cos(q\xi_\infty)\big[\cos(j\xi_\infty)\cos(j\Delta_j(\xi_\infty)) - \sin(j\xi_\infty)\sin(j\Delta_j(\xi_\infty))\big] \\
&\qquad\qquad \cdot\left(1 + \frac{d}{d\xi_\infty}\Delta_j(\xi_\infty)\right)\,d\xi_\infty \\
&= \frac{K(k_j)}{\pi}\int_0^{2\pi} \Big[\big[\cos((q+j)\xi_\infty) + \cos((q-j)\xi_\infty)\big]\cos(j\Delta_j(\xi_\infty)) \\
&\qquad\qquad - \big[\sin((q+j)\xi_\infty) - \sin((q-j)\xi_\infty)\big]\sin(j\Delta_j(\xi_\infty))\Big] \\
&\qquad\qquad \cdot\left(1 + \frac{d}{d\xi_\infty}\Delta_j(\xi_\infty)\right)\,d\xi_\infty
\end{aligned}$$

$$
\begin{aligned}
&= 2K(k_j)\delta_{q,j} \\
&\quad + \frac{K(k_j)}{\pi}\int_0^{2\pi} \Big[\big[\cos((q+j)\xi_\infty)+\cos((q-j)\xi_\infty)\big]\big(\cos(j\Delta_j(\xi_\infty))-1\big) \\
&\qquad\qquad - \big[\sin((q+j)\xi_\infty)-\sin((q-j)\xi_\infty)\big]\sin(j\Delta_j(\xi_\infty))\Big] \\
&\qquad\qquad \cdot \left(1+\frac{d}{d\xi_\infty}\Delta_j(\xi_\infty)\right) d\xi_\infty.
\end{aligned}
$$

Since Δ_j is analytic in the strip of width ρ, then also $\frac{d}{d\xi_\infty}\Delta_j$, $\sin(j\Delta_j)$ and $\cos(j\Delta_j)-1$ are analytic in the same strip and their analytic norm in the strip of width ρ is at most $O_{e_0,c}(1/j)$; hence, their Fourier coefficients decay exponentially. It suffices to notice that the above integral consists of a combination of their Fourier coefficients. Therefore,

$$
\tilde{a}_{2q,2j} = 2K(k_j)\,\delta_{q,j} + O_{e_0,c}\left(\frac{1}{j}e^{-\rho\,|q-j|}\right). \qquad \square
$$

Appendix F. From local to global Birkhoff conjecture

In this appendix we want to outline some ideas on how to use our local results to prove the global Birkhoff conjecture. Roughly speaking, we would like to use the Affine Length Shortening (ALS) PDE flow, which evolves any convex domain into an ellipse [42], in order to extend our result from a small neighborhood of ellipses to all strictly convex domains. The idea we outline here is to find a *Lyapunov function* for the flow, which measures the *non-integrability* of a domain. Moreover, we propose to reduce the analysis to glancing periodic orbits, which stay in a nearly integrable zone during the whole ALS evolution.

F.1. *Affine length and affine curvature of a plane curve.* Let us first recall some definitions (see, for instance, [42]). Let $C : \mathbb{T} \to \mathbb{R}^2$ be an embedded closed curve with curve parameter p. A reparametrization s can be chosen so that in the new parameter s one has (hereafter we will use the shorthand to use subscripts to denote derivatives)

$$
[C_s, C_{ss}] = 1,
$$

where $[X, Y]$ stands for the determinant of the 2×2 matrix whose columns are given by vectors $X, Y \in \mathbb{R}^2$. Notice that the relation is invariant under the $\mathrm{SL}_2(\mathbb{R})$-transformations. Call the parameter s the *affine arc-length*; in particular, if

$$
g(p) = [C_p, C_{pp}]^{1/3}
$$

then the parameter s is explicitly given by

$$
s(p) = \int_0^p g(\xi)\, d\xi.
$$

Assume $g(\xi)$ is non-vanishing, which is automatically satisfied for strictly convex curves.

Call *the affine curvature* $\nu(s)$ the function given by

$$\nu = [C_{ss}, C_{sss}].$$

The affine perimeter for the closed curve C is then defined by

$$L := \oint g(p)\, dp.$$

Remark 51. In analogy with what happens for the Euclidean curvature, the curves of constant affine curvature ν are precisely all non-singular conics. More specifically, those with $\nu > 0$ are ellipses, those with $\nu = 0$ are parabolas, and those with $\nu < 0$ are hyperbolas.

To conclude this subsection, let us point out the relation between the (constant) affine curvature of an ellipse ν_0 and its instant eccentricity μ_0 (in elliptic coordinates). One can easily show that

$$\mu_0 = \operatorname{arsinh}(2\nu_0^{-3/2}/c^2)/2.$$

Moreover, if we consider a domain Ω, which is ε-close to an ellipse $\mathcal{E}$ (of instant eccentricity μ_0 and affine curvature ν_0), and we denote by $\nu(s)$ the affine curvature of $\partial\Omega$ and by $\mu(s) = \mu_0 + \varepsilon\mu_1(s)$ the instant eccentricity in the elliptic coordinate frame associated to $\mathcal{E}$, as in (11); then,

$$\mu(s) = f(\nu(s)) = f(\nu_0 + \varepsilon\nu_1) = f(\nu_0) + \varepsilon f'(\nu_0)\nu_1 + O(\varepsilon^2),$$

where $f(a) = \operatorname{arsinh}(2a^{-3/2}/c^2)/2$. Thus, Fourier expansion of μ_1 coincides with Fourier expansion of ν_1 up to $O(\varepsilon^2)$-error.

F.2. *Affine Length Shortening* (ALS) *flow.* The study of evolution of plane curves in the direction of the Euclidean normal with speed proportional to the Euclidean curvature (also known as *curve-shortening flow*) has been intensively studied; see, for example, [15] and references therein. The classical result says that the Euclidean curvature evolution is a "*Euclidean curve shortening*" and flows every convex domain toward a circle.[11] More specifically, for any closed convex curve, the isoperimetric ratio (i.e., the ratio between the squared curve length and the area) decreases monotonically (and in finite time) to 4π, i.e., the value of this ratio for circles.

Adapting this idea, Sapiro and Tannenbaum [42] developed an analogous flow describing the evolution of plane curves in the direction of the *affine* normal, with speed proportional to the *affine* curvature; this flow is generally

[11] Actually it shrinks every curve to a point. However, rescaling of either the perimeter (or the area) the curve will converge to a circle.

called the *affine length shortening* (ALS) *flow* (or *affine curvature flow*) and, analogously to the Euclidean one, it is "affine length shortening." Similarly to the Euclidean curvature evolution, in fact, this flow evolves every convex domain to an ellipse. More specifically, the isoperimetric ratio (i.e., the ratio between the squared affine curve length and the area) decreases monotonically to $8\pi^2$, which is the ratio for ellipses[12].

F.3. *Application to billiards.* Our idea is to apply the above geometric flow to deduce the non-integrability of a domain, by means of a suitable *Lyapunov function*. Let us describe this construction more specifically.

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain with a sufficiently smooth boundary $\partial\Omega$. Let s be the arc-length parameter of the boundary, and let us denote by $|\partial\Omega|$ its Euclidean perimeter.

For each $q > 2$ and for every point s on the boundary, let us denote by $L_{1/q}(s)$ the maximal perimeter of a q-gon starting at this point. For each $q > 2p > 1$ and for every point s on the boundary, let us denote by $L_{p/q}(s)$ the maximal perimeter of a star shape q-gon starting at this point whose rotation number is p/q and the points are ordered on the boundary in the same cyclical order as the rotation by p/q. Notice that if there exists an integrable rational caustic of rotation number p/q, then $L_{p/q}(s)$ is constant or, equivalently,

(74)
$$\Delta_{p/q} := \int_0^{|\partial\Omega|} (L_{p/q}(s) - \langle L_{p/q}\rangle)^2\, ds = 0, \quad \text{where } \langle L_{p/q}\rangle = \int_0^{|\partial\Omega|} L_{p/q}(s) ds.$$

Suppose now that for any strictly convex domain Ω that is sufficiently close to an ellipse, but not an ellipse, the billiard map in Ω satisfies one of the two conditions:

(1) either it has caustics for all rotation numbers in $(0, 1/q_0]$ for some $q_0 > 2$;
(2) or it has a sequence $q_k \to \infty$, as $k \to \infty$, and a sequence p_k such that $\frac{p_k}{q_k} \to 0$ and there is no integrable rational caustic of rotation number $\frac{p_k}{q_k}$ or, equivalently, $\Delta_{p_k/q_k} \neq 0$.

This situation corresponds to a stronger version of the local Birkhoff conjecture than the one proved in the present article. So far, this picture has been proven to hold true only for ellipses of small eccentricities (see [21]).

Recall that, as we explained in Section F.2, for any convex domain Ω, different from an ellipse, its evolution Ω_t under the ALS flow brings it into a neighborhood of the ellipses. Thus, for some $T > 0$, we have that

- Ω_T belongs to a neighborhood of ellipses;

[12]Also in this case the flow shrinks every curve to a point. However, under rescaling of either the perimeter (or the area) the curve will converge to an ellipse.

- there is a sequence $q_k \to \infty$ as $k \to \infty$ and a sequence p_k such that $p_k/q_k \to 0$ and the billiard map associated to Ω_T has no integrable rational caustics of rotation number p_k/q_k or, equivalently, $\Delta_{p_k/q_k} \neq 0$.

We conjecture the following.

CONJECTURE. *Let Ω_t be the evolution of the domain Ω under the normalized affine curvature flow (i.e., we keep the perimeter, or the area, of the domain fixed along the flow), and let $\Delta^t_{p/q}$ be the $\Delta_{p/q}$–function associated Ω_t. Then, there exists $q_0 = q_0(\Omega) > 2$ such that for some rational $0 < p/q < 1/q_0$, we have that $\Delta^t_{p/q}$ is monotone in t.*

Hereafter we verify a local version of this conjecture when Ω is the unit circle. See Lemma 52.

F.4. ALS *flow evolution*. Let us first describe some results on the ALS flow. In [42, (32), p. 96], the formula for the evolution of the affine curvature ν is derived:

$$\frac{\partial \nu}{\partial t} = \frac{4}{3}\nu^2 + \frac{1}{3}\nu_{ss}. \tag{75}$$

Let us describe what happens in the case of ellipses, i.e., $\nu \equiv \nu_0$; in particular, we want to point out a subtlety of this flow, namely, certain blow up in a finite time. Then

$$\frac{\partial \nu}{\partial t} = \frac{4}{3}\nu^2$$

becomes an ODE. If we make a substitution $\nu = \chi^{-1}$, then

$$\frac{4}{3}\nu^2 = \frac{\partial \nu}{\partial t} = -\frac{1}{\chi^2}\,\frac{\partial \chi}{\partial t} = -\nu^2\,\frac{\partial \chi}{\partial t}.$$

Thus, $\chi(t) = \chi_0 - \frac{4t}{3}$ and $\nu_0(t) = \frac{3}{3\chi_0 - 4t}$. Notice that in finite time $\nu_0(t)$ blows up. It corresponds to the area of the corresponding curve converging to zero. See discussions in [42, §7.1]. In [42, §8.1] bounds on the time of blow up are presented in terms of minimal and maximal affine curvature ν.

Denote the above solution as $\nu_0(t)$. Notice that one needs to rescale ν, e.g., to keep the area inside the domain fixed. If no rescaling is done, then the domain collapses to a point. Indeed, let $\nu(s,t) = \nu_0(t) + \varepsilon\Delta\nu(s,t)$ for small ε. Then we get

$$\begin{aligned}\frac{\partial \nu}{\partial t} &= \frac{\partial \nu_0}{\partial t} + \varepsilon\frac{\partial \Delta\nu}{\partial t}\\ &= \frac{4}{3}(\nu_0^2 + 2\varepsilon\nu_0\Delta\nu + \varepsilon^2\Delta\nu^2) + \frac{1}{3}(\nu_0)_{ss} + \frac{1}{3}\varepsilon\Delta\nu_{ss}.\end{aligned}$$

Simplifying,

$$\frac{\partial \Delta\nu}{\partial t} = \frac{4}{3}(2\nu_0\Delta\nu + \varepsilon\Delta\nu^2) + \frac{1}{3}\Delta\nu_{ss}.$$

Rewriting as a Fourier expansion

$$\Delta\nu(s,t) = \sum_{k\in\mathbb{Z}} \Delta\nu_k(t) e^{iks},$$

we obtain

$$\frac{\partial \Delta\nu_k}{\partial t} = \left(\frac{8}{3}\nu_0(t) - \frac{k^2}{3}\right)\Delta\nu_k + O(\varepsilon(\Delta\nu^2)_k),$$

where $(\Delta\nu^2)_k$ is the k'th Fourier coefficient of $\Delta\nu^2$. This shows that for each $|k| > \sqrt{8\,\nu_0(t)}$ such that $\Delta\nu_k \neq 0$, for ε small enough, this Fourier coefficient decays along the ALS flow. We will use this fact to prove that locally in time the functional Δ_q decays monotonically; see Lemma 52.

F.5. *Preservation of rational caustics.* In this section we relate the presence of an integrable rational caustic of rotation number $1/q$ to properties of resonant Fourier coefficients, i.e., those with index divisible by q.

Let us first recall the following facts. In [37, §4] they study small perturbation of ellipses. Following notation of [37, §4], we have that the perimeter

$$L_\varepsilon = L_0 + \varepsilon L_1 + O(\varepsilon^2),$$

is given by [37, formula (5)]. (Here we the drop subindex q.) Then by [37, Prop. 4.1] (see also Proposition 20), the linear term in ε has the form

$$L_1(\varphi) = 2\lambda \sum_{k=0}^{q-1} \mu_1(\varphi_q^k), \tag{76}$$

where λ is the parameter associated to a given caustic (see also Section 3.2) and μ_1 represents the first-order perturbation (in ε) of the boundary (see Section 5).

Let us now consider the usual polar coordinates and let $\Omega = \{(\rho,\varphi) : \rho = \rho_0\}$ be the circle centered at the origin and radius ρ_0. We are interested in studying small perturbations given by

$$\Omega_\varepsilon = \{(\rho_\varepsilon,\varphi) :\ \rho_\varepsilon = \rho_0 + \varepsilon\rho_1(\varphi) + O(\varepsilon^2)\}, \tag{77}$$

where ρ_1 is a C^r smooth function for $r \geq 2$. Assume by rescaling that $\rho_1(0) = 1$. Expand the perturbation in the Fourier series:

$$\rho_1(\varphi) = \sum_{j\in\mathbb{Z}} \rho_1^{(j)} e^{ij\varphi}.$$

We show that for perturbations of the circle and for an appropriate choice of q, the existence of integrable rational caustics depends on resonant Fourier coefficients, i.e., those divisible by q. In fact, plug the rigid rotation $\varphi \mapsto \varphi + \frac{2\pi}{q}$ into (76). Denote by $\Delta_q(\rho_1,\varepsilon)$ the value of the function $\Delta_{1/q}$ associated to

the domain Ω_ε, as defined in (74). Using (76) we have that for some $c > 0$ independent of ε,

$$\begin{aligned} L_\varepsilon &= L_0 + c\varepsilon \sum_{k=0}^{q-1} \rho_1\left(\varphi + \frac{2\pi k}{q}\right) + O(\varepsilon^2) \\ &= L_0 + c\varepsilon \sum_{k=0}^{q-1} \sum_{j\in\mathbb{Z}} \rho_1^{(j)} e^{ij\left(\varphi + \frac{2\pi k}{q}\right)} + O(\varepsilon^2). \end{aligned}$$

Thus,

$$\Delta_q(\rho_1, \varepsilon) := c^2\varepsilon^2\, q^2 \sum_{j\in\mathbb{Z}\setminus\{0\}} (\rho_1^{(jq)})^2 + O(\varepsilon^3). \tag{78}$$

Consider the domain Ω_ε defined in (77). The vanishing of the function $\Delta_q(\rho_1, \varepsilon)$ detects the existence of an integrable rational caustic of rotation number $1/q$. According to our computations this function has an asymptotic expansion (78). Denote by Ω_ε^t the image of Ω_ε under the ALS flow (75).

LEMMA 52. *Let $\rho_\varepsilon(\varphi)$, $\varepsilon \geq 0$ be the family of domains in* (77). *Assume that $q > 2$ and that $\rho_1^{(q)} \neq 0$. Then, for ε sufficiently small, the family of domains Ω_ε^t for $0 \leq t \leq \varepsilon$ satisfies*

$$\frac{\partial \Delta_q^t(\rho_1, \varepsilon)}{\partial t} < 0.$$

Proof. Notice that up to a affine-length parametrization, s and polar angle φ are the same. Consider the derivative with respect to the affine length shortening flow (75) of $\Delta_q^t(\rho_1, \varepsilon)$. According to (78), this leads to the derivative of the resonant Fourier coefficients. For each $j > 0$, we have that

$$\frac{\partial \rho_1^{(jq)}}{\partial t} = \varepsilon \left[\left(\frac{8}{3}\rho_1^{(0)} - \frac{j^2q^2}{3} \right) \rho_1^{(jq)} + \varepsilon \sum_{p\in\mathbb{Z}\setminus\{0\}} \rho_1^{(jq-p)} \rho_1^{(p)} + O(\varepsilon^2) \right].$$

It follows from (76) and (78) that

$$\Delta_{q,\varepsilon}^t = c^2\varepsilon^2 q^2 \sum_{j\in\mathbb{Z}\setminus\{0\}} \left(\rho_1^{(jq)}\right)^2 + O(\varepsilon^3).$$

Consider

$$\frac{\partial}{\partial t} \sum_{j\in\mathbb{Z}\setminus\{0\}} \left(\rho_1^{(jq)}\right)^2 =$$

$$= \varepsilon \sum_{j\in\mathbb{Z}\setminus 0} \left[\left(\frac{8}{3}\rho_1^{(0)} - \frac{j^2q^2}{3} \right) \left(\rho_1^{(jq)}\right)^2 + \varepsilon \sum_{p\in\mathbb{Z}\setminus\{0\}} \rho_1^{(jq-p)} \rho_1^{(p)} \rho_1^{(jq)} + O(\varepsilon^2) \right].$$

Since $\rho_1^{(q)} \neq 0$, for ε small enough, the last expression is negative. □

Concluding remarks. This lemma is certainly only an example of, what we believe, is a much more general phenomenon. More specifically, we conjecture

Monotonicity of the functional Δ_q *along the* ALS *flow* (75).[13]

The next step would be a local analysis of the ALS flow in a neighborhood of ellipses. It would be more challenging to extend this local analysis to the space of strictly convex domains, and this will be an important step to prove the global Birkhoff conjecture.

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