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Deformational spectral rigidity of axially symmetric symplectic billiards

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Abstract

Symplectic billiards are discrete dynamical systems which were introduced by Albers and Tabachnikov and take place in a strongly convex bounded planar domain with smooth boundary. They are described by the *symplectic law of reflection*, in contrast to the elastic reflection law of Birkhoff billiards. In this paper, we prove a version of dynamical spectral rigidity for symplectic billiards which is a counterpart to previous results on classical billiards by De Simoi, Kaloshin and Wei. Namely, we show that close to an ellipse, any sufficiently smooth one-parameter family of axially symmetric domains either contains domains with different area spectra or is trivial, in the sense that the domains differ by area-preserving affine transformations of the plane. We also prove that in general—that is, even if the domains are not close to an ellipse—any sufficiently smooth one-parameter family of axially symmetric domains which preserves the area-spectrum is tangent to a finite-dimensional space.

Keywords: symplectic billiards, spectral rigidity, action spectrum, hamiltonian dynamics, periodic orbits, isospectrality

Mathematics Subject Classification numbers: 37E40, 37J35, 37J50, 53C24

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1. Introduction

In this paper we consider *symplectic billiards*, which were introduced by Albers and Tabachnikov [1]. These systems involve bounded, strongly convex, planar domains with smooth boundaries. The symplectic billiard map concatenates line segments according to the *symplectic law of reflection*. This law can be described as follows (see [1] for more details): if a particle is emitted from a point q_1 on the boundary of the domain and strikes the boundary again at q_2, then following the line segment q_1q_2, it will bounce at q_2 and continue along a new trajectory until it hits the boundary at point q_3. The point q_3 is uniquely determined by the condition that the line q_1q_3 is parallel to the tangent line to the boundary at q_2 (see figure 1).

The term *symplectic billiard* is derived from the fact that the dynamics of such systems can be described by a variational principle related to the standard symplectic (area) form. Let ω denote the standard symplectic form on the plane, and let q_1, q_2, q_3 be distinct points on the domain's boundary. A symplectic billiard bounce from q_1 to q_2 to q_3, as previously described, occurs if and only if q_2 is a critical point of the quantity

$$\omega(q_1, q_2) + \omega(q_2, q_3).$$

More precisely, the billiard map associated to symplectic billiards is an exact-symplectic twist-map of an annulus, whose generating function is given by ω [1].

1.1. Main results

In this paper, we establish a rigidity result for symplectic billiards, described as follows. Corresponding to the symplectic billiard in a domain Ω, we define the area spectrum $\mathcal{A}(\Omega)$ to be the closure of the set of actions of its periodic trajectories. This area spectrum is related to the set of areas enclosed by periodic trajectories. Due to the underlying symplectic structure, the area spectrum remains unchanged if we apply an area-preserving affine transformation of the plane to the domain [1]. The inverse problem asks whether or not it is possible to uniquely determine a domain Ω, up to affine transformation, from its area spectrum. Specifically, we provide a partial answer to the following question:

Question. *Let $r > 0$ be an integer, I an open interval containing 0 and $(\Omega_\tau)_{\tau\in I}$ a $\mathscr{C}^r$-smooth one-parameter family of strongly convex planar domains with $\mathscr{C}^r$-smooth boundaries such that $\mathcal{A}(\Omega_\tau) = \mathcal{A}(\Omega_0)$ for any $\tau \in I$. Is it true that $\Omega_\tau = f_\tau(\Omega_0)$, where f_τ is an area-preserving affine transformation of the plane?*

A deformation $(\Omega_\tau)_{\tau\in I}$ of Ω_0 is said to be *affine* if for any $\tau \in I$, there is an area-preserving affine transformation $f_\tau : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Omega_\tau = f_\tau(\Omega_0)$. If $\mathcal{D}$ is a subset of all possible domains and $\Omega \in \mathcal{D}$, we say that Ω is *area spectrally rigid in $\mathcal{D}$* if the answer to the previous question is positive for any sufficiently smooth family $(\Omega_\tau)_{\tau\in I}$ with $\Omega_0 = \Omega$ and $\Omega_\tau \in \mathcal{D}$ for all $\tau \in I$.

To state the result, we consider the set $\mathcal{S}$ of bounded, axially symmetric, strongly convex domains whose boundary is a $\mathscr{C}^8$-smooth embedded submanifold. We endow $\mathcal{S}$ with the $\mathscr{C}^8$-topology on the set of embeddings of domains' boundaries up to reparameterization by a $\mathscr{C}^8$-smooth diffeomorphism.

Theorem 1.1 (Perturbative result). *Any domain $\Omega \in \mathcal{S}$ which is sufficiently $\mathscr{C}^8$-close to an ellipse is area spectrally rigid in $\mathcal{S}$.*

Remark 1.2. Theorem 1.1 was proved independently by Baracco *et al* [4] using similar ideas and estimates.

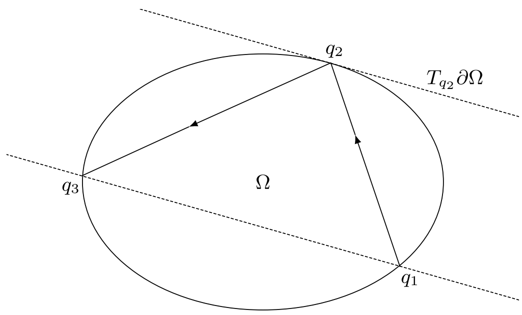

Figure 1. A symplectic bounce at q_2 in a domain Ω. The line q_1q_3 and the tangent line to $\partial\Omega$ at q_2 are parallel.

We refer to theorem 1.1 as *perturbative* because it applies to domains which are *close* to ellipses in a suitable topology. For a domain $\Omega \in \mathcal{S}$ which is not necessarily $\mathscr{C}^8$-close to an ellipse, we can still provide a condition on the one-parameter families $(\Omega_\tau)_{\tau\in I}$ in $\mathcal{S}$ for which the area spectrum $\mathcal{A}(\Omega_\tau)$ remains constant.

A one-parameter family Ω_τ which is $\mathscr{C}^8$-smooth in τ can be described using a collection of $\mathscr{C}^6$-smooth maps $(n_\tau)_\tau$ known as *deformation maps*. Given τ, the map n_τ roughly corresponds to local variation of the boundary of Ω_τ. More precisely, the map $n_\tau : \partial\Omega_\tau \to \mathbb{R}$ assigns to each point $q_\tau \in \partial\Omega_\tau$ a value that approximates the area of the parallelogram formed by the unit tangent vector to $\partial\Omega_\tau$ at q_τ and the vector $\partial_\tau q_\tau$. The regularity requirement for n_τ comes from the observation that the first variation of a quantity defined with respect to a $\mathscr{C}^7$-smooth parametrization of the boundaries is $\mathscr{C}^6$-smooth.

Theorem 1.3 (Non perturbative result). *Let $(\Omega_\tau)_{\tau\in I}$ be a one-parameter family of domains in $\mathcal{S}$ such that $\mathcal{A}(\Omega_\tau) = \mathcal{A}(\Omega_0)$ for all τ. Then, there is a continuous family $(V_\tau)_\tau$ of finite-dimensional vector spaces V_τ of the space $\mathscr{C}^6(\partial\Omega_\tau, \mathbb{R})$ such that $n_\tau \in V_\tau$. Moreover, V_0 is uniquely determined by Ω_0.*

Remark 1.4. A similar result in the case of smooth integrable deformations of a strictly convex integrable Birkhoff billiard preserving the integrability near the boundary has been proven in [9].

Remark 1.5. The proof of theorem 1.1 relies on the existence of symmetric periodic orbits of rotation number $1/q$ for each $q \geqslant 3$: studying how their lengths change within a τ-deformation (Ω_τ) provides information on the family of domains itself. This information is enough to prove theorem 1.1 when Ω_0 is close to an ellipse, but this seemingly does not suffice for a general domain. We still believe that the statement of theorem 1.1 holds even if Ω_0 is not close to an ellipse, but to prove it, we would need to consider a larger family of periodic orbits.

1.2. Background

This work was inspired by a breakthrough by De Simoi *et al* [6] in the context of classical Birkhoff billiards, namely, billiard models in which the reflection law at impact points is given by the optical law: the angle of incidence equals the angle of reflection. In that case, isospectrality is with respect to the *lengths* of periodic orbits as opposed to areas. One also studies the spectrum of the Laplacian with suitable (e.g. Dirichlet) boundary conditions on a billiard table. In 1967, Mark Kac asked the famous question *'Can one hear the shape of a drum?'*, i.e. are Laplace isospectral sets (modulo Euclidean isometries) multiplicity free? While it has been shown that the general answer to this question is negative [7], it remains unresolved for the class of strongly convex domains with smooth boundaries. Melrose [14] and Osgood, Phillips, and Sarnak [16–18] demonstrated that Laplace-isospectral sets of planar domains are compact in the $\mathscr{C}^\infty$ topology. The third author proved an analogous result for the marked length spectrum [23]. In the setting of one-parameter families of domains, Hezari and Zelditch [8] provided a positive answer for analytic Laplace isospectral deformations of ellipses which preserve biaxial reflectional symmetries, as well as flatness of the corresponding variations in the $\mathscr{C}^\infty$ setting. These results were further extended by Popov and Topalov [20].

1.3. Outline of the paper

The paper is organized as follows. In section 2 we introduce more precisely the symplectic billiard map as well as its associated area spectrum. Section 3 is devoted to the construction of an operator which we call *linear isospectral operator*—following the terminology in [6]: the latter is associated with a specific domain and its injectivity is related to the rigidity of the corresponding domain. We prove our main theorems 1.1 and 1.3 using the key estimates given by proposition 3.10. Appendix A recalls the notion of affine parametrization of planar curves. Appendix B presents explicit formulae for periodic orbits of the symplectic billiard in an ellipse. Appendix C gives asymptotic estimates for periodic orbits of the symplectic billiard in a domain which is sufficiently $\mathscr{C}^8$-close to an ellipse; these estimates generalize estimates which can be found in [3]. Appendix D states invertibility properties of operators acting on Sobolev spaces.

2. Preliminaries on symplectic billiards

2.1. Symplectic billiard map

Let Ω be a bounded, strongly convex planar domain and $\gamma : \mathbb{R}/L\mathbb{Z} \longrightarrow \mathbb{R}^2$ a parametrization of $\partial\Omega$, where $L > 0$. Given a point $\gamma(t)$, denote by $\gamma(t^*)$ the other point on $\partial\Omega$ where the tangent line is parallel to that of $\partial\Omega$ at $\gamma(t)$. According to [1], the phase space of the symplectic billiard map in Ω is then the set of the oriented chords $\gamma(t_0)\gamma(t_1)$ where $t_0 < t_1 < t_0^*$, with respect to the orientation of γ; alternatively, the phase space can be described as the set

$$\mathcal{X}_\Omega := \{(t_0,t_1) :\ \omega(\dot\gamma(t_0),\dot\gamma(t_1)) > 0\}$$

where ω denotes the standard area form in the plane (i.e. the determinant of a matrix formed by two vectors). The vertical foliation consists of all chords with a fixed initial point. The symplectic billiard map is given by:

$$\begin{aligned} B_\Omega : \mathcal{X}_\Omega &\longrightarrow \mathcal{X}_\Omega, \\ (t_0,t_1) &\longmapsto (t_1,t_2), \end{aligned} \tag{1}$$

where (t_1,t_2) is uniquely determined by the condition that the tangent line to $\partial\Omega$ at $\gamma(t_1)$ is parallel to the line $\gamma(t_0)\gamma(t_2)$ (see figure 1). The map B_Ω is an exact symplectic twist map and can be extended to the boundary of its phase space by continuity: $B_\Omega(t,t) := (t,t)$ and $B_\Omega(t,t^*) := (t^*,t)$.

Remark 2.1. The symplectic billiard map B_Ω can also be defined on the set of points $(q_0,q_1) \in \partial\Omega \times \partial\Omega$. In this setting, it is known that it commutes with affine transformations: if F

is such a transformation, then $B_{F(\Omega)}(F(q_0,q_1)) = F(B_\Omega(q_0,q_1))$, where $F(q_0,q_1)$ stands for $(F(q_0),F(q_1))$.

2.2. Generating function and area spectrum

The symplectic billiard map is an exact-symplectic twist map and hence, is associated with a generating function $S_\Omega : \{(t_0,t_1) :\ t_0 < t_1 < t_0^*\} \to \mathbb{R}$ defined for all (t_0,t_1) by

$$S_\Omega\left(t_0,t_1\right) = \omega\left(\gamma\left(t_0\right),\gamma\left(t_1\right)\right).$$

Here, $\omega = \mathrm{d}x \wedge \mathrm{d}y$ is the standard area form of $\mathbb{R}^2(x,y)$. It has the property that given three distinct points $t_0 < t_1 < t_2$, $B_\Omega(t_0,t_1) = (t_1,t_2)$ if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\left(S_\Omega\left(t_0,t_1\right) + S_\Omega\left(t_1,t_2\right)\right) = 0. \tag{2}$$

A sequence $\underline{t} = \{(t_k,t_{k+1})\}_k$ is called a *periodic orbit* of B_Ω with period $q > 0$ if for any integer k, $B_\Omega(t_k,t_{k+1}) = (t_{k+1},t_{k+2})$ and $(t_q,t_{q+1}) = (t_0,t_1)$. We define its action by

$$A\left(\underline{t}\right) = \sum_{k=0}^{q-1} S_\Omega\left(t_k,t_{k+1}\right).$$

When the points $\gamma(t_0),\ldots,\gamma(t_{q-1})$ form a polygon (i.e. without intersection of edges), then $A(\underline{t})$ corresponds to twice its area. We define the *area spectrum* of Ω, denoted by $\mathcal{A}(\Omega)$, to be the *closure of the set*

$$\left\{A\left(\underline{t}\right)\ \middle|\ \underline{t} \text{ is a periodic orbit of } \Omega\right\}.$$

Remark 2.2. Following remark 2.1, if Ω' is a domain obtained from Ω by applying an area-preserving affine transformation, then each periodic orbit of Ω is transformed into a periodic orbit of Ω' with the same action, and necessarily the area spectra of both domains coincide, namely $\mathcal{A}(\Omega') = \mathcal{A}(\Omega)$.

We say that a periodic orbit $\underline{t}$ has *rotation number* $p/q \in \mathbb{Q}$ if for any integer k, it satisfies $t_{k+q} = t_k + p$ when lifted to the universal cover (a strip). Given an integer $q \geqslant 2$, let A_q be the maximal area of a q-periodic orbit having rotation number $1/q$ in Ω. It was proven [1] that there exists a sequence $(a_n)_{n\geqslant 0}$ of real numbers such that for any $q \geqslant 2$,

$$A_q = a_0 + \frac{a_1}{q^2} + \frac{a_2}{q^4} + \ldots + \mathcal{O}\left(\frac{1}{q^{2s}}\right) \tag{3}$$

with $s > 0$ an integer depending on the regularity of the domain. In the following, we will be mostly interested in a_0, which corresponds to the area of Ω, together with a_1 and a_2, which are given by the integral expressions:

$$a_1 = \frac{L^3}{12} \qquad \text{and} \qquad a_2 = -\frac{L^4}{240}\int_0^L k\left(t\right)\mathrm{d}t.$$

Here, k is the affine curvature of $\partial\Omega$ expressed in an affine parametrization t of the boundary and L is its affine-perimeter. These are defined more precisely in appendix A.

Remark 2.3. In the context of classical billiards, where the action of periodic orbits corresponds to their lengths, a similar analysis was carried out by Marvizi and Melrose in [15], where they construct a sequence of spectral invariants associated with the asymptotic expansion of the length of periodic billiard trajectories in smooth, strictly convex planar domains. These quantities are nowadays called *Marvizi–Melrose invariants*; see [22, 23] for more details.

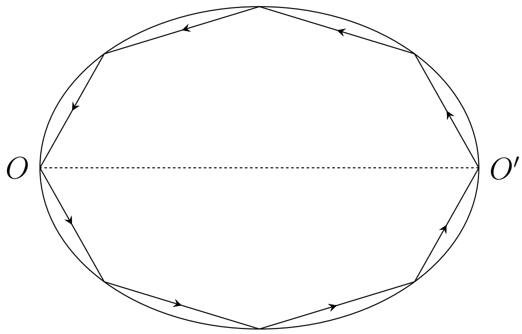

Figure 2. An axially symmetric symplectic billiard orbit of rotation number $\omega = 1/8$.

3. Construction of an isospectral operator

Let $(\Omega_\tau)_{\tau\in I}$ be an axially symmetric $\mathscr{C}^8$-smooth deformation of $\Omega = \Omega_0$. By applying affine transformations to each domain in this family, we may assume that there are two distinct points O and O' on the plane such that for each τ, the boundary $\partial\Omega_\tau$ contains both O and O' and the line OO' is an axis of symmetry of Ω_τ (figure 2). Hence, we consider a map $\gamma : I\times\mathbb{R}\to\mathbb{R}^2$ such that for any $\tau\in I$, $\gamma(\tau,\cdot)$ parametrizes $\partial\Omega_\tau$ with $\gamma(\tau,0) = O$ and the points $\gamma(\tau,-t)$, $\gamma(\tau,t)$ are symmetric with respect to OO'. Throughout the paper, we will assume that for each $\tau\in I$, the map $\gamma(\tau,\cdot)$ is an affine parametrization of $\partial\Omega_\tau$. In particular γ is $\mathscr{C}^7$-smooth and satisfies $\det(\partial_t\gamma(\tau,t),\partial_t^2\gamma(\tau,t)) = 1$ for any $\tau\in I$ and $t\in\mathbb{R}$—see appendix A for a discussion of affine parametrizations. Moreover, if L_τ is the affine perimeter of $\partial\Omega_\tau$, $\gamma(\tau,\cdot)$ is L_τ-periodic and we require that $\gamma(\tau,0) = O$ and $\gamma(\tau,L_\tau/2) = O'$.

Remark 3.1. Note that in the proof of theorem 1.1, we do not assume *a priori* that L_τ is constant in τ. Indeed, according to the value of a_1 in (3), if the area spectrum is preserved throughout the deformation, that is $\mathcal{A}(\Omega_\tau) = \mathcal{A}(\Omega_0)$ for any $\tau\in I$, then the affine perimeter does not depend on τ. Namely, $L_\tau = L_0$ for any $\tau\in I$ and the periodicity in t is uniform in τ.

Remark 3.2. Throughout the paper, we consider domains with $\mathscr{C}^8$-smooth boundaries. This implies that their affine parametrizations are $\mathscr{C}^7$-smooth.

3.1. Deformation map

Given a family of parameters $\underline{t} = (t_0,\dots,t_{q-1})$ and $\tau\in I$, we define the action

$$A_q(\tau,\underline{t}) = \sum_{k=0}^{q-1}\omega\left(\gamma(\tau,t_k),\gamma(\tau,t_{k+1})\right).$$

We also introduce the *deformation map* $n_\tau(t)$ of the family $(\Omega_\tau)_{\tau\in I}$, which is defined for any $\tau\in I$ and $t\in\mathbb{R}$ by

$$n_\tau(t) = \omega\left(\partial_\tau\gamma(\tau,t),\partial_t\gamma(\tau,t)\right).$$

Note that since the map γ is $\mathscr{C}^7$-smooth, n is only $\mathscr{C}^6$-smooth. It satisfies the following property:

Lemma 3.3. *Let $\tau_0 \in I$ and assume that $(\gamma(\tau_0, t_k))_k$ corresponds to a q-periodic symplectic billiard orbit in Ω_{τ_0}. Then*

$$\partial_\tau A_q(\tau_0, \underline{t}) = \sum_{k=0}^{q-1} n_{\tau_0}(t_k)\, \varrho_{\tau_0}^{-1/3}(t_k)\, \ell_k^{(q)},$$

where $\varrho_\tau(t)$ is the radius of curvature of $\partial\Omega_\tau$ at $\gamma(\tau, t)$, $\ell_k^{(q)}$ is the length of the segment between the two points $\gamma(\tau_0, t_{k-1})$ and $\gamma(\tau_0, t_{k+1})$, and $n_\tau(s)$ is the deformation map of the family $(\Omega_\tau)_{\tau\in I}$.

Remark 3.4. Since $A_q(\tau, \underline{t})$ is unchanged when we apply an area-preserving affine transformation, so is the expression of $\partial_\tau A_q(\tau_0, \underline{t})$ calculated in lemma 3.3. Note that by construction, n_τ is also invariant under such transformations and although $\varrho(t_k)$ and $\ell_k^{(q)}$ are not invariant, the proof of lemma 3.3 will reveal that the quantity $\varrho_{\tau_0}^{-1/3}(t_k)\ell^{(q)}$ is indeed invariant.

Proof. Using bilinearity of the action, we see that

$$\partial_\tau A_q(\tau_0, \underline{t}) = \sum_k \omega\left(\partial_\tau\gamma(\tau_0, t_k), \gamma(\tau_0, t_{k+1})\right) + \omega\left(\gamma(\tau_0, t_k), \partial_\tau\gamma(\tau_0, t_{k+1})\right).$$

Changing index summation in the second term, we obtain

$$\partial_\tau A_q(\tau_0, \underline{t}) = \sum_k \omega\left(\partial_\tau\gamma(\tau_0, t_k), \gamma(\tau_0, t_{k+1}) - \gamma(\tau_0, t_{k-1})\right).$$

Now, by definition of the reflection law,

$$\gamma(\tau_0, t_{k+1}) - \gamma(\tau_0, t_{k-1}) = \alpha_k \partial_t\gamma(\tau_0, t_k),$$

where $\alpha_k > 0$. Thus, we conclude that

$$\alpha_k = \|\gamma(\tau_0, t_{k+1}) - \gamma(\tau_0, t_{k-1})\|\,\|\partial_t\gamma(\tau_0, t_k)\|^{-1} = \varrho_{\tau_0}^{-1/3}(t_k)\,\ell_k^{(q)},$$

since $\|\partial_t\gamma(\tau, t_k)\| = \varrho_\tau^{1/3}(t_k)$. □

Proposition 3.5. *Assume that the deformation map $n_\tau(s)$ vanishes for any (τ, s). Then $\Omega_\tau = \Omega_0$ for any τ.*

Proof. This is classical and follows immediately from [6]. The fact that $n \equiv 0$ implies that for any (s, τ), the vectors $\partial_\tau\gamma(\tau, s)$ and $\partial_t\gamma(\tau, s)$ are collinear. In other words, $D\gamma(s, \tau)$ has rank one. By the constant rank theorem, $\mathrm{Im}\,\gamma$ is contained in $\partial\Omega_0$, whence $\partial\Omega_0 = \partial\Omega_\tau$ for any τ. □

3.2. A family of nearly glancing orbits

Let Ω be a strongly convex axially symmetric domain with $\mathscr{C}^1$-smooth boundary and $q \geqslant 2$ an integer. Denote by L the affine perimeter of Ω. Assume that one of the two points in $\partial\Omega$ which belong to the symmetry axis is marked and call it the origin, which we will denote by O. Consider the set $\mathcal{P}^{(q)}$ of all convex q-gons

$$\underline{p}^{(q)} = \left(p_0^{(q)}, \ldots, p_{q-1}^{(q)}\right)$$

inscribed in Ω which are symmetric with respect to the axis of symmetry and have $p_0^{(q)}$ fixed at the origin.

Proposition 3.6. *Let $\underline{p}^{(\Omega,q)}$ be a polygon of maximal area amongst all polygons in $\mathcal{P}^{(q)}$. Then $\underline{p}^{(\Omega,q)}$ is a q-periodic orbit of the symplectic billiard map in Ω.*

Proof. Let $\underline{t} = (t_1 \ldots, t_{q-1})$ be such that $p_j^{(q)} = \gamma(t_j)$ is the jth point of the polygon $\underline{p}^{(\Omega,q)}$ for any $j \in \{0, \ldots, q-1\}$. Because $\underline{p}^{(\Omega,q)}$ is maximal, $\underline{t}$ must be a critical point of the functional

$$\underline{t} \mapsto \sum_{j=0}^{q-1} S_\Omega\left(t_j, t_{j+1}\right), \tag{4}$$

where $t_0 = t_q = 0$. For $j \in \{1, \ldots, q-1\}$, differentiating (4) with respect to t_j and setting it equal to zero, we see that the chord $p_{j-1}p_j$ is sent to p_jp_{j+1} by the symplectic billiard map. It is also true for $j = 0$ as a consequence of the symmetry assumption. □

Given an L-periodic map $n : \mathbb{R} \to \mathbb{R}$ and an integer $q \geqslant 2$, we define its *discrete X-ray transform* along the orbit $\underline{t}^{(q)}$ to be the quantity

$$a_{\Omega,q}(n) = \sum_{k=0}^{q-1} n\left(t_k^{(q)}\right) \varrho^{-1/3}\left(t_k^{(q)}\right) \ell_k^{(q)}, \tag{5}$$

where $t_k^{(q)}$ is the affine arclength coordinate of $p_k^{(q)}$ on $\partial\Omega$, $\varrho(t)$ is the radius of curvature of $\partial\Omega$ at a point of affine arclength t and $\ell_k^{(q)}$ is the Euclidean distance between $p_{k-1}^{(q)}$ and $p_{k+1}^{(q)}$.

Note that no regularity on n is required for $a_{\Omega,q}(n)$ to be well-defined. Later on, however, we will assume that n is in the Sobolev space H^γ with $\gamma \in (3,4)$, meaning that its Fourier coefficients $(\widehat{n}_p)_{p\in\mathbb{Z}}$ satisfy $|p|^\gamma \widehat{n}_p \to 0$ as $|p| \to \infty$.

Remark 3.7. In the case when $q = 2$, we have the trivial identity $a_{\Omega,2}(n) = 0$ for any map n, as $\ell_0^{(2)} = \ell_1^{(2)} = 0$. This follows from the fact that 2-periodic symplectic billiard orbits have zero action.

Proposition 3.8. *Assume that $\mathcal{A}(\Omega_\tau) = \mathcal{A}(\Omega_0)$ for any $\tau \in I$. Then for any $\tau \in I$ and any $q \geqslant 2$, we have*

$$a_{\Omega_\tau,q}(n_\tau) = 0,$$

where n_τ is the deformation map. Moreover, the affine perimeter L_τ of $\partial\Omega_\tau$ is independent of τ. I.e., for any $\tau \in I$,

$$L_\tau = L_0.$$

Proof. We observe, as in [6], that the set $\mathcal{A}(\Omega_0)$ has measure zero. We now describe an argument found in unpublished lecture notes of De Simoi. Let Ω be a symplectic billiard with a $\mathscr{C}^r$-smooth boundary, $r \geqslant 2$. Define the maps $\Theta_q : \mathbb{R}^2 \to \mathbb{R}^q$ and $\mathscr{A} : \mathbb{R}^q \to \mathbb{R}$ as follows: 1) Θ_q is the $\mathscr{C}^{r-1}$-smooth map which associates to a pair $(t_0, t_1) \in \mathbb{R}^2$ the q-tuple $(t_0, t_1, \ldots, t_{q-1}) \in \mathbb{R}^q$ such that the symplectic billiard map sends (t_k, t_{k+1}) to (t_{k+1}, t_{k+2}) for any $k \in \{0, \ldots, q-2\}$; 2) $\mathscr{A}$ is the $\mathscr{C}^r$-smooth map defined by $\mathscr{A}(t_0, \ldots, t_{q-1}) = \sum_{j=0}^{q-1} S_\Omega(t_j, t_{j+1(\text{mod q})})$ for any q-tuple $(t_0, t_1, \ldots, t_{q-1}) \in \mathbb{R}^q$. It follows from the variational property (2) that the action of a periodic orbit is a critical value of the $\mathscr{C}^{r-1}$-smooth map of two variables $\mathscr{A} \circ \Theta : \mathbb{R}^2 \to \mathbb{R}$. Hence if $r - 1 > 1$, i.e. if $r > 2$, then by Sard's theorem, the critical values of $\mathscr{A} \circ \Theta$ have measure zero. Hence, the same holds for the area spectrum $\mathcal{A}(\Omega_0)$.

For $\tau \in I$ and $q \geqslant 2$, denote by

$$\underline{p}^{(\Omega_\tau,q)} = \left(p_0^{(q,\tau)}, \ldots, p_{q-1}^{(q,\tau)}\right)$$

the maximal symplectic billiard orbit in Ω_τ which was introduced in proposition 3.6. The map $\tau \mapsto A_q(\tau, \underline{t}^{(q,\tau)}) \in \mathcal{A}(\Omega_\tau) = \mathcal{A}(\Omega_0)$ is continuous and hence constant. We denote this constant by α_q. Let us now fix $\tau_0 \in I$. Since each polygon $\underline{p}^{(q,\tau)}$ maximizes the area, for any $\tau \in I$, we have

$$A_q\left(\tau, \underline{t}^{(q,\tau_0)}\right) \leqslant \alpha_q = A_q\left(\tau, \underline{t}^{(q,\tau)}\right),$$

with equality at $\tau = \tau_0$. The left-hand side is differentiable in τ and admits a maximal value at $\tau = \tau_0$. Hence $\partial_\tau A_q(\tau_0, \underline{t}^{(q,\tau_0)}) = 0$. By lemma 3.3, the latter quantity is $a_{\Omega_{\tau_0},q}(n_{\tau_0})$. Moreover, for any $\tau \in I$, it was shown in [1] that

$$A_q\left(\tau, \underline{t}^{(q,\tau)}\right) \sim \frac{L_\tau^3}{12}$$

as $q \to \infty$. Since $A_q(0, \underline{t}^{(q,0)}) = A_q(\tau, \underline{t}^{(q,\tau)})$, we conclude that $L_0 = L_\tau$. □

3.3. Domains close to ellipses

The quantities $a_{\Omega,q}(n)$ can be computed explicitly when Ω is an ellipse $\mathscr{E}$. Moreover, when Ω is close to an ellipse $\mathscr{E}$ in the $\mathscr{C}^8$ topology, $a_{\Omega,q}(n)$ is well-approximated by $a_{\mathscr{E},q}(n)$. These estimates are presented in this section, namely in propositions 3.9 and 3.10.

Given an integer $r > 0$, the distance between strongly convex planar domains with $\mathscr{C}^r$-smooth boundaries can be defined as follows: given such domain Ω, we consider the set $\mathscr{I}^r_\Omega \subset \mathscr{C}^r(\mathbb{S}^1, \mathbb{R}^2)$ of $\mathscr{C}^r$-smooth immersions $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ such that $\gamma(\mathbb{S}^1) = \partial\Omega$. The $\mathscr{C}^r$-distance $d_{\mathscr{C}^r}(\Omega, \Omega')$ between two domains Ω and Ω' is the infimum of the distances between immersions of the corresponding domains in the Whitney $\mathscr{C}^r$-smooth topology, i.e.

$$d_{\mathscr{C}^r}(\Omega, \Omega') = \inf\{\|\gamma_{\Omega'} - \gamma_\Omega\|_{\mathscr{C}^r} : (\gamma_\Omega, \gamma_{\Omega'}) \in \mathscr{I}^r_\Omega \times \mathscr{I}^r_{\Omega'}\}.$$

In this paper, we consider *affine parametrizations* of domains. These are $\mathscr{C}^{r-1}$-smooth immersions such that two domains which are close in the $\mathscr{C}^r$ topology admit sufficiently close affine parametrizations in the $\mathscr{C}^{r-1}$-topology. Moreover, the correspondence $\Omega \mapsto k_\Omega$, which associates to a domain Ω its affine curvature, is continuous. Here, the space of affine curvature functions is endowed with the metric on $\mathscr{C}^{r-4}$-smooth maps. In particular, if Ω converges to an ellipse, then k_Ω converges to a constant.

Let Ω be a strongly convex axially symmetric domain with $\mathscr{C}^8$-smooth boundary which is δ-close to an ellipse $\mathscr{E}$ in the $\mathscr{C}^8$-topology for some $\delta > 0$.

Proposition 3.9 (Elliptical symplectic billiards). *If the ellipse $\mathscr{E}$ has affine curvature $k_{\mathscr{E}}$ and affine perimeter $L_{\mathscr{E}}$, then $a_{\mathscr{E},q}(n)$ can be expressed for any $q \geqslant 2$ and any $L_{\mathscr{E}}$-periodic map $n : \mathbb{R} \to \mathbb{R}$ as*

$$a_{\mathscr{E},q}(n) = \mu_q [n]_q,$$

where $\mu_q = 2k_{\mathscr{E}}^{-1/2} q \sin(2\pi/q)$ and $[n]_q$ denotes the cyclic sum

$$[n]_q = \frac{1}{q}\sum_{j=0}^{q-1} n(j/qL_{\mathscr{E}}).$$

Proof. The proof relies on proposition B.2. Fix $q \geqslant 3$. Suppose that the ellipse is described by the following affine parametrization

$$\gamma(t) = \left(a\cos\left(\sqrt{k_{\mathscr{E}}}t\right), b\sin\left(\sqrt{k_{\mathscr{E}}}t\right)\right),$$

where $a > 0$ is the length of the major axis of $\mathscr{E}$ and $b \in (0,a)$ is the length of its minor axis. From proposition B.2, we have

$$t_j := t_j^{(q)} = \frac{2\pi}{\sqrt{k_{\mathscr{E}}}} \frac{j}{q}.$$

It follows that

$$\ell_j^{(q)} = \|\gamma(t_{j+1}) - \gamma(t_{j-1})\| = 2\sin\left(\frac{2\pi}{q}\right)\left(a^2\sin^2\left(\frac{2\pi j}{q}\right) + b^2\cos^2\left(\frac{2\pi j}{q}\right)\right)^{1/2}.$$

Moreover, since γ is an affine parametrization,

$$\varrho(t_j)^{1/3} = \|\gamma'(t_j)\| = \sqrt{k_{\mathscr{E}}}\left(a^2\sin^2\left(\frac{2\pi j}{q}\right) + b^2\cos^2\left(\frac{2\pi j}{q}\right)\right)^{1/2}.$$

The result follows. □

Any smooth, 1-periodic, even map $n : \mathbb{R} \to \mathbb{R}$ can be decomposed into Fourier modes as

$$n(\theta) = \sum_{p \geqslant 0} \widehat{n}_p \cos(p\theta).$$

Given $\gamma > 0$, define the Sobolev space H^γ of 1-periodic even maps $n : \mathbb{R} \to \mathbb{R}$ such that the sequence

$$(p^\gamma \widehat{n}_p)_p$$

is bounded. On this space, we consider the Banach norm

$$\|n\|_\gamma = \sup\{\{p^\gamma|\widehat{n}_p|, p \geqslant 1\} \cup \{|\widehat{n}_0|\}\}.$$

The following proposition is the core result of this paper, allowing us to prove theorems 1.1 and 1.3. It applies to any domain Ω and roughly says that the first variation in τ of a q-periodic orbit's action tends to that of an ellipse for any given domain Ω when $q \to +\infty$, or for any given q when Ω is close to an ellipse.

Proposition 3.10 (Perturbative regime). *Let $\Omega \in \mathcal{S}$, $\gamma \in (3,4)$ and $n \in H^\gamma$. For any $q \geqslant 2$, the quantity $a_{\Omega,q}(n)$ can be expressed as*

$$a_{\Omega,q}(n) = a_{\mathscr{E},q}(n) + \alpha_0(n) + \frac{\alpha_1(n)}{q^2} + \frac{\|n\|_\gamma R_{\Omega,q}(n)}{q^4}, \tag{6}$$

where $\alpha_0 = \lambda_\Omega \widehat{n}_0$ and $\lambda_\Omega = \mathcal{O}(\delta)$ depend on Ω, α_1 is a real-valued linear functional of n and the remainder $R_{\Omega,q}(n)$ is real valued, bounded uniformly in q and n, and tending to 0 uniformly in both q and n as Ω converges to the ellipse $\mathscr{E}$ in the $\mathscr{C}^8$-smooth topology.

The proofs of propositions 3.9 and 3.10 follow from estimates given in appendix C. Note that the proximity of $\partial\Omega$ to an ellipse only plays a role in the smallness of $R_{\Omega,q}$; but the latter exists and is bounded for any given domain.

3.4. *Isospectral operator and proof of theorem 1.1*

In this subsection, we show how the proof of theorem 1.1 can be deduced from propositions 3.9 and 3.10.

Let Ω be a strongly convex domain with $\mathscr{C}^8$-smooth boundary and affine perimeter 1. Assume that $\partial\Omega$ contains the two points O and O' and that it is symmetric with respect to the line OO'. Let us define an operator T_Ω from the space of 1-periodic even maps $n : \mathbb{R} \to \mathbb{R}$

to the space of real valued sequences $(u_q)_{q\geqslant 0}$. In the notation of propositions 3.9 and 3.10, we write, for any integer $q \geqslant 0$ and 1-periodic even map $n : \mathbb{R} \to \mathbb{R}$,

$$T_\Omega(n)_q = \begin{cases} \widehat{n}_0 & \text{if } q = 0; \\ n(0) & \text{if } q = 1; \\ n(1/2) & \text{if } q = 2; \\ a_{\Omega,q}(n) - (\mu_q + \lambda_\Omega)\widehat{n}_0 - \frac{\alpha_1(n)}{q^2} & \text{if } q \geqslant 3. \end{cases} \tag{7}$$

This defines a sequence $T_\Omega(n) = (T_\Omega(n)_q)_{q\geqslant 0}$ of real numbers.

Given $\gamma > 1$, we consider the operator T_Ω restricted to the previously defined space of maps H^γ. This space has an analogue for sequences, namely the space h^γ of sequences $(u_q)_{q\geqslant 0}$ such that $q^\gamma u_q$ is bounded. We now show that $T_\Omega(H^\gamma) \subset h^\gamma$ whenever $\gamma \in (3,4)$.

Proposition 3.11. *Let $\gamma \in (3,4)$ $\Omega \in \mathcal{S}$. The map T_Ω defines a bounded operator from H^γ to h^γ called linear isospectral operator.*

Proof. Let $n \in H^\gamma$. By proposition 3.10, for $q \geqslant 3$, $T_\Omega(n)_q$ can be expressed as

$$T_\Omega(n)_q = \mu_q [n]_q^* + \frac{\|n\|_\gamma R_{\Omega,q}(n)}{q^4},$$

where $[n]_q^* = [n]_q - \widehat{n}_0$ and there exists a $K > 0$ such that $|R_{\Omega,q}(n)| \leqslant K$ for all q and n. As shown in lemma D.2, $n \in H^\gamma$ implies that $[n]_q^* \in h^\gamma$ together with $\|[n]^*\|_\gamma \leqslant C\|n\|_\gamma$ for some universal constant $C > 0$. Hence,

$$\|T_\Omega(n)\|_\gamma = \sup_{q>0} q^\gamma |T_\Omega(n)_q| \leqslant C \sup_q \mu_q + K < +\infty,$$

which concludes the proof of the proposition. □

The idea of calling T_Ω a *linear isospectral operator* goes back to [6], where T_Ω has an analogue for classical (Birkhoff) billiards. The name refers to the following property of T_Ω, which is related to length isospectral deformations of Ω:

Proposition 3.12. *If $(\Omega_\tau)_\tau$ is a $\mathscr{C}^8$-smooth one-parameter family of domains in $\mathcal{S}$ such that $\mathcal{A}(\Omega_\tau) = \mathcal{A}(\Omega_0)$ for any τ, then $T_{\Omega_\tau}(n_\tau) = 0$ for all τ.*

Proof. Fix τ and write $\Omega = \Omega_\tau$ and $n = n_\tau$. By proposition 3.8, for any $q \geqslant 2$, we can write

$$0 = a_{\Omega,q}(n) = a_{\mathscr{E},q}(n) + \alpha_0(n) + \frac{\alpha_1(n)}{q^2} + \frac{\|n\|_\gamma R_{\Omega,q}(n)}{q^4}. \tag{8}$$

Since $\gamma > 3$, the terms in $1/q^0$, $1/q^2$ and $1/q^3$ should vanish in equation (8). Using the expressions of $a_{\mathscr{E},q}(n)$ given in proposition 3.9 and of $\alpha_0(n)$ given in proposition 3.10, we deduce that

$$\widehat{n}_0 = 0, \qquad \text{and} \qquad \alpha_1(n) = 0.$$

This automatically implies that $T_\Omega(n)_q = 0$ for $q \geqslant 3$ and $q = 0$. The cases $q = 1$ and $q = 2$ also hold since each domain $\partial\Omega_\tau$ contains the points O and O'. □

As stated in proposition 3.12, for each $\tau \in I$, the deformation map n_τ associated to an isospectral deformation lies in the kernel of the operator T_{Ω_τ}. We now prove that in fact, if Ω is sufficiently close to an ellipse in the $\mathscr{C}^8$ metric, the operator $T_\Omega : H^\gamma \to h^\gamma$ is invertible and hence, has trivial kernel.

Proposition 3.13. *Let $\gamma \in (3,4)$ and $\mathscr{E}$ be an ellipse. There exists $\delta = \delta(\mathscr{E}) > 0$ such that if Ω is δ-close to $\mathscr{E}$ in the $\mathscr{C}^8$-topology, then the operator $T_\Omega : H^\gamma \to h^\gamma$ is invertible.*

Proof. The proof relies on the following two points: 1) for the ellipse $\mathscr{E}$, the operator $T_{\mathscr{E}}$: $H^\gamma \to h^\gamma$ is bounded and invertible; 2) the operators T_Ω and $T_{\mathscr{E}}$ are arbitrarily close to each other in norm if δ is sufficiently small. The conclusion will follow from 1) and 2).

Invertibility of $T_{\mathscr{E}}$ is proved in proposition D.6. To prove 2), let us fix $\varepsilon > 0$ and choose $\delta_0 > 0$ so that if Ω is δ-close to $\mathscr{E}$ in the $\mathscr{C}^8$ topology, then for any $n \in H^\gamma$ and $q \geqslant 3$, we have $|R_{\Omega,q}(n)| \leqslant \varepsilon$. The formula for $T_\Omega(n)_q$ given in equation (7) implies that

$$q^\gamma |T_\Omega(n)_q - T_{\mathscr{E}}(n)_q| = \mathcal{O}(\varepsilon \|n\|_\gamma).$$

Note that by definition, $T_\Omega(n)$ and $T_{\mathscr{E}}(n)_q$ coincide for $q \in \{0,1,2\}$. As a consequence, $\|T_\Omega - T_{\mathscr{E}}\|_\gamma = \mathcal{O}(\varepsilon)$ in the operator norm $\|\cdot\|_\gamma$ associated to the Banach spaces H^γ and h^γ. □

We can now prove theorem 1.1.

Proof of theorem 1.1. Let $\delta > 0$ be such that for any domain Ω which is δ-close to $\mathscr{E}$ in the $\mathscr{C}^8$-topology, the operator T_Ω is invertible. In particular, if $(\Omega_\tau)_\tau$ is a one-parameter family of domains which are δ-close to $\mathscr{E}$, then the operators T_{Ω_τ} are invertible. Yet for any $\tau \in I$, the deformation map n_τ lies in the kernel of T_{Ω_τ}, as stated in proposition 3.12. Hence, n_τ vanishes identically which, as a consequence of proposition 3.5, implies that $\Omega_\tau = \Omega_0$ for all τ. □

Proof of theorem 1.3. For arbitrary $\Omega \in \mathcal{S}$, it is not clear whether or not T_Ω is invertible. However, we can still define an invertible operator on subspaces. Namely, for a given integer $q_0 \geqslant 2$ and $\gamma \in (3,4)$, consider the spaces $H^\gamma_{q_0}$ and $h^\gamma_{q_0}$ given by

$$h^\gamma_{q_0} = \left\{(u_q)_{q>q_0} : q^\gamma u_q \to_{+\infty} 0\right\}$$

and

$$H^\gamma_{q_0} = \{n \in H^\gamma : \forall q \in \{0,\dots,q_0\} \quad \widehat{n}_q = 0\}.$$

We endow them with the corresponding norms defined for $u \in h^\gamma_{q_0}$ and $n \in H^\gamma_{q_0}$ by

$$\|u\|_\gamma = \sup_{q>q_0} q^\gamma |u_q|, \qquad \|n\|_\gamma = \sup_{q>q_0} q^\gamma |\widehat{n}_q|.$$

Consider the operator $\tilde{T}$ defined for any $n \in H^\gamma$ and $q > q_0$ by

$$\tilde{T}(n)_q = T_\Omega(n)_q.$$

By proposition 3.11, it is an operator from H^γ to $h^\gamma_{q_0}$. Now consider the operator $T_2^\Omega : H^\gamma_{q_0} \to h^\gamma_{q_0}$ obtained by restricting $\tilde{T}$ to $H^\gamma_{q_0}$, namely $T_2^\Omega = \tilde{T}_{|H^\gamma_{q_0}}$. Since T_Ω is bounded, so is T_2^Ω, and the two are related for all $n \in H^\gamma$ by

$$\tilde{T}_\Omega(n) = \tilde{T}\left(P^-_{q_0}(n)\right) + T_2^\Omega\left(P^+_{q_0}(n)\right), \tag{9}$$

where $P^+_{q_0} : H^\gamma \to H^\gamma_{q_0}$ and $P^-_{q_0} : H^\gamma \to H^\gamma$ are the maps defined for any $n \in H^\gamma$ by

$$P^-_{q_0}(n) = \sum_{q=0}^{q_0} \widehat{n}_q \cos(2\pi qt) \qquad P^+_{q_0}(n) = n - P^-_{q_0}(n).$$

The result will be proven if we show that T_2^Ω is invertible for a certain q_0: indeed from (9), if this is the case we can deduce that if n is in the kernel of T_Ω, then

$$P^+_{q_0}(n) = -\left(T_2^\Omega\right)^{-1}\left(\tilde{T}\left(P^-_{q_0}(n)\right)\right).$$

Hence, $P^{+}_{q_0}(n)$ is a linear function of $P^{-}_{q_0}(n)$: the kernel of T_Ω is contained in a graph over a finite-dimensional space generated by $\{1, \cos(2\pi t), \ldots, \cos(2\pi q_0 t)\}$[3].

To prove that T_2^Ω is invertible, we fix any ellipse $\mathscr{E}$ and apply proposition 3.10. For $q > q_0$ and $n \in H^\gamma_{q_0}$, $T_2^\Omega(n)_q$ can be expressed as

$$T_2^\Omega(n)_q = T_2^{\mathscr{E}}(n)_q + \frac{\|n\|_\gamma R_{\Omega,q}(n)}{q^4}, \tag{10}$$

where $T_2^{\mathscr{E}}(n)_q = \mu_q [n]^*_q$ and there exists a constant $K > 0$ such that for any n, $R_{\Omega,q}(n) < K$ whenever $q > q_0$. The result will follow if we show that $T_2^{\mathscr{E}}$ is invertible and $\|(T_2^{\mathscr{E}})^{-1}(T_2^\Omega - T_2^{\mathscr{E}})\|_\gamma < 1$, in which case $(T_2^\Omega)^{-1}$ can be expanded via a Neumann series. The invertibility of $T_2^{\mathscr{E}}$ follows from the proof of proposition D.6, where an inverse for the operator $[\cdot]^*$ is computed explicitly—see equation (28). The same also holds for the existence of a bounded inverse $(T_2^{\mathscr{E}})^{-1}$ of $T_2^{\mathscr{E}}$. Therefore, there exists a $C > 0$ depending only on $k_{\mathscr{E}}$, independently of q_0, such that

$$\| \left(T_2^{\mathscr{E}}\right)^{-1} \|_\gamma \leqslant C.$$

From equation (10) follows that

$$\|T_2^\Omega - T_2^{\mathscr{E}}\|_\gamma \leqslant \frac{K}{q_0^{4-\gamma}}.$$

Therefore, for any sufficiently large q_0, namely $q_0 > (CK)^{\frac{1}{4-\gamma}}$,

$$\| \left(T_2^{\mathscr{E}}\right)^{-1} \left(T_2^\Omega - T_2^{\mathscr{E}}\right) \|_\gamma \leqslant \frac{CK}{q_0^{4-\gamma}} < 1.$$

Hence, T_2^Ω is invertible and the result follows. □

Data availability statement

No new data were created or analysed in this study.

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[3] One can show that this space is generated by the family indexed by $k = 0, \ldots, q_0$ of maps $\cos(2\pi kt) + n_k$, where $n_k \in H^\gamma_{q_0}$ is *small* in the sense that it satisfies $\|n_k\|_\gamma \leqslant \frac{M}{q_0^{4-\gamma}}$, for a certain $M > 0$ independent on q_0 and k.

authors of [3] arrived at a similar result using different techniques. We shared our ideas and mutually acknowledged that these results were obtained independently from one another.

Appendix A. Affine curvature

Given an integer $r \geqslant 2$, consider a strongly convex and closed $\mathscr{C}^r$-smooth curve $\Gamma \subset \mathbb{R}^2$ whose radius of curvature, denoted at each point by ϱ, never vanishes. The *affine length* (or *perimeter*) of Γ is defined to be

$$L = \int_0^{|\partial\Omega|} \varrho^{-1/3}(s)\,\mathrm{d}s,$$

where s denotes a parametrization by arc-length.

Given two vectors u, v in $\mathbb{R}^2$, denote by $[u,v] = \omega(u,v)$ the determinant of u and v in a positively oriented orthonormal basis of $\mathbb{R}^2$; equivalently, $\omega = \mathrm{d}x \wedge \mathrm{d}y$, where (x,y) are Darboux coordinates corresponding to the canonical basis of $\mathbb{R}^2$.

Proposition A.1 (See [21]). *Γ can be parametrized by a $\mathscr{C}^{r-1}$-smooth map $\gamma : \mathbb{R} \to \mathbb{R}^2$ which is L-periodic and satisfies $[\gamma'(t), \gamma''(t)] = 1$ for each $t \in \mathbb{R}$. Such a map is called an affine arclength parametrization of γ.*

Proof. Consider an arc-length parametrization $\delta(s)$ of Γ and define $\gamma(t) = \delta(\varphi(t))$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism given implicitly by

$$\varphi^{-1}(s) = \int_0^s [\delta'(\sigma), \delta''(\sigma)]^{1/3}\,\mathrm{d}\sigma.$$

Direct computations give

$$\begin{aligned}[\gamma'(t), \gamma''(t)] &= \varphi'(t)^3 [\delta'(\varphi(t)), \delta''(\varphi(t))] \\ &= \frac{1}{(\varphi^{-1})'(\varphi(t))^3} [\delta'(\varphi(t)), \delta''(\varphi(t))]\end{aligned}$$

and the result follows by construction of φ. □

It can be deduced that $[\gamma'(t), \gamma^{(3)}(t)] = 0$ for any t and hence, there is a $\mathscr{C}^{r-4}$-smooth map $k : \mathbb{R} \to \mathbb{R}$, called the *affine curvature* of Γ, which satisfies

$$\gamma^{(3)} = -k\gamma'.$$

The following proposition gives formulas for the first three affine arclength derivatives of γ, which we will use in the paper:

Proposition A.2. *Let u(t) be the unit tangent vector to $\partial\Omega$ at $\gamma(t)$ and denote by N(t) the outward pointing unit normal to $\partial\Omega$ at $\gamma(t)$. Then,*

$$\begin{aligned}\gamma'(t) &= \varrho^{1/3}(t)\,u(t), \\ \gamma''(t) &= \frac{1}{3}\varrho'(t)\,\varrho(t)^{-2/3}u(t) - \varrho(t)^{-1/3}N(t), \\ \gamma^{(3)}(t) &= -k(t)\,\varrho(t)^{1/3}u(t),\end{aligned}$$

where

$$k(t) = \varrho(t)^{-4/3} - \frac{1}{3}\varrho(t)^{-1/3}\left(\varrho'\varrho^{-2/3}\right)'(t)$$
$$= \varrho(t)^{-4/3} - \frac{1}{3}\varrho''(t)\,\varrho^{-1} + \frac{2}{9}\varrho'(t)^2\,\varrho(t)^{-2}.$$

Proof. The radius of curvature is given in general by the formula

$$\varrho(t) = \frac{\|\gamma'(t)\|^3}{[\gamma'(t),\gamma''(t)]}.$$

Since $[\gamma',\gamma''] \equiv 1$, we obtain that $\|\gamma'(t)\| = \varrho^{1/3}(t)$ and the first formula follows.

To obtain the second formula, we differentiate the first one to obtain

$$\gamma''(t) = \frac{1}{3}\varrho'(t)\,\varrho(t)^{-2/3}u(t) + \varrho^{1/3}(t)\,u'(t).$$

Since u' is orthogonal to u (because u has unit length), we can write

$$\gamma''(t) = \frac{1}{3}\varrho'(t)\,\varrho(t)^{-2/3}u(t) + \lambda(t)N(t),$$

for a certain $\lambda(t) \in \mathbb{R}$. Again, the relation $[\gamma',\gamma''] \equiv 1$ together with the orientation of $N(t)$ give $\lambda(t) = -\varrho^{-1/3}(t)$, which implies the second formula.

Note that using the previous relation, we can write

$$\gamma''(t) = \frac{1}{3}\varrho'(t)\,\varrho(t)^{-1}\gamma'(t) - \varrho^{-2/3}(t)R\gamma'(t),$$

where R is a rotation by $\frac{\pi}{2}$ which transforms u into N. Differentiating this relation, we obtain

$$\gamma^{(3)}(t) = \left(\frac{1}{3}\varrho'(t)\,\varrho(t)^{-1}\right)'\gamma'(t) + \frac{1}{3}\varrho'(t)\,\varrho(t)^{-1}\gamma''(t)$$
$$- \left(\varrho^{-2/3}(t)\right)'R\gamma'(t) - \varrho^{-2/3}(t)R\gamma''(t). \tag{11}$$

Now differentiating the relation $[\gamma',\gamma''] = 1$, we deduce that $\gamma^{(3)}$ is collinear with γ'. Hence, it is enough to gather the terms in $\gamma'(t)$ from the previous relation, namely

$$\gamma^{(3)}(t) = \left(\left(\frac{1}{3}\varrho'(t)\varrho(t)^{-1}\right)' + \frac{1}{3}\varrho'(t)\varrho(t)^{-1}\cdot\frac{1}{3}\varrho'(t)\varrho(t)^{-1}\right.$$
$$\left. - \varrho^{-2/3}(t)\cdot\varrho^{-2/3}(t)\right)\gamma'(t), \tag{12}$$

from which the last formula follows. □

It is known [21] that conic sections Γ are characterized uniquely by constant affine curvature k_Γ. In particular, $k > 0$ if Γ is an ellipse, $k = 0$ if Γ is a parabola and $k < 0$ if Γ is a hyperbola.

Proposition A.3. *Assume that $\partial\Omega$ has affine constant curvature $k > 0$. Then it is an ellipse.*

Proof. If $\gamma(t)$ is a parametrization of $\partial\Omega$ by affine arclength and $k > 0$ is a constant, the following linear differential equation

$$\gamma^{(3)}(t) = -k\gamma'(t)$$

can be solved explicitly: there exist vectors $u, v, w \in \mathbb{R}^2$ such that for all $t \in \mathbb{R}$

$$\gamma(t) = \cos\left(\sqrt{k}t\right)u + \sin\left(\sqrt{k}t\right)v + w,$$

which immediately implies that $\partial\Omega$ is an ellipse. □

The converse is a simple exercise.

Appendix B. Symplectic billiard in an ellipse

Let $\mathscr{E}$ be an ellipse of affine curvature $k > 0$ which bounds a domain Ω. Assume that $\mathscr{E}$ is parametrized with respect to affine arclength by $\gamma(t)$. The following proposition shows that the affine arclength can be considered as an angle coordinate in action angle coordinates for the symplectic billiard map on an ellipse.

Proposition B.1. *Given $(t,\varepsilon) \in \mathbb{R}^2$, the points*

$$\gamma(t-\varepsilon), \gamma(t), \gamma(t+\varepsilon)$$

on $\mathscr{E}$ constitute successive reflection points of the symplectic billiard map on Ω. Namely, the line $\gamma(t-\varepsilon)\gamma(t+\varepsilon)$ is parallel to the tangent line of $\mathscr{E}$ at $\gamma(t)$.

Proof. Since the property of two lines being parallel is preserved under affine transformations, we can show the property for any other domain obtained by an affine transformation of $\mathscr{E}$, for example a disk. In this case the result is trivial using the symmetries of the disk. □

It follows that periodic symplectic billiard orbits in Ω equidistribute (with respect to the affine arclength parameter), as described in the following.

Corollary B.2. *Assume that Ω is bounded by an ellipse which is parametrized by an affine arclength coordinate t on the boundary, vanishing at some point on its major axis. Then for any integer $q > 0$, the polygon whose successive vertices are given in affine arclength coordinates by $(t_0, \dots, t_{q-1})$ with*

$$t_j = \frac{2\pi}{\sqrt{k}}\frac{j}{q} \qquad j = 0, \dots, q-1$$

constitutes a q-periodic symplectic billiard orbit in Ω.

Proof. For a fixed q, consider any integer $j \geqslant 0$, viewed modulo q. Introduce

$$\varepsilon := \frac{2\pi}{q\sqrt{k}}.$$

Considering also $j+1$ and $j-1$ modulo q, we compute

$$t_j - \varepsilon = t_{j-1} \qquad \text{and} \qquad t_j + \varepsilon = t_{j+1},$$

so that according to proposition B.1, $\gamma(t_{j-1})$, $\gamma(t_j)$ and $\gamma(t_{j+1})$ are consecutive points of the symplectic billiard in Ω. □

Appendix C. Estimates of periodic orbits

This section is devoted to the proof of proposition 3.10. Fix a strongly convex, bounded and axially symmetric domain Ω with $\mathscr{C}^r$-smooth boundary. The core of the proof amounts to estimating the quantities $t_k^{(q)}$, which are the affine arclength coordinates of impact points of a q-periodic orbit in equation (5), defined for all integers $q \geqslant 2$ and $k \in \{0, \ldots, q-1\}$. In what follows, we assume that the boundary $\partial\Omega$ is parametrized with respect to affine arclength by a curve $\gamma(t)$, $t \in \mathbb{R}/L\mathbb{Z}$, where $L > 0$ is the affine perimeter of $\partial\Omega$.

C.1. Lazutkin coordinates for the symplectic billiard map

The symplectic billiard map inside Ω defines a map

$$(t, t_1) \in \mathcal{P} \mapsto (t_1, t_2) \in \mathcal{P},$$

where t, t_1 and t_2 are the affine arclength coordinates of three consecutive impact points and $\mathcal{P}$ is the appropriately coordinatized phase space for the symplectic billiard map (see [1] for more details). Following [1], we consider ε and ε_1 defined by

$$\varepsilon = t_1 - t \qquad \text{and} \qquad \varepsilon_1 = t_2 - t_1.$$

The pairs (t, ε) define new coordinates in which the symplectic billiard map becomes

$$(t, \varepsilon) \mapsto (t_1, \varepsilon_1).$$

Proposition C.1. *Assume that Ω is a domain with $\mathscr{C}^8$-smooth boundary. In (t, ε)-coordinates, the symplectic billiard map is a $\mathscr{C}^6$-smooth map which admits the following asymptotic expansion*

$$\begin{cases} t_1 &= t + \varepsilon, \\ \varepsilon_1 &= \varepsilon + \frac{1}{30} k'(t_1)\varepsilon^4 + R_\Omega(\varepsilon, t_1)\varepsilon^6, \end{cases} \tag{13}$$

where k is the affine curvature of $\partial\Omega$ and $R_\Omega(\varepsilon, t_1)$ is a uniformly bounded remainder which is continuous in (ε, t_1) and converges to 0 as $\partial\Omega$ converges to an ellipse in the $\mathscr{C}^8$-topology (i.e. as the affine curvature tends to a positive constant).

Proof. The proof is inspired by [1]. We first assume that $\partial\Omega$ is $\mathscr{C}^{r+1}$-smooth for a general $r > 0$ sufficiently large and we consider an affine arclength parametrization $\gamma(t)$ of $\partial\Omega$ whose affine perimeter is $L = 1$. By construction, γ is $\mathscr{C}^r$-smooth. Let

$$t = t_1 - \varepsilon, \quad t_1, \text{ and } \quad t_2 = t_1 + \varepsilon_1$$

be the parameters of three successive points of reflection on $\partial\Omega$ corresponding to a symplectic billiard orbit. They satisfy

$$I_\Omega(t_1, \varepsilon, \varepsilon_1) := [\gamma(t_1 + \varepsilon_1) - \gamma(t_1 - \varepsilon), \gamma'(t_1)] = 0, \tag{14}$$

where $[u\ v]$ denotes the determinant of the vectors $u, v \in \mathbb{R}^2$. We take (ε, t_1) as coordinates so that the billiard map can be expressed as $(\varepsilon, t_1) \mapsto (\varepsilon_1, t_1 + \varepsilon_1)$.

We first justify regularity properties of the symplectic billiard map by means of the implicit function theorem for Banach spaces applied to a suitably chosen functional. Assume that $\varepsilon, \varepsilon_1 \in \mathbb{R}$ satisfy $|\varepsilon_1 + \varepsilon| < 1$.

Note that I_Ω is $\mathscr{C}^{r-1}$-smooth in all three variables. In the case when $\varepsilon + \varepsilon_1 \neq 0$, differentiating the left-hand side of equation (14) with respect to ε_1 gives

$$\partial_{\varepsilon_1} I_\Omega(t_1, \varepsilon, \varepsilon_1) = [\gamma'(t_1 + \varepsilon_1), \gamma'(t_1)].$$

The expression $\partial_{\varepsilon_1} I_\Omega(t_1,\varepsilon,\varepsilon_1)=0$ is equivalent to having parallel tangent lines to Ω at $\gamma(t_1)$ and at $\gamma(t_1+\varepsilon_1)$. The symplectic law of reflection then implies that either $t_1-\varepsilon=t_1$ or $t_1-\varepsilon=t_1+\varepsilon_1$. However, neither of these is possible since $|\varepsilon+\varepsilon_1|\notin\{0,1\}$. Hence, by the implicit function theorem in Banach spaces, the symplectic billiard map is locally $\mathscr{C}^{r-1}$-smooth.

In the case when $\varepsilon+\varepsilon_1=0$, which is equivalent to $\varepsilon=\varepsilon_1=0$, we consider another functional F given by

$$F_\gamma(\varepsilon,\varepsilon_1)=\frac{1}{\varepsilon_1+\varepsilon}\left[\gamma(t_1+\varepsilon_1)-\gamma(t_1-\varepsilon),\gamma'(t_1)\right].$$

Note that if γ is $\mathscr{C}^r$-smooth, then F_γ is $\mathscr{C}^{r-1}$-smooth for $(\varepsilon,\varepsilon_1)$ in a neighbourhood of $(0,0)$, as one can see by writing $\gamma(t_1+\varepsilon_1)-\gamma(t_1-\varepsilon)=(\varepsilon_1+\varepsilon)\int_0^1\gamma'((\varepsilon_1+\varepsilon)\theta+t_1-\varepsilon)\mathrm{d}\theta$. Moreover, it satisfies $\partial_{\varepsilon_1}F_\gamma(0,0)=-1/2\neq 0$. By the implicit function theorem, there exist a $\mathscr{C}^{r-1}$-smooth map $\varphi_\gamma: I\to\mathbb{R}$, where I is a small interval containing 0, such that $F_\gamma(\varepsilon,\varepsilon_1)=0$ if and only if $\varepsilon_1=\varphi_\gamma(\varepsilon)$. As a consequence of the implicit function theorem for Banach spaces, it follows that φ is continuous in (γ,ε), where γ is considered as an element of the set of $\mathscr{C}^r$-smooth maps endowed with the corresponding topology. In particular, we have an expansion of the symplectic billiard map $(t,\varepsilon)\mapsto(t_1=t+\varepsilon,\varepsilon_1)$ given by

$$\varepsilon_1=a_0(t_1)+a_1(t_1)\varepsilon+\ldots+\ldots+a_{r-2}(t_1)\varepsilon^{r-2}+R_\Omega^{r-1}(\varepsilon,t_1)\varepsilon^{r-1},$$

where each $a_k(t_1)$ depends on the k-jet of F_γ at $(0,0)$ and $R_\Omega^{r-1}(\varepsilon,t_1)$ is a real valued remainder which is continuous in (ε,t_1) and depends only on the $(r-1)$-jet of F_γ restricted to the graph of φ_γ. In particular, R_Ω^{r-1} depends on the r-jet of γ on the graph of φ_γ. If Ω is bounded by an ellipse, then $\varepsilon_1=\varepsilon$. It follows that if $\partial\Omega$ converges to an ellipse in the $\mathscr{C}^r$-topology, then R_Ω^{r-1} converges uniformly to 0.

Assume that $r=7$, so that $\partial\Omega$ is $\mathscr{C}^8$-smooth. We now compute $a_0,\ldots,a_5$ by asymptotically expanding F_γ. It is clear that $a_0=0$ by continuity, since the glancing region (with zero angle of reflection) corresponds to fixed points of the symplectic billiard map. To compute a_1, let us write

$$\begin{aligned}\int_0^1\gamma'((\varepsilon_1+\varepsilon)\theta+t_1-\varepsilon)\mathrm{d}\theta&=\int_0^1(\gamma'(t_1)+((a_1+1)\theta-1)\varepsilon\gamma''(t_1))\,\mathrm{d}\theta\;+\mathcal{O}\left(\varepsilon^2\right)\\&=\gamma'(t_1)+\frac{1}{2}(a_1-1)\varepsilon\gamma''(t_1)+\mathcal{O}\left(\varepsilon^2\right).\end{aligned}$$

Since $\det(\gamma'(t_1),\gamma''(t_1))=1$, the equality $F_\gamma(\varepsilon,\varepsilon_1)=0$ implies that $a_1=1$. Let us now show that $a_2=a_3=0$ by expanding to fourth order. To simplify, we omit the parameter t_1. Note also that $\det(\gamma^{(3)},\gamma')=0$ and hence, it is unnecessary to compute the coefficient multiplying $\gamma^{(3)}$. Using the identity $\int_0^1(2\theta-1)\mathrm{d}\theta=\int_0^1(2\theta-1)^3\mathrm{d}\theta=0$, we obtain

$$\begin{aligned}&\int_0^1\gamma'((\varepsilon_1+\varepsilon)\theta+t_1-\varepsilon)\,\mathrm{d}\theta\\&\quad=\int_0^1\gamma'+(2\theta-1)\varepsilon\gamma''+\left(\theta a_2\varepsilon^2+\theta a_3\varepsilon^3\right)\gamma''\\&\qquad+\frac{1}{6}(2\theta-1)^3\varepsilon^3\gamma^{(4)}+b_\varepsilon(\theta)\gamma^{(3)}+\mathcal{O}\left(\varepsilon^4\right)\\&\quad=\gamma'+\frac{1}{2}\left(a_2\varepsilon^2+a_3\varepsilon^3\right)\gamma''+\tilde{b}_\varepsilon\gamma^{(3)}+\mathcal{O}\left(\varepsilon^4\right),\end{aligned}$$

where b and $\tilde{b}$ are maps depending on (ε,θ) and ε respectively. Considering that $\det(\gamma'',\gamma') = -1$ and $\det(\gamma^{(3)},\gamma') = 0$, we now compute

$$F_\gamma(\varepsilon,\varepsilon_1) = -\frac{1}{2}a_2\varepsilon^2 - \frac{1}{2}a_3\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right).$$

The condition $F_\gamma(\varepsilon,\varepsilon_1) = 0$ immediately implies that $a_2 = a_3 = 0$. Let us finally compute a_4 and a_5 by writing

$$(\varepsilon_1+\varepsilon)\theta + t_1 - \varepsilon = t_1 + (2\theta-1)\varepsilon + \theta a_4\varepsilon^4 + \theta a_5\varepsilon^5 + \mathcal{O}\left(\varepsilon^6\right)$$

and then expanding

$$\int_0^1 \gamma'((\varepsilon_1+\varepsilon)\theta + t_1 - \varepsilon)\,\mathrm{d}\theta = \gamma' + \frac{1}{2}\left(a_4\varepsilon^4 + a_5\varepsilon^5\right)\gamma'' + c_\varepsilon\gamma^{(3)} + \frac{1}{120}\varepsilon^4\gamma^{(5)} + \mathcal{O}\left(\varepsilon^6\right),$$

where c is a function depending on ε. Therefore, $F_\gamma(\varepsilon,\varepsilon_1)$ is given by

$$F_\gamma(\varepsilon,\varepsilon_1) = -\frac{1}{2}\left(a_4\varepsilon^4 + a_5\varepsilon^5\right) + \frac{1}{120}\varepsilon^4\det\left(\gamma^{(5)},\gamma'\right) + \mathcal{O}\left(\varepsilon^6\right).$$

Hence,

$$a_4 = \frac{1}{60}\det\left(\gamma^{(5)},\gamma'\right) \qquad \text{and} \qquad a_5 = 0.$$

Differentiating twice the equality $\gamma^{(3)} = -k\gamma$, we obtain $\det(\gamma^{(5)},\gamma') = 2k'$, which, by taking the limit as $q\to\infty$, implies that

$$a_4 = \frac{1}{30}k'(t_1).$$

This completes the proof. □

C.2. Estimates of q-periodic symmetric orbits

Now consider an axially symmetric domain Ω whose boundary is parametrized by an affine arclength coordinate t such that $t = 0$ corresponds to a point on the axis of symmetry. Given $q \geqslant 2$, consider the coordinates

$$(t_0 = 0, t_1, \ldots, t_{q-1})$$

of consecutive points $p_0,\ldots,p_{q-1}$ on the boundary $\partial\Omega$ such that $p_0\ldots p_{q-1}$ form a polygon of maximal area. We omit the dependence on q of the t_j's and estimate their asymptotic behaviour, as well that of each $\varepsilon_j = t_{j+1} - t_j$. These estimates were first computed in [3, proposition 4.3], but we give an alternate proof with the additional result that the remainder tends to 0 for domains which are $\mathscr{C}^8$-close to ellipses.

Proposition C.2 (See [3] proposition 4.3). *Assume that $\partial\Omega$ is $\mathscr{C}^8$-close to an ellipse $\mathscr{E}$. Then*

$$t_j = L\frac{j}{q} + \frac{1}{q^2}a_0(j/q) + \frac{R^0_\Omega(j,q)}{q^4},$$
$$\varepsilon_j = \frac{L}{q} + \frac{1}{q^3}b_0(j/q) + \frac{1}{q^4}b_1(j/q) + \frac{R^1_\Omega(j,q)}{q^5},$$

where L is the affine perimeter of $\partial\Omega$, $R^0_\Omega(j,q)$ *and* $R^1_\Omega(j,q)$ *are bounded remainders which converge uniformly to 0 as* Ω *converges to* $\mathscr{E}$ *in the* $\mathscr{C}^8$*-topology, satisfying*

$$R^i_\Omega(j+1,q) - R^i_\Omega(j,q) = \mathcal{O}\left(\frac{1}{q}\right) \tag{15}$$

uniformly in j, and

$$a_0(\theta) = \frac{L^2}{30}\int_0^{L\theta} k(t) - (k)_0 \,\mathrm{d}t,$$
$$b_0(\theta) = \frac{L^3}{30}\left(k(L\theta) - (k)_0\right), \qquad b_1(\theta) = \frac{L^4}{60}k'(L\theta),$$

for any $\theta \in \mathbb{R}/\mathbb{Z}$*, where k is the affine curvature of* $\partial\Omega$ *with mean value*

$$(k)_0 = \frac{1}{L}\int_0^L k(t)\,\mathrm{d}t.$$

Remark C.3. Note that a_0, a_1, b_0 and b_1 vanish if $\partial\Omega$ is an ellipse.

In order to prove proposition C.2, we will need the following lemma on a preliminary bound for ε_j.

Lemma C.4. *There exist* $C = C(\Omega) > 0$ *such that for any* $j \in \{0,\ldots,q-1\}$*, the quantity* ε_j *is bounded by*

$$\varepsilon_j \leqslant \frac{C}{q}.$$

Proof. This proof is based on an analogous proof for classical billiards found in [2]. Note however that some arguments there are missing. We will provide them here. As a consequence of (13), one obtains

$$1 = \varepsilon_0 + \ldots + \varepsilon_{q-1}$$

and hence, there is $j_* \in \{0,\ldots,q-1\}$ such that

$$\varepsilon_{j_*} \leqslant \frac{1}{q}.$$

Since the orbit is cyclic, we will consider quantities ε_j with $j \geqslant q$ by considering j modulo q. Again using equation (13), we deduce the existence of an $M = M(\Omega) > 1$ such that for any $j \in \{0,\ldots,q-1\}$,

$$|\varepsilon_{j+1} - \varepsilon_j| \leqslant M\varepsilon_j^4.$$

Let us choose $q > 0$ sufficiently large; in fact $q \geqslant q_0 := e^{3/2}M^2$ is enough. By induction on j, we see that for any $j \in \{j_*,\ldots,j_*+q-1\}$,

$$\varepsilon_j \leqslant \frac{M}{q}\left(1+\frac{1}{q}\right)^j.$$

It is obviously true for $j = j_*$ and if it is true for $j \geqslant j_*$, we estimate

$$\varepsilon_{j+1} \leqslant \varepsilon_j + |\varepsilon_{j+1} - \varepsilon_j| \leqslant \varepsilon_j + M\varepsilon_j^4.$$

Hence,

$$\varepsilon_{j+1} \leqslant \frac{M}{q}\left(1+\frac{1}{q}\right)^j\left(1+\frac{M^4}{q^3}\left(1+\frac{1}{q}\right)^{3j}\right),$$

where

$$\frac{M^4}{q^3}\left(1+\frac{1}{q}\right)^{3j} \leqslant \frac{M^4 e^3}{q^3} \leqslant \frac{1}{q}$$

by our assumptions on q. This concludes the inductive step. In particular, the statement of lemma C.4 is true with $C = eM$ for $q \geqslant q_0$. To make the proof work for all q, we can take C larger so that $\varepsilon_j \leqslant C/q$ is satisfied even if $q < q_0$; the latter only requires a finite set of inequalities to be satisfied. □

Proof of proposition C.2. To prove the result, we will assume that the affine perimeter of Ω is $L = 1$. The general case can be reduced this by homothety of the domain. Substituting the inequality given in lemma C.4 into the expansion (13), we see that

$$\varepsilon_j = \varepsilon_0 + \mathcal{O}\left(\frac{1}{q^3}\right)$$

uniformly in j. Moreover, from the equality

$$\varepsilon_0 + \ldots + \varepsilon_{q-1} = 1, \tag{16}$$

we deduce that

$$\varepsilon_j = \frac{1}{q} + \mathcal{O}\left(\frac{1}{q^3}\right) \qquad \text{and} \qquad t_j = \frac{j}{q} + \mathcal{O}\left(\frac{1}{q^2}\right)$$

uniformly in j.

Hence, let us write

$$\varepsilon_j = \frac{1}{q} + \frac{r_q(j)}{q^3}$$

for some quantity $r_q(j)$, yet to be determined. Substituting the equations for t_j and ε_j into (13), we obtain

$$\varepsilon_{j+1} - \varepsilon_j = \frac{r_q(j+1) - r_q(j)}{q^3} = \frac{\alpha(j/q)}{q^4} + \frac{\alpha'(j/q)}{q^5} + \mathcal{O}\left(\frac{1}{q^6}\right) \tag{17}$$

uniformly in j. Hence we deduce that, uniformly in j,

$$r_q(j) = r_q(0) + S_q(\alpha) + \frac{1}{q} S_q(\alpha') + \mathcal{O}\left(\frac{1}{q^2}\right), \tag{18}$$

where $S_{q,j}(f) = q^{-1}\sum_{i=0}^{j-1} f(i/q)$. Note that by comparing these sums to integral terms, we can deduce that if f is $\mathscr{C}^2$, then

$$S_{q,j}(f) = u(j/q) + \frac{v(j/q)}{q} + \mathcal{O}\left(\frac{1}{q^2}\right)$$

uniformly in j, where u and v are differentiable and satisfy $u' = f$ and $v' = -f'/2$. From this remark applied to equation (18), there exists b_0, b_1 differentiable such that $b_0' = \alpha$ and $b_1' = \alpha'/2$ and

$$r_q(j) = b_0(j/q) + \frac{b_1(j/q)}{q} + \mathcal{O}\left(\frac{1}{q^2}\right)$$

uniformly in j. Moreover, if $q^{-2}R^1_\Omega(j,q)$ is the remainder, applying again equation (17), we obtain

$$R^1_\Omega(j+1,q)-R^1_\Omega(j,q)=\mathcal{O}\left(\frac{1}{q}\right).$$

uniformly in j. Taking into account the periodicity and initial conditions on the maps, we obtain the expressions for b_0 and b_1.

Now applying the same method to the relation

$$t_{j+1}-t_j=\varepsilon_j=\frac{1}{q}+\frac{1}{q^3}b_0(j/q)+\frac{1}{q^4}b_1(j/q)+\frac{R^1_\Omega(j,q)}{q^5},$$

we obtain the desired result. □

We deduce the following asymptotic estimates.

Lemma C.5. *Assume that $\partial\Omega$ is $\mathscr{C}^8$-close to an ellipse $\mathscr{E}$ of affine perimeter $L_\mathscr{E}$ and affine curvature $k_\mathscr{E}$. We then have the following estimates for all $q>1$ and $j\in\{0,\ldots,q-1\}$:*

$$\varrho(t_j)^{-1/3}\ell^{(q)}_j=2k_\mathscr{E}^{-1/2}\sin\left(\frac{2\pi}{q}\right)+\frac{c_0}{q}+\frac{c_1(Lj/q)}{q^3}+\frac{R_\Omega(j,q)}{q^5},\tag{19}$$

where $\ell^{(q)}_j$ is the distance between the points $\gamma(t_{j-1})$ and $\gamma(t_{j+1})$, $R_\Omega(j,q)$ is a uniformly bounded remainder which converges to 0 as Ω converges to $\mathscr{E}$ in the $\mathscr{C}^8$-topology,

$$c_0=2(L-L_\mathscr{E}),$$
$$c_1(t)=\frac{1}{3}\left(k_\mathscr{E}L^3_\mathscr{E}-k(t)L^3\right)+\frac{L^3}{15}\left(k(\theta)-(k)_0\right),\quad\forall t\in\mathbb{R}/L\mathbb{Z},$$

L is the affine perimeter of $\partial\Omega$ and k is its affine curvature.

Remark C.6. Note that c_0, c_1 and c_2 are invariant by affine area-preserving transformations since affine curvature and perimeter are. This was to be expected by the invariance of $\varrho(t_j)^{-1/3}\ell^{(q)}_j$.

Proof. In the following, any subscript or superscript $\mathscr{E}$ will indicate that the underlying quantity is related to dynamics in an ellipse. We first compute the quantity $\varrho(t_j)^{-1/3}\ell^{(q)}_j$ for an ellipse and then proceed for a general domain.

For the ellipse, since it is invariant by area-preserving affine transformations, we can assume that $\mathscr{E}$ is a disk with the same affine curvature. By proposition A.2, $\varrho_\mathscr{E}=k_\mathscr{E}^{-3/4}$ and simple trigonometric considerations in a disk of radius $\varrho_\mathscr{E}$ imply that $\ell^{(q)}_j=2\varrho_\mathscr{E}\sin(2\pi/q)$. Hence,

$$\varrho(t_j)^{-1/3}\ell^{(q)}_j=2k_\mathscr{E}^{-1/2}\sin\left(\frac{2\pi}{q}\right).\tag{20}$$

We now compute similar estimates for the domain Ω. Consider an affine parametrization γ of $\partial\Omega$ which is δ-$\mathscr{C}^6$-close to the one of $\mathscr{E}$, which we denote by $\gamma_\mathscr{E}$. We first estimate $\ell^{(q)}_j$ and then $\varrho(t_j)^{-1/3}$.

For simplicity, given an L-periodic map $f:\mathbb{R}\to\mathbb{R}$, we write $f=f(Lj/q)$, $f^+=f(L(j+1)/q)$ and $f^-=f(L(j-1)/q)$. Then, considering proposition C.2 and doing Taylor expansions at Lj/q, the following estimates hold uniformly in j:

$$\gamma(t_{j\pm1})=\gamma^\pm+\frac{a_0\gamma'}{q^2}\pm\frac{a_0\gamma''+a_0'\gamma'}{q^3}+\frac{u_{qj}+R^0_\Omega(j\pm1,q)\gamma'}{q^4}+\mathcal{O}\left(\frac{1}{q^5}\right),$$

where $u_{qj} \in \mathbb{R}^2$ depends on the 2-jet of a_0 and γ at Lj/q. Hence, by equation (15),

$$\gamma(t_{j+1}) - \gamma(t_{j-1}) = \gamma^+ - \gamma^- + \frac{2(a_0'\gamma' + La_0\gamma'')}{q^3} + \frac{R_q}{q^5},$$

where R_q is uniformly bounded in j and q and tends to 0 as Ω converges to an ellipse. Taylor expanding $\gamma^\pm$ at Lj/q and using the expression of γ'' computed in proposition A.2, we obtain

$$\gamma^+ - \gamma^- = \left(\frac{2L}{q} - \frac{kL^3}{3q^3}\right)\gamma' + \frac{R_q'}{q^5},$$

where R_q' has the same properties as R_q. With the same arguments, we also have

$$a_0'\gamma' + La_0\gamma'' = \left(a_0' + \frac{L}{3}\varrho'\varrho^{-1}a_0\right)\gamma' - La_0\varrho^{-1/3}N,$$

where N is the unit outward normal vector to $\partial\Omega$ at the corresponding point on $\partial\Omega$. From the preceding equations, we deduce that

$$\gamma(t_{j+1}) - \gamma(t_{j-1}) = \left(\frac{2L}{q} - \frac{kL^3}{3q^3} + \frac{1}{q^3}\left(a_0' + \frac{L}{3}\varrho'\varrho^{-1}a_0\right) + \mathcal{O}\left(\frac{1}{q^5}\right)\right)\gamma' + \mathcal{O}\left(\frac{1}{q^3}\right)N. \tag{21}$$

Taking the norm of previous expression, we obtain

$$\ell_j^{(q)} = \frac{2L}{q}\varrho^{1/3} + \frac{\varrho^{1/3}}{q^3}\left(2\left(a_0' + \frac{L}{3}\varrho'\varrho^{-1}a_0\right) - \frac{kL^3}{3}\right) + \frac{R_q''}{q^5}, \tag{22}$$

where we used the formula $\|\gamma'\| = \varrho^{1/3}$ (see proposition A.2) and where R_q'' is a functional depending on $\gamma^{(5)}$.

Now estimating $\varrho(t_j)^{-1/3}$ using proposition C.2, we obtain:

$$\varrho(t_j)^{-1/3} = \varrho^{-1/3} - \frac{1}{3}\varrho'\varrho^{-4/3}\frac{a_0}{q^2} - \frac{1}{3}\varrho'\varrho^{-4/3}\frac{a_1}{q^3} + \mathcal{O}\left(\frac{1}{q^4}\right). \tag{23}$$

Therefore, using the two estimates (22) and (23),

$$\varrho(t_j)^{-1/3}\ell_j^{(q)} = \frac{2L}{q} + \frac{1}{3q^3}\left(6a_0' - kL^3\right) + \frac{r_q}{q^5},$$

where r_q is bounded uniformly in j.

Using equation (20) and the expressions for a_0 given in proposition C.2, formulas for the coefficients c_j follow from the previous expansion and estimates on each asymptotic term when $\partial\Omega$ is $\mathscr{C}^8$-close to an ellipse. □

We can now present the

Proof of proposition 3.10. Let Ω be a domain which is close to an ellipse $\mathscr{E}$ in the $\mathscr{C}^8$-smooth topology. For any integer $q \geqslant 3$ and $j \in \{0, \ldots, q-1\}$, the estimates on the jth impact point t_j of a nearly glancing q-periodic orbit given by proposition C.2 imply that

$$n(t_j) = n(Lj/q) + n'(Lj/q)\frac{a_0(j/q)}{q^2} + \frac{\|n\|_{\mathscr{C}^2}R_\Omega(j,q)}{q^4}, \tag{24}$$

where $R_\Omega(j,q)$ is a uniformly bounded remainder which converges to 0 as Ω converges to $\mathscr{E}$ in the $\mathscr{C}^8$ topology. Multiplying the expressions in equation (19) by those in equation (24), we obtain

$$n(t_j)\,\varrho(t_j)^{-1/3}\,\ell_j^{(q)} = 2k_{\mathscr{E}}^{-1/2}\sin\left(\frac{2\pi}{q}\right)n(Lj/q) + \frac{c_0}{q}n(Lj/q) + \frac{c_1(Lj/q)}{q^3}n(Lj/q) + \frac{\left(4\pi k_{\mathscr{E}}^{-1/2}+c_0\right)a_0(Lj/q)}{q^3}n'(Lj/q) + \frac{\|n\|_{\mathscr{C}^2}R_\Omega^*(j,q)}{q^5}, \tag{25}$$

where $R_\Omega^*(j,q)$ is a uniformly bounded remainder which converges uniformly to 0 as Ω converges to $\mathscr{E}$ in the $\mathscr{C}^8$ topology.

We now sum over j in the equation (25). First, note that lemmas D.2 and D.3 imply that

$$\frac{1}{q}\sum_{j=0}^{q-1}c(Lj/q)\,n(Lj/q) = (cn)_0 + \mathcal{O}\left(\frac{\|n\|_\gamma}{q^\gamma}\right)$$

and

$$\frac{1}{q}\sum_{j=0}^{q-1}c(Lj/q)\,n'(Lj/q) = (cn')_0 + \mathcal{O}\left(\frac{\|n\|_\gamma}{q^{\gamma-1}}\right)$$

for any L-periodic map c in the space H^γ, where $(\cdot)_0$ returns the average value of a function. Following this remark, we conclude the proof by setting

$$\alpha_0(n) = (c_0 n)_0 = c_0\widehat{n}_0$$

and

$$\alpha_1(n) = \left(c_1 n + \left(4\pi k_{\mathscr{E}}^{-1/2}+c_0\right)a_0 n'\right)_0.$$

□

Appendix D. Operators acting on Sobolev spaces

For any $\gamma > 0$, denote by h^γ the space of sequences $(u_q)_{q\geqslant 0}$ such that

$$q^\gamma u_q = \mathcal{O}(1)$$

and define the norm $\|u\|_\gamma = \sup\{q^\gamma|u_q|,\ q\in\mathbb{N}\}\cup\{|u_0|\}$. Given a continuous 1-periodic even map $n:\mathbb{R}\to\mathbb{R}$, consider its Fourier decomposition

$$n = \sum_{j\geqslant 0}\widehat{n}_j e_j,$$

where e_j denotes the jth Fourier mode given by $e_j(\theta) = \cos(2\pi j\theta)$, $\theta\in\mathbb{R}$. The space of even 1-periodic maps $n:\mathbb{R}\to\mathbb{R}$ such that

$$q^\gamma\widehat{n}_q = \mathcal{O}(1)$$

will be denoted by H^γ with the norm $\|\cdot\|_\gamma$ defined by

$$\|n\|_\gamma = \sup\{\{q^\gamma|\widehat{n}_q|,\ q\in\mathbb{N}\}\cup\{|\widehat{n}_0|\}\}.$$

Note that we can extend $\|n\|_\gamma$ to $\mathbb{R}\cup\{+\infty\}$ when the sup involved in the definition is infinite. In this case, $n\in H^\gamma$ if and only if $\|n\|_\gamma<+\infty$.

We can also define analogous notions for odd maps by considering their decomposition in the sin basis. We will sometimes identify h^γ with H^γ via the isometry

$$n=\sum_{j\geqslant 0}\widehat{n}_j e_j\in H^\gamma\leftrightarrow(\widehat{n}_j)_j\in h^\gamma.$$

Given a 1-periodic even map $F:\mathbb{R}\to\mathbb{R}$, define for any $q\geqslant 1$ the quantities

$$[F]_q=\frac{1}{q}\sum_{k=0}^{q-1}F(k/q)\qquad\text{and}\qquad [F]_q^*=[F]_q-\widehat{F}_0.$$

Remark D.1. $[F]_q$ and $[F]_q^*$ are related to the Fourier coefficients of F whose order is divisible by q, namely

$$[F]_q=\sum_{m\geqslant 0}\widehat{F}_{mq}\qquad\text{and}\qquad [F]_q^*=\sum_{m>0}\widehat{F}_{mq}.$$

Lemma D.2. *Assume that $F\in H^\gamma$ with $\gamma>1$. Then we have the following bound on $[F]_q^*$:*

$$|\,[F]_q^*\,|\leqslant\frac{\zeta(\gamma)}{q^\gamma}\|F\|_\gamma,$$

where $\zeta(\gamma)$ is the Riemann ζ function evaluated at γ.

Proof. The Fourier decomposition of F induces the following expansion

$$[F]_q=\sum_{m\geqslant 0}\widehat{F}_{mq}=\widehat{F}_0+\sum_{m>0}\widehat{F}_{mq}.$$

By definition, we have the bound

$$|\widehat{F}_{mq}|\leqslant\frac{1}{(mq)^\gamma}\|F\|_\gamma.$$

The estimates follow by summing over m and recalling that $\zeta(\gamma)=\sum_{m\geqslant 0}m^{-\gamma}$. $\square$

Lemma D.3. *Let $\gamma>1$ and α,β be 1-periodic maps of the same parity. Then the following inequality holds in $[0,+\infty]$:*

$$\|\alpha\beta\|_\gamma\leqslant C_\gamma\|\alpha\|_\gamma\|\beta\|_\gamma,$$

where $C_\gamma=2(\zeta(\gamma)+1)>0$.

Remark D.4. In particular, if $\alpha,\beta\in H^\gamma$ have the same parity, then $\alpha\beta\in H^\gamma$.

Proof. Without loss of generality, assume that α and β are even. Computing the jth Fourier coefficient of $\alpha\beta$, we find

$$\widehat{(\alpha\beta)}_j=\frac{1}{2}\sum_{k=0}^{j}\widehat{\alpha}_k\widehat{\beta}_{j-k}+\frac{1}{2}\sum_{|k-\ell|=j}\widehat{\alpha}_k\widehat{\beta}_\ell.$$

It follows that

$$j^\gamma|\widehat{(\alpha\beta)}_j|\leqslant\|\alpha\|_\gamma\|\beta\|_\gamma\left(\frac{j^\gamma}{2}\sum_{k=0}^{j}\frac{1}{k^\gamma(j-k)^\gamma}+\frac{j^\gamma}{2}\sum_{|k-\ell|=j}\frac{1}{k^\gamma\ell^\gamma}\right)$$

To conclude, we use the bound

$$\sum_{k=1}^{j-1} \frac{1}{k^{\gamma}(j-k)^{\gamma}} \leqslant \frac{2\zeta(\gamma)}{j^{\gamma}},$$

which comes from the inequality

$$\frac{1}{x^{\gamma}(1-x)^{\gamma}} \leqslant \frac{1}{x^{\gamma}} + \frac{1}{(1-x)^{\gamma}}$$

for any $x \in (0,1)$. □

Lemma D.5. *Let $r > 1$ and $\gamma > r+1$. Assume that $n : \mathbb{R} \to \mathbb{R}$ is a 1-periodic even map which belongs to H^{γ}. Then, n is $\mathscr{C}^r$-smooth and*

$$\|n\|_{\mathscr{C}^r} \leqslant (2\pi)^r \zeta(\gamma - r) \|n\|_{\gamma}.$$

Proof. Expanding n in Fourier modes, we see that for any integer $0 \leqslant k \leqslant r$,

$$\|n^{(k)}\|_{\mathscr{C}^0} \leqslant \sum_{j \geqslant 0} (2\pi j)^k |\widehat{n}_j| \leqslant (2\pi)^k \sum_{j \geqslant 0} j^{k-\gamma} \|n\|_{\gamma} = (2\pi)^k \zeta(\gamma - k) \|n\|_{\gamma}.$$

The result follows by taking the maximum over all possible k. □

Proposition D.6. *Let $\gamma > 0$ and consider the following operator. For $n \in H^{\gamma}$, take $T(n) = (u_q)_{q \geqslant 0}$ to be the sequence defined by*

$$u_q = \begin{cases} \widehat{n}_0 & \text{if } q = 0; \\ n(0) & \text{if } q = 1; \\ n(1/2) & \text{if } q = 2; \\ \mu_q [n]_q^* & \text{if } q > 2, \end{cases} \tag{26}$$

where $\mu_q \in \mathbb{R}^*$ has a nonzero limit as $q \to \infty$. Then T defines a bounded invertible operator between H^{γ} and h^{γ}.

Proof. We first prove that T is bounded. This comes from estimates on the different expressions of u_q: if $q = 0$, then $|u_0| \leqslant \|n\|_{\gamma}$ by definition of the H^{γ} norm. Moreover, by expanding n in Fourier modes, one sees that $|1^{\gamma} u_1| = |n(0)| \leqslant \zeta(\gamma)\|n\|_{\gamma}$. Finally, from lemma D.2, we conclude that $q^{\gamma}|u_q| \leqslant \zeta(\gamma)\|n\|_{\gamma}$ for any $q \geqslant 2$.

To prove that T is invertible, we may assume that $\mu_q = 1$ for any $q \geqslant 3$. Indeed, if we take $|\mu_q|$ to be bounded away from 0 and ∞, then the operator $S_{\mu} : h^{\gamma} \to h^{\gamma}$ defined for any $q \geqslant 0$ and $u \in h^{\gamma}$ by

$$S_{\mu}(u)_q = \mu_q u_q$$

is well-defined, continuous, invertible and its inverse is given by $S_{\mu^{-1}}$, where μ^{-1} is the sequence defined for any $q \geqslant 0$ by $(\mu^{-1})_q = \mu_q^{-1}$. Hence, considering $S_{\mu}^{-1} \circ T$, we can assume that $\mu_q = 1$ for any $q \geqslant 1$.

Let us define an inverse for this operator. Fix $u = (u_q)_q \in h^{\gamma}$ and consider the so-called *Möbius function* $\mu : \mathbb{Z}^{>0} \to \{-1, 0, 1\}$, which is defined as follows. Given an integer $k \geqslant 0$, consider its decomposition into s distinct primes and set

$$\mu(k) = \begin{cases} 0 & \text{if } k \text{ has a square in its decomposition;} \\ 1 & \text{if } k \text{ has a no squares in its decomposition and } s \text{ is even;} \\ -1 & \text{if } k \text{ has a no squares in its decomposition and } s \text{ is odd.} \end{cases}$$

It is known to satisfy the following formula

$$\sum_{d|k} \mu(d) = \delta_{k,1}, \tag{27}$$

where the sum ranges over all divisors $d > 0$ of k and $\delta_{k,1}$ is the Kronecker delta. For $j > 2$, define

$$\widehat{n}_j = \sum_{\substack{q>0 \\ j|q}} \mu\left(\frac{q}{j}\right) u_q. \tag{28}$$

Set $\widehat{n}_0 = u_0$ and $\widehat{n}_1, \widehat{n}_2 \in \mathbb{R}$ so that they satisfy

$$\sum_{j\geqslant 0} \widehat{n}_j = u_1 \qquad \text{and} \qquad \sum_{j\geqslant 0} (-1)^j \widehat{n}_j = u_2.$$

Defining n to be the even map whose Fourier coefficients are given by the sequence $(\widehat{n}_j)_j$, one can easily show that n lies in H^γ, with $\|n\|_\gamma \leqslant \zeta(\gamma)\|u\|_\gamma$. Furthermore, equation (27) together with the choices of $\widehat{n}_0$, $\widehat{n}_1$ and $\widehat{n}_2$ implies that $T(n) = u$, which concludes the proof. □

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