On the structure of action-minimizing sets for Lagrangian systems

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Abstract

In this thesis we investigate the topological properties of the action-miniziming sets that appear in the study of Tonelli Lagrangian and Hamiltonian systems.

In the first part, we will focus on understanding the topology of the quotient Aubry set, and in particular its total disconnectedness. This property, in fact, plays a key role in the variational methods developed for constructing orbits with a prescribed behavior or connecting different regions of the state space. We will show how this problem may be related to a Sard-like property for certain subsolutions of Hamilton-Jacobi equation and use this approach to show total disconnectedness under suitable assumptions on the Lagrangian.

In the second part, we will discuss some relations between the dynamics of the system and the underlying symplectic geometry of the space. In particular, we will point out how to deduce from weak KAM theory the symplectic invariance of the Aubry set and the quotient Aubry set, and we will study the action minimizing properties of invariant measures supported on Lagrangian graphs. We will then use these results to deduce uniqueness of invariant Lagrangian graphs in a fixed homology or cohomology class, with particular attention to the case of KAM tori and Herman's Tori.

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Ad *Uno, Nessuno e Centomila.*

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Or se' giunta, anima fella!

(Dante, Inferno VIII, 18)

Contents

Introduction

La Nature est un temple où de vivants piliers
laissent parfois sortir de confuses paroles.
(Charles Baudelaire, *Correspondances*)

It was about forty years ago when the striking, and somehow unexpected, phenomenon of "diffusion" pointed out by V. I. Arnol'd [3] (that now bears his name) shed a new light on the stable picture drawn by the works of Kolmogorov [44], Moser [68] and Arnol'd himself [2] few years earlier (from whose initials the acronym KAM). This new insight led to a change of perspectives and in order to make sense of the complex balance between stable and unstable motions that was looming out, new approaches needed to be exploited. Amongst these, variational methods turned out to be particularly successful. Mostly inspired by the so-called *least action principle*, a sort of widely accepted thriftiness of the Nature in all its actions, they provide the natural setting to get over the local view given by the analytical methods and make towards a global understanding of the dynamics.

Aubry-Mather theory represents probably one of the biggest triumphs in this direction. Developed independently by Serge Aubry [6] and John Mather [51] in the eighties, this novel approach to the study of the dynamics of twist diffeomorphisms of the annulus (which correspond to Poincaré maps of 1-dimensional Hamiltonian systems [68]) pointed out the existence of many action minimizing sets that generalize invariant rotational curves and continue to exist even after these curves disappear.

Besides providing a detailed structure theory for these new sets, this powerful approach yielded to a better understanding of the destiny of invariant rotational curves and to the construction of interesting chaotic orbits as a result of their destruction [53, 38, 58].

Motivated by these achievements, John Mather [57, 59] - and later Ricardo Mañé [48, 23] and Albert Fathi [32] in different ways - developed a generalization of this theory to higher dimensional systems. Positive definite superlinear Lagrangians on compact manifolds, also called *Tonelli Lagrangians* (see Definition 1.1.1), were the appropriate setting to work in. Although it was still possible to show the existence of interesting action minimizing sets, the situation turned out to be more complicated and, even today, very little is known about their structure. This lack represents one of the biggest restraints to the potentiality of such approaches, in particular in view of proving the existence of chaotic orbits. In fact, these sets play a twofold role. Whereas on the one hand they may provide an obstruction to the existence of diffusion, on the other hand they have a fundamental importance in the variational methods developed for constructing orbits with a prescribed behavior (see for instance [11, 12, 25, 61, 63, 65, 78]). Most of these methods, in fact, are based on strong assumptions on the structure of these sets; however, understanding such properties is a very difficult task. It is in the light of this observation that our interest in the structural properties of these sets can be better explained and understood.

This thesis is organized as follows:

- In chapter 1, we will present a brief - but comprehensive - introduction to Mather's theory for Lagrangian systems and Fathi's weak KAM theory, providing the necessary ground for understanding the main results and techniques in the following chapters.

- In chapter 2, we will focus on understanding the topology of the *quotient Aubry set*.

This set plays a key role in the variational method developed by John Mather [61, 63, 65] for showing the generic existence of Arnol'd diffusion in two-and-half degrees of freedom. In order to extend this approach to higher dimensions, it would be useful to understand when this set is totally disconnected or small in some dimensional sense. In this chapter, we will address this problem and the results in [75], exploiting an existing relation with a Sard-like lemma for certain subsolutions of Hamilton-Jacobi equation.

- In chapter 3, we will discuss the relation between the dynamics of the system and the underlying symplectic geometry of the phase space. In particular, we will prove the symplectic invariance of the Aubry set and the quotient Aubry set and study the action minimizing properties of Lagrangian graphs. We will use these results to deduce some uniqueness properties [34] of invariant Lagrangian graphs in a fixed homology or cohomology class. Besides a dynamical and symplectic topological interest, these results were motivated by a question on the global uniqueness of KAM tori (and Herman's tori) with a given rotation vector.

Description of the main results.

Chapter 2: On the structure of the quotient Aubry set

The *quotient Aubry set* (see definition 2.1.1) and its topological properties play, as we already mentioned, an important role in the variational methods developed for the construction of orbits connecting different regions of the phase space. In particular, in the light of John Mather's studies on Arnol'd diffusion [61, 63, 65], it is important to understand when this set is totally disconnected. In fact, as we will see, one can think of it as the set of α and ω limits of global action-minimizers: each of these curves is asymptotic in the past and in the future to an element of this quotient set, also called *static class* [48, 59] (see also remark 1.3.13). From here the need for a clear understanding of its topology in order to control where the constructed orbit comes from or is heading to.

In [62] Mather showed that if the state space has dimension $\leq$ 2 (in the non-autonomous case) or the Lagrangian is the kinetic energy associated to a Riemannian metric and the state space has dimension $\leq$ 3, then the quotient Aubry set is totally disconnected for all cohomology classes (see the addendum to section 2.2).

Unfortunately in higher dimension the situation turns out to be more complicated and in general total disconnectedness might fail. In [64], for instance, for each $d \geq 3$ and $0 < \varepsilon < 1$, John Mather constructed a $C^{2d-3,1-\varepsilon}$ *mechanical Lagrangian* on $\mathrm{T}\mathbb{T}^d$ (*i.e.,* a Lagrangian which is the sum of the kinetic energy and a potential) whose associated quotient Aubry set - corresponding to the zero cohomology class - is isometric to a closed interval (see also the addendum to section 2.2 for similar counterexamples due to Albert Fathi [31]). What emerges from this construction is that the machinery breaks down when one tries to obtain a smoother Lagrangian, as if there were an intrinsic relation between the regularity of the Lagrangian, the dimension of the state space and the dimension of the quotient Aubry set.

In [75] we proved the following result.

Theorem 2.3.1. *Let M be a compact connected manifold of dimension $d \geq 1$, L a Tonelli Lagrangian on $\mathrm{T}M$ and let $\mathcal{L} : \mathrm{T}M \longrightarrow \mathrm{T}^*M$ denote the Legendre transform (see (1.4)). Suppose that :*

i) $\Lambda_L := \mathcal{L}(M \times \{0\})$ *is a C^2 Lagrangian submanifold of T^*M, with the canonical symplectic form, and let $c_L \in \mathrm{H}^1(M;\mathbb{R})$ be its Liouville (or cohomology) class.*

ii) $L(\,\cdot\,,0) \in C^r(M)$, *with $r \geq 2d-2$.*

Then, the quotient Aubry set $(\bar{\mathcal{A}}_{c_L}, \bar{\delta}_{c_L})$ is totally disconnected.

It is worthy mentioning that a similar result has been also proven independently by Albert Fathi, Alessio Figalli and Ludovic Rifford ([33], in preparation). Observe that this result clearly applies to mechanical Lagrangians and, more generally, to *symmetric* (or *reversible*) Lagrangians (*i.e.,* $L(x,v) = L(x,-v)$ for all $(x,v) \in \mathrm{T}M$).

Corollary 2.3.4. *Let M be a compact connected manifold of dimension $d \geq 1$ and let $L(x,v)$ be a symmetrical Tonelli Lagrangian on $\mathrm{T}M$, such that $L(x,0) \in C^r(M)$, with $r \geq 2d-2$. Then, the quotient Aubry set $(\bar{\mathcal{A}}_0,\, \bar{\delta}_0)$ is totally disconnected.*

More specifically,

Corollary 2.3.5. *Let M be a compact connected manifold of dimension $d \geq 1$ and let $L(x,v) = \frac{1}{2}\|v\|_x^2 + U(x)$ be a mechanical Lagrangian on $\mathrm{T}M$, such that the potential $U \in C^r(M)$, with $r \geq 2d-2$. Then, the quotient Aubry set $(\bar{\mathcal{A}}_0,\, \bar{\delta}_0)$ is totally disconnected.*

In these cases, in fact, Λ_L is the zero section of T^*M and $c_L = 0$. Moreover, because of Mather's counterexamples [64], the regularity condition is optimal (one can easily modify the proof of the theorem to include also the case $L(\,\cdot\,,0) \in C^{2d-3,1}(M)$).

Observe that this result also applies to Mañé's Lagrangians (see section 1.1 for a definition) associated to *irrotational* vector fields.

Corollary 2.3.7. *Let M be a compact connected manifold of dimension $d \geq 1$, equipped with a C^∞ Riemannian metric g. Let X be a C^r vector field on M and consider the associated Mañé's Lagragian $L_X(x,v) = \frac{1}{2}\|v - X(x)\|_x^2$. If $r \geq 2d-2$ then the quotient Aubry set $(\bar{\mathcal{A}}_{c_{L_X}},\, \bar{\delta}_{c_{L_X}})$ is totally disconnected, where $X^\flat = g(X,\cdot)$ is the associated 1-form and $c_{L_X} = [X^\flat] \in \mathrm{H}^1(M;\mathbb{R})$.*

The main idea behind the proof of theorem 2.3.1 is that there exists a *liaison* between this problem and Sard's Lemma and this relation becomes more clear if one considers the Weak KAM theory setting (see section 1.4). Such an approach is based on the concept of *critical subsolutions* and *weak solutions* of Hamilton-Jacobi equation and consider the Hamiltonian setting, rather than the Lagrangian one. From a symplectic geometric point of view, it can be equivalently interpreted as the study of particular Lagrangian graphs and their non-removable intersections (see [69] and chapter 3).

Critical subsolutions (see definition 1.4.6) are a particular class of subsolutions that correspond to what is called the *Mañé's critical energy level*. They carry important information about the dynamics of the system and allow one to recover most of the results previously obtained via Mather and Mañé's approaches. It is a non-trivial fact that such Lipschitz subsolutions exist; moreover, Albert Fathi and Antonio Siconolfi [37] showed that C^1 critical subsolutions do also exist and are "dense" (see theorem 1.4.24). Patrick Bernard [13] extended this result to $C^{1,1}$ critical subsolutions.

The relation with Sard's Lemma that we were mentioning above can be easily expressed in terms of a Sard-like property for differences of these critical subsolutions. Let us denote this set by $\mathcal{D}_c^{1,1}$ the set of differences of $C^{1,1}$ c-critical subsolutions (see (2.1)). In section 2.2 we showed:

Proposition 2.2.6. *Let M be a compact connected manifold of dimension $d \geq 1$, L a Tonelli Lagrangian and $c \in H^1(M;\mathbb{R})$. If each $w \in \mathcal{D}_c^{1,1}$ is of Morse-Sard type, then the quotient Aubry set $(\bar{\mathcal{A}}_c, \bar{\delta}_c)$ is totally disconnected.*

Observe that this lemma immediately implies Mather's result in dimension $d \leq 2$ (for the autonomous case). In fact, in dimension $d \leq 2$ Sard's lemma holds for $C^{1,1}$ functions [10].

As a result of this, the original problem can be rephrased as follows: under which conditions on L and c, are these differences of subsolutions (or a "dense" subset of it) of Sard type? Unfortunately, this is not so easy to tackle either. It is not possible to use directly the classical Sard's lemma (or one of the available versions), due to a lack of regularity of these critical subsolutions, since in general they will be at most $C^{1,1}$. In fact, although it is always possible to smooth them out of the Aubry set and obtain functions in $C^\infty(M \setminus \mathcal{A}_c) \cap C^{1,1}(M)$, the presence of the Aubry set (where the value of their differential is prescribed) represents an obstacle that in some cases becomes impossible to overcome (see remark 1.4.25). On the other hand, one could try to control the *complexity* of these functions near their critical value set, using the rigid structure provided by Hamilton-Jacobi equation and the smoothness of the Hamiltonian itself, rather than the regularity of the subsolutions. While there are several difficulties in pursuing this approach in the general case - mostly related to the nature of the Aubry set - under the hypotheses of theorem 2.3.1 we managed to prove the following result, which can be seen a version of Sard's lemma for critical subsolutions of certain Hamilton-Jacobi equations:

Proposition 2.3.10. *Under the hypotheses of the theorem 2.3.1, if u is a η_L-critical subsolution, then $|u(\mathcal{A}_{c_L})| = 0$. In particular, for any $w \in \mathcal{D}_{c_L}$ we have that $|w(\mathcal{A}_{c_L})| = 0$.*

Chapter 3: Dynamics and Symplectic geometry

In this chapter we want to highlight some relations between the dynamics of the system and the underlying symplectic geometry of the phase space. We start in section 3.2, by discussing purely symplectic definitions of the Aubry set and the quotient Aubry set, which follow quite easily from weak KAM theory approach. In fact, one can reinterpret the notion of critical subsolution from a more geometric perspective, and introduce what we called *c-subcritical Lagrangian graphs* (see definition 3.2.1). It will follow immediately from weak KAM theory (in particular theorem 1.4.19), that for any given cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$ the Aubry set $\mathcal{A}^*_c$ can be obtained as the intersection of all these Lagrangian graphs (see also [69]). In a similar way we will get a symplectic characterization for Mather's pseudodistance δ_c, the quotient

Aubry set $\bar{\mathcal{A}}_c$ and the Mañé critical energy level $\mathcal{E}_c^*$. Using these characterizations, one can deduce the following result concerning the invariance of these sets under exact symplectomorphisms. Let us first recall that a diffeomorphism $\Psi : \mathrm{T}^*M \longrightarrow \mathrm{T}^*M$ is a *symplectomorphism* if it preserves the symplectic form ω, *i.e.,* $\Psi_*\omega = \omega$; in particular this is equivalent to say that $\Psi_*\lambda - \lambda$ is a closed 1-form (where λ denotes the *Liouville form* and $\omega = -d\lambda$; see section 3.1 and the addendum to section 1.1). We will say that Ψ is an *exact symplectomorphism* if $\Psi_*\lambda - \lambda$ is exact.

Theorem 3.2.3. *Let H be an optical Hamiltonian on T^*M and let $\Psi : \mathrm{T}^*M \longrightarrow \mathrm{T}^*M$ be an exact symplectomorphism. Consider the new Hamiltonian $H' = H \circ \Psi^{-1}$. Then for all $c \in \mathrm{H}^1(M;\mathbb{R})$:* [1]

- $\mathcal{E}_c^{*\prime} = \Psi(\mathcal{E}_c^*)$ *and therefore* $\alpha'(c) = \alpha(c)$;
- $\mathcal{A}_c^{*\prime} = \Psi(\mathcal{A}_c^*)$;
- $\delta_c(x,y) = \delta_c'(\Psi(x), \Psi(y))$; *therefore it maps c-static classes of H' into c-static classes of H and the induced map $\overline{\Psi} : \bar{\mathcal{A}}_c \rightarrow \bar{\mathcal{A}}_c'$ is an isometry.*

A similar result has been also proven, with a different approach, by Patrick Bernard in [15].

In section 3.3 we will analyze the minimizing properties of invariant measures supported on Lagrangian graphs and use them to deduce some uniqueness results for invariant Lagrangian graphs within a fixed homology or cohomology class. These results are based on a joint work [34] with Albert Fathi and Alessandro Giuliani, originally motivated by a question on the global uniqueness of KAM tori with a given rotation vector.

We start with the following characterization of minimizing measures (or *Mather's measures*, see section 1.2):

[1] We will indicate with a *prime* all quantities associated to H' (*e.g.,* $\mathcal{E}_c^{*\prime}, \mathcal{A}', \delta_c'$, etc...)

Lemma 3.3.1. *Let μ be an invariant probability measure on $\mathrm{T}M$ and $\mu^* = \mathcal{L}_*\mu$ its push-forward to T^*M, via the Legendre transform $\mathcal{L}$. Then, μ is a Mather's measure if and only if $\operatorname{supp}\mu^*$ is contained in the critical part of a subcritical Lagrangian graph. In particular, any invariant probability measure μ^* on T^*M, whose support is contained in an invariant Lagrangian graph with Liouville class c, is the image of a c-action minimizing measure, via the Legendre transform.*

Note that the fact that the orbits on an invariant Lagrangian graph are action-minimizing can be also deduced from a classical result of calculus of variations due to Weierstrass, as already pointed out by Jürgen Moser (see remark in [57]). In fact, Weierstrass method or the use of the Hamilton-Jacobi Equation (that we are using in our proof) are essentially two sides of the same coin.

This result easily implies an already-known uniqueness result for Lagrangian graphs supporting invariant measures of full support, in a fixed cohomology class (see also [57], in which a different proof is presented).

Theorem 3.3.3. *If $\Lambda \subset \mathrm{T}^*M$ is a Lagrangian graph on which the Hamiltonian dynamics admits an invariant measure μ^* with full support, then $\Lambda = \mathcal{L}\big(\widetilde{\mathcal{M}}_c\big) = \mathcal{A}_c^*$, where c is the cohomology class of Λ. Therefore, if Λ_1 and Λ_2 are two Lagrangian graphs as above, with the same cohomology class, then $\Lambda_1 = \Lambda_2$. In other words, for any given $c \in \mathrm{H}^1(M;\mathbb{R})$, there exists at most one invariant Lagrangian graph Λ with cohomology class c, that carries an invariant measure whose support is the whole of Λ.*

Moreover,

Theorem 3.3.4. *If Λ and μ are as in Theorem 3.3.3 and ρ is the rotation vector of $\mu = \mathcal{L}^{-1}\mu^*$, then $\Lambda = \mathcal{L}\big(\widetilde{\mathcal{M}}^\rho\big)$. Therefore, if Λ_1 and Λ_2 are two Lagrangian graphs supporting measures of full support and the same rotation vector ρ, then $\Lambda_1 = \Lambda_2$. Moreover, Mather's β-function is differentiable at ρ with $\partial\beta(\rho) = c$, where c is the* *cohomology class of* Λ.

Using the notion of *rotation vector* or *Schwartzman asymptotic cycle* of an invariant measure, which describes how asymptotically an average orbit winds around $\mathrm{T}M$ (see sections 1.2, 3.5 and [57, 74]), we introduce the notion of *Schwartzman ergodicity*. We shall say that a Lagrangian graph Λ is *Schwartzman uniquely ergodic* if all invariant measures supported on Λ have the same rotation vector ρ, which will be called *homology class of* Λ (see definition 3.5.5). Moreover, if there exists an invariant measure with full support, Λ will be called *Schwarztman strictly ergodic* (see definition 3.5.9). We will give a detailed description of these flows and their properties in section 3.5.

We will prove the following uniqueness result.

Theorem 3.3.6. *Let* Λ *be a Schwartzman strictly ergodic invariant Lagrangian graph with homology class* ρ. *The following properties are satisfied:*

(i) *if* $\Lambda \cap \mathcal{A}^*_c \neq \emptyset$, *then* $\Lambda = \mathcal{A}^*_c$ *and* $c = c_\Lambda$, *where* c_Λ *is the cohomology class of* Λ.

(ii) *the Mather function* α *is differentiable at* c_Λ *and* $\partial\alpha(c_\Lambda) = \rho$.

Therefore,

(iii) *any invariant Lagrangian graph that carries a measure with rotation vector* ρ *is equal to the graph* Λ*;*

(iv) *any invariant Lagrangian graph is either disjoint from* Λ *or equal to* Λ.

If $M = \mathbb{T}^d$, it is natural to ask what this implies in the setting of KAM theory. In section 3.4 we will use these results to discuss the problem of uniqueness of KAM tori (see definition 3.4.1) and, more generally, of the invariant tori belonging to the closure of the set of KAM tori (that we will call *Herman's tori*, [43]). In fact, while the proof of KAM theorem is constructive and the invariant torus one finds is locally unique (see for instance [72, 73]), the issue of global uniqueness of such tori is still an object of some debate and study (see, for instance, [17]).

Corollary 3.4.6. *Every optical Hamiltonian H on $\mathrm{T}^*\mathbb{T}^d$ possesses at most one Lagrangian KAM torus for any given rotation vector ρ.*

The property of being Lagrangian plays a crucial role. When ρ is *rationally independent* (*i.e.,* $\langle \rho, \nu \rangle \neq 0$, $\forall \nu \in \mathbb{Z}^d \setminus \{0\}$), every KAM torus with frequency ρ is automatically Lagrangian (this is a remark due to Michael Herman; see proposition 3.1.7).

We also extend this uniqueness result to generic invariant tori contained in the C^0-closure $\overline{\Upsilon}$ of the set Υ of all Lagrangian KAM tori. These "new" tori were first studied by Michael Herman [43], who showed that generically they are not conjugated to rotations. Moreover, they represent the majority, in the sense of topology, and hence most invariant tori cannot be obtained by the KAM algorithm. More precisely, Herman showed that in $\overline{\Upsilon}$ there exists a dense G_δ set (*i.e.,* a dense countable intersection of open sets) of invariant Lagrangian graphs on which the dynamics is strictly ergodic and weakly mixing, and for which the rotation vector is not Diophantine. These invariant graphs are therefore not obtained by the KAM theorem, however our uniqueness result do still apply to these graphs since strict ergodicity implies Schwartzman strict ergodicity.

More generally, given any Tonelli Lagrangian on $\mathbb{T}^d$, we consider the set $\tilde{\Upsilon}$ of invariant Lagrangian graphs on which the dynamics of the flow is topologically conjugated to an *ergodic* linear flow on $\mathbb{T}^d$. We show the following proposition.

Proposition 3.4.7. *There exists a dense G_δ set $\mathcal{G}$ in the C^0 closure of $\tilde{\Upsilon}$ consisting of strictly ergodic invariant Lagrangian graphs. Any $\Lambda \in \mathcal{G}$ satisfies the following properties:*

(i) *the invariant graph Λ has a well-defined rotation vector $\rho(\Lambda)$.*

(ii) *Any invariant Lagrangian graph that intersects* Λ *coincides with* Λ.

(iii) *Any Lagrangian invariant graph that carries an invariant measure whose rotation is* $\rho(\Lambda)$ *coincides with* Λ.

Chapter 1

Mather-Fathi theory for Lagrangian systems

1.1 Tonelli Lagrangians and optical Hamiltonians on compact manifolds

In this section we want to introduce the basic setting that we will be considering hereafter. Let M be a compact and connected smooth manifold without boundary. Denote by $\mathrm{T}M$ its tangent bundle and T^*M the cotangent one. A point of $\mathrm{T}M$ will be denoted by (x, v), where $x \in M$ and $v \in \mathrm{T}_xM$, and a point of T^*M by (x, p), where $p \in \mathrm{T}_x^*M$ is a linear form on the vector space T_xM. Let us fix a Riemannian metric g on it and denote by d the induced metric on M; let $\|\cdot\|_x$ be the norm induced by g on T_xM; we will use the same notation for the norm induced on T_x^*M.

We will consider functions $L : \mathrm{T}M \longrightarrow \mathbb{R}$ of class C^2, which are called *Lagrangians*. Associated to each Lagrangian, there is a flow on $\mathrm{T}M$ called the *Euler-Lagrange flow*, defined as follows. Let us consider the action functional A_L from the space of continuous piecewise C^1 curves $\gamma : [a,b] \to M$, with $a \le b$, defined by:

$$A_L(\gamma) := \int_a^b L(\gamma(t), \dot\gamma(t))\, dt.$$

Curves that extremize this functional among all curves with the same end-points are solutions of the *Euler-Lagrange equation*:

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t)) = \frac{\partial L}{\partial x}(\gamma(t), \dot\gamma(t)) \qquad \forall\, t \in [a,b]\,. \tag{1.1}$$

Observe that this equation is equivalent to

$$\frac{\partial^2 L}{\partial v^2}(\gamma(t), \dot\gamma(t))\ddot\gamma(t) = \frac{\partial L}{\partial x}(\gamma(t), \dot\gamma(t)) - \frac{\partial^2 L}{\partial v \partial x}(\gamma(t), \dot\gamma(t))\dot\gamma(t)\,,$$

therefore, if the second partial vertial derivative $\partial^2 L/\partial v^2(x,v)$ is non-degenerate at all points of $\mathrm{T}M$, we can solve for $\ddot\gamma(t)$. This condition

$$\det \frac{\partial^2 L}{\partial v^2} \neq 0$$

is called *Legendre condition* and allows one to define a vector field X_L on $\mathrm{T}M$, such that the solutions of $\ddot\gamma(t) = X_L(\gamma(t), \dot\gamma(t))$ are precisely the curves satisfying the Euler-Lagrange equation. This vector field X_L is called the *Euler-Lagrange vector field* and its flow Φ_t^L is the *Euler-Lagrange flow* associated to L. It turns out that Φ_t^L is C^1 even if L is only C^2.

Definition 1.1.1 (Tonelli Lagrangian). *A function $L : \mathrm{T}M \longrightarrow \mathbb{R}$ is called a Tonelli Lagrangian if:*

i) $L \in C^2(\mathrm{T}M)$*;*

ii) *L is strictly convex in the fibers, in the C^2 sense, i.e., the second partial vertical derivative $\partial^2 L/\partial v^2(x,v)$ is positive definite, as a quadratic form, for all (x,v);*

iii) L is superlinear in each fiber, i.e.,

$$\lim_{\|v\|_x \to +\infty} \frac{L(x,v)}{\|v\|_x} = +\infty;$$

this condition is equivalent to ask that for each $A \in \mathbb{R}$ there exists $B(A) \in \mathbb{R}$ such that

$$L(x,v) \geq A\|v\| - B(A) \qquad \forall\,(x,v) \in \mathrm{T}M\,.$$

Observe that since all Riemannian metrics on a compact manifold are equivalent, then condition *iii)* is independent of the choice of the Riemannian metric g.

Remark 1.1.2. More generally, one can consider the case of a *time-periodic Tonelli Lagrangian* $L:\ \mathrm{T}M \times \mathbb{T} \longrightarrow \mathbb{R}$ (also called *non-autonomous case*), as it was originally done by John Mather [57]. As it was pointed out to him by Jürgen Moser, this was the right setting to generalize Aubry and Mather's results for twist maps to higher dimensions; in fact, every twist map can be seen as the time one map associated to the flow of a periodic Tonelli Lagrangian on the one dimensional torus (see for instance [68]). In this case, a further condition on the Lagrangian is needed:

iv) The Euler-Lagrange flow is complete, i.e., every maximal integral curve of the vector field X_L has all $\mathbb{R}$ as its domain of definition.

In the non-autonomous case, in fact, this condition is necessary in order to have that action-minimizing curves (or *Tonelli minimizers*, see section 1.3) satisfy the Euler-Lagrange equation. Without such an assumption Ball and Mizel [7] have constructed example of Tonelli minimizers that are not C^1 and therefore are not solutions of the Euler-Lagrange flow. The role of the completeness hypothesis can be explained as follows. It is possible to prove, under the above conditions, that action minimizing curves not only exist and are absolutely continuous, but they are C^1 on an open and dense full measure subset of the interval in which they are defined; it is easy to check that they satisfy the Euler-Lagrange equation on this set, while their velocity goes to infinity on the exceptional set on which they are not C^1. Asking the flow to be complete, therefore, implies that Tonelli minimizers are C^1 everywhere and are actual solutions of the Euler-Lagrange equation. A sufficient condition for the completeness of the Euler-Lagrange flow, for example, can be expressed in terms of a growth condition for $\partial L/\partial t$:

$$-\frac{\partial L}{\partial t}(x,v,t) \leq C\left(1+\frac{\partial L}{\partial v}(x,v,t)\cdot v - L(x,v,t)\right) \quad \forall\,(x,v,t)\in \mathrm{T}M\times\mathbb{T}.$$

Examples of Tonelli Lagrangians.

- **Riemannian Lagrangians.** Given a Riemannian metric g on $\mathrm{T}M$, the *Riemannian Lagrangian* on (M,g) is given by the *Kinetic energy*:

  $$L(x,v)=\frac{1}{2}\|v\|_x^2\,.$$

  Its Euler-Lagrange equation is the equation of the geodesics of g:

  $$\frac{D}{dt}\dot{x}\equiv 0\,,$$

  and its Euler-Lagrange flow coincides with the geodesic flow.

- **Mechanical Lagrangians.** These Lagrangians play a key-role in the study of classical mechanics. They are given by the sum of the kinetic energy and a *potential* $U: M \longrightarrow \mathbb{R}$:

  $$L(x,v)=\frac{1}{2}\|v\|_x^2+U(x)\,.$$

The associated Euler-Lagrange equation is given by:

$$\frac{D}{dt}\dot{x} = \nabla U(x)\,,$$

where ∇U is the gradient of U with respect to the riemannian metric g, *i.e.,*

$$d_x U \cdot v = \langle \nabla U(x), v\rangle_x \quad \forall\,(x,v) \in \mathrm{T}M\,.$$

- **Mañé's Lagrangians.** This is a particular class of Tonelli Lagrangians, introduced by Ricardo Mañé in [46] (see also [33]). If X is a C^k vector field on M, with $k \geq 2$, one can embed its flow φ_t^X into the Euler-Lagrange flow associated to a certain Lagrangian, namely

$$L_X(x,v) = \frac{1}{2}\,\|v - X(x)\|_x^2\,.$$

It is quite easy to check that the integral curves of the vector field X are solutions to the Euler-Lagrange equation. In particular, the Euler-Lagrange flow $\Phi_t^{L_X}$ restricted to $\mathrm{Graph}(X) = \{(x, X(x)),\ x \in M\}$ (that is clearly invariant) is conjugated to the flow of X on M and the conjugation is given by $\pi|\mathrm{Graph}(X)$, where $\pi : \mathrm{T}M \longrightarrow M$ is the canonical projection. In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Graph}(X) & \xrightarrow{\Phi_t^{L_X}} & \mathrm{Graph}(X) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\varphi_t^X]{} & M \end{array}$$

that is, for every $x \in M$ and every $t \in \mathbb{R}$, $\Phi_t^{L_X}(x, X(x)) = (\gamma_x^X(t), \dot{\gamma}_x^X(t))$, where $\gamma_x^X(t) = \varphi_t^X(x)$.

In the study of classical dynamics, it turns often very useful to consider the associated *Hamiltonian system*, which is defined on the cotangent space T^*M. Let us describe how to define this new system and what is its relation with the Lagrangian one.

A standard tool in the study of convex functions is the so-called *Fenchel transform*, which allows one to transform functions on a vector space into functions on the dual space (see for instance [32, 71] for excellent introductions to the topics). Given a Lagrangian L, we can define the associated *Hamiltonian*, as its Fenchel transform (or *Fenchel-Legendre transfrom*):

$$\begin{array}{rcl} H:\ \mathrm{T}^*M & \longrightarrow & \mathbb{R} \\ (x,p) & \longmapsto & \sup_{v\in \mathrm{T}_xM} \{\langle p,\, v\rangle_x - L(x,v)\} \end{array}$$

where $\langle\,\cdot,\,\cdot\,\rangle_x$ denotes the canonical pairing between the tangent and cotangent space.

If L is a Tonelli Lagrangian, one can easily prove that H is finite everywhere (as a consequence of the superlinearity of L), C^2, superlinear and strictly convex in each fiber (in the C^2 sense). Such a Hamiltonian is called *optical* (or *Tonelli*) *Hamiltonian.*

Definition 1.1.3 (Optical Hamiltonian). *A function $H:\ \mathrm{T}^*M \longrightarrow \mathbb{R}$ is called an optical (or Tonelli) Hamiltonian if:*

i) H is of class C^2;

ii) H is strictly convex in each fiber in the C^2 sense, i.e., the second partial vertical derivative $\partial^2 H/\partial p^2(x,p)$ is positive definite, as a quadratic form, for any $(x,p)\in \mathrm{T}^*M$;

iii) H is superlinear in each fiber, i.e.,

$$\lim_{\|p\|_x\to+\infty} \frac{H(x,p)}{\|p\|_x} = +\infty\,.$$

Examples of optical Hamiltonians.

Let us see what are the optical Hamiltonians associated to the Tonelli Lagrangians that we have introduced in the previous examples.

- **Riemannian Hamiltonians.** If $L(x,v) = \frac{1}{2}\|v\|_x^2$ is the Riemannian Lagrangian associated to a Riemannian metric g on M, the corresponding Hamiltonian will be

$$H(x,p) = \frac{1}{2}\|p\|_x^2,$$

  where $\|\cdot\|$ represents - in this last expression - the induced norm on the cotangent space T^*M.

- **Mechanical Hamiltonians.** If $L(x,v) = \frac{1}{2}\|v\|_x^2 + U(x)$ is a mechanical Lagrangian, the associated Hamiltonian is:

$$H(x,p) = \frac{1}{2}\|p\|_x^2 - U(x),$$

  that it is sometimes referred to as *mechanical energy*.

- **Mañé's Hamiltonians.** If X is a C^k vector field on M, with $k \geq 2$, and $L_X(x,v) = \|v - X(x)\|_x^2$ is the associated Mañé Lagrangian, one can check that the corresponding Hamiltonian is given by:

$$H(x,p) = \frac{1}{2}\|p\|_x^2 + \langle p, X(x)\rangle .$$

Given a Hamiltionian one can consider the associated *Hamiltonian flow* Φ_t^H on T^*M. In local coordinates, this flow can be expressed in terms of the so-called

Hamilton's equations:

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t)) \\ \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t)) \,. \end{cases} \tag{1.2}$$

It is easy to check that - only in the autonomous case - the Hamiltonian is a *prime integral of the motion*, *i.e.,* it is constant along the solutions of these equations.

Now, we would like to explain what is the relation between the Euler-Lagrange flow and the Hamiltonian one. It follows easily from the definition of Hamiltonian (and Fenchel transform) that for each $(x, v) \in \mathrm{T}M$ and $(x, p) \in \mathrm{T}^*M$ the following inequality holds:

$$\langle p,\, v\rangle_x \leq L(x, v) + H(x, p)\,; \tag{1.3}$$

this is called *Fenchel inequality* and plays a crucial role in the study of Lagrangian and Hamiltonian dynamics and, in particular, the variational methods that we are going to describe. In particular, equality holds if and only if $p = \partial L/\partial v(x, v)$. One can therefore introduce the following diffeomorphism between $\mathrm{T}M$ and T^*M:

$$\begin{aligned} \mathcal{L}:\ \mathrm{T}M &\longrightarrow \mathrm{T}^*M \\ (x, v) &\longmapsto \left(x,\, \frac{\partial L}{\partial v}(x, v)\right)\,. \end{aligned} \tag{1.4}$$

This is called the *Legendre transform* and the following relation with the Hamiltonian holds:

$$H \circ \mathcal{L}(x, v) = \left\langle \frac{\partial L}{\partial v}(x, v),\, v \right\rangle_x - L(x, v)\,.$$

A crucial observation is that this diffeomorphism represents a conjugation between the two flows, namely the Euler-Lagrange flow on $\mathrm{T}M$ and the Hamiltonian one on

T^*M; in other words, the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{\Phi_t^L} & TM \\ \mathcal{L}\big\downarrow & & \big\downarrow\mathcal{L} \\ T^*M & \xrightarrow[\Phi_t^H]{} & T^*M \end{array}$$

Therefore one can equivalently study the Euler-Lagrange flow or the Hamiltonian flow, obtaining in both cases information on the dynamics of the system. Each of these equivalent approaches will provide different tools and advantages, which may prove very useful to understand the dynamical properties of the system. For instance, the tangent space is the natural setting for the classical calculus of variations and for Mather and Mañé's approaches (sections 1.2 and 1.3); on the other hand, the cotangent space can be equipped with a canonical symplectic structure (see the addendum at the end of this section), which allows one to use many fine symplectic topological results, coming from the study of Lagrangian graphs, Hofer's theory, Floer Homology, *etc* ... Moreover, a particular fruitful approach in T^*M is the so-called *Hamilton-Jacobi method* (or *Weak KAM theory*), which concerns with the study of *solutions* and *subsolutions* of Hamilton-Jacobi equations and, in a certain sense, represents the functional analytical counterpart of the above-mentioned variational approach (section 1.4). In the following sections we will provide a complete description of these methods and their implications to the study of the dynamics of the system.

ADDENDUM

Symplectic structure of the cotangent space and Hamiltonian flows

There is a more geometric and intrinsic way to define the Hamiltonian flow. Remember that T^*M, as a cotangent bundle, may be equipped with a *canonical* symplectic structure. Namely, if $(\mathcal{U},\, x_1, \dots, x_d)$ is a local coordinate chart for M and $(\mathrm{T}^*\mathcal{U},\, x_1, \dots, x_d, p_1, \dots, p_d)$ the associated cotangent coordinates, one can define the 2-form

$$\omega = \sum_{i=1}^{d} dx_i \wedge dp_i \,.$$

It is easy to show that ω is a symplectic form (*i.e.,* it is non-degenerate and closed). In particular, one can check that ω is intrinsically defined (*i.e.,* it does not depend on the choice of coordinate charts), by considering the 1-form on $\mathrm{T}^*\mathcal{U}$

$$\lambda = \sum_{i=1}^{d} p_i \, dx_i \,,$$

which satisfies $\omega = -d\lambda$ and is coordinate-independent; in fact, in terms of the natural projection

$$\begin{aligned} \pi : \mathrm{T}^*M \quad &\longrightarrow \quad M \\ (x,p) \quad &\longmapsto \quad x \end{aligned}$$

the form λ may be equivalently defined pointwise by

$$\lambda_{(x,p)} = (d\pi_{(x,p)})^* p \ \in \mathrm{T}^*_{(x,p)}\mathrm{T}^*M \,.$$

The 1-form λ is called the *Liouville form* (or the *tautological form*).

Since ω is non-degenerate and closed, the following relation determines a unique vector field X_H on T^*M:

$$\omega\left(X_H(x,p), \cdot\right) = d_x H(\cdot)\,.$$

This vector field is the *Hamiltonian vector field* and one can easily check that, in local coordinates, it coincides with the one given by Hamilton's equation (1.2).

We will discuss more in depth the relation between the dynamics of the system and the symplectic properties of the underlying space in chapter 3.

1.2 Action-minimizing measures: Mather sets

This and the following sections are meant to provide a comprehensive introduction to Mather's theory for Lagrangian systems and Fathi's weak KAM theory. We will recall most of the results that we are going to use, trying to give - unless where it is needed - general ideas rather than rigourous proofs (for which we refer to [32, 57] and references therein).

Remark 1.2.1. Before entering into the details of Mather's theory, let us make a crucial remark, which is at the base of this approach. Observe that if η is a 1-form on M, we can see it as a function on the tangent space (linear on each fiber)

$$\begin{aligned} \hat{\eta} : \mathrm{T}M &\longrightarrow \mathbb{R} \\ (x,v) &\longmapsto \langle \eta(x),\, v\rangle_x \end{aligned}$$

and consider a new Tonelli Lagrangian $L_\eta := L - \hat{\eta}$; the associated Hamiltonian will be $H_\eta(x,p) = H(x, \eta(x) + p)$. Moreover, if η is closed, then $\int L\,dt$ and $\int L_\eta dt$ will have the same extremals and therefore the Euler-Lagrange flows on $\mathrm{T}M$ associated to L and L_η will be the same. This last conclusion may be easily deduced observing that, since η is closed, the variational equations $\delta\big[\int L\,dt\big] = 0$ and $\delta\big[\int (L - \hat{\eta})\,dt\big] = 0$ for the fixed end-point problem have the same solutions. Although the extremals are the same, this is not generally true for the orbits "minimizing the action" (we will give a precise definition of "minimizers" later in section 1.3). What one can say is that they stay the same when we change the Lagrangian by an exact 1-form. Thus, for a fixed L, the minimizers will depend only on the de Rham cohomology class $c = [\eta] \in \mathrm{H}^1(M;\mathbb{R})$. Therefore, instead of studying the action minimizing properties of a single Lagrangian, one can consider a family of such “modified” Lagrangians, parameterized over $H^1(M;\mathbb{R})$. This idea represents the keystone of the approach that we are going to describe.

In order to generalize to more degrees of freedom Aubry and Mather's variational approach to twist maps, a first important notion is that of *minimal measure*, which replaces that of action minimizing orbit. Aubry-Mather theory in higher dimension, in fact, cannot deal with such orbits, due to a lack of them: a classical example due to Hedlund [40] shows the existence of a Riemannian metric on a three-dimensional torus, for which minimal geodesics exist only in three directions. Instead, Mather proposed to look at the closely related notion of action minimizing invariant probability measures. Let us try to describe this idea. Let $\mathfrak{M}(L)$ be the space of probability measures on $\mathrm{T}M$ that are invariant under the Euler-Lagrange flow of L. To each $\mu \in \mathfrak{M}(L)$, we may associate its *average action*

$$A_L(\mu) = \int_{\mathrm{T}M} L \, d\mu \, .$$

Since L is bounded below (because of the superlinear growth condition), this integral exists although it might be $+\infty$. In [57] Mather showed the existence of $\mu \in \mathfrak{M}(L)$ such that $A_L(\mu) < +\infty$. The argument is mainly the same as Krylov-Bogoliubov's theorem concerning existence of invariant measures for flows on compact spaces. This argument is applied to a one-point compactification of $\mathrm{T}M$, and the main step consists in showing that the measure provided by this construction has no atomic part supported at ∞ (which is a fixed point for the extended system).

Remark 1.2.2. Note that Mather's approach works also for periodic time-dependent Lagrangians. For time independent Lagrangians, finding such a μ is much easier. By conservation of energy, the levels of the energy function are compact and invariant under the Euler-Lagrange flow Φ_t^L, and therefore carry such measures.

In case $A_L(\mu) < \infty$, thanks to the superlinearity of L, the integral $\int_{\mathrm{T}M} \hat{\eta} d\mu$ is well defined and finite for any closed 1-form η on M (see [57]). Moreover, it is quite easy to show (again see [57]) that since μ is invariant by the Euler-Lagrangian flow Φ_t^L, if $\eta = df$ is an exact 1-form, then $\int \widehat{df} d\mu = 0$. Therefore, we can define a linear functional

$$\begin{array}{rcl} \mathrm{H}^1(M;\mathbb{R}) & \longrightarrow & \mathbb{R} \\ c & \longmapsto & \displaystyle\int_{\mathrm{T}M} \hat{\eta} d\mu \,, \end{array}$$

where η is any closed 1-form on M with cohomology class c. By duality, there exists $\rho(\mu) \in \mathrm{H}_1(M;\mathbb{R})$ such that

$$\int_{\mathrm{T}M} \hat{\eta}\, d\mu = \langle c, \rho(\mu) \rangle \qquad \forall\, c \in \mathrm{H}^1(M;\mathbb{R})$$

(the bracket on the right–hand side denotes the canonical pairing between cohomology and homology). We call $\rho(\mu)$ the *rotation vector* of μ. It is the same as the Schwartzman's asymptotic cycle of μ (see section 3.5 and [74] for more details). A natural question is whether there exist invariant probability measures for any given rotation vector. The answer turns out to be affirmative. In fact, using that the action functional $A_L : \mathfrak{M}(L) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, one can prove the following:

Proposition 1.2.3. [57]. *For every $h \in \mathrm{H}_1(M;\mathbb{R})$ there exists $\mu \in \mathfrak{M}(L)$ with $A_L(\mu) < \infty$ and $\rho(\mu) = h$. In other words, the map $\rho : \mathfrak{M}(L) \longrightarrow \mathrm{H}_1(M;\mathbb{R})$ is surjective.*

Amongst the all probability measures with a prescribed rotation vector, a peculiar role - from a dynamical systems point of view - will be played by those minimizing the average action. Following Mather, let us consider the minimal value of the average action A_L over the probability measures with rotation vector h:

$$\begin{aligned} \beta : \mathrm{H}_1(M;\mathbb{R}) &\longrightarrow \mathbb{R} \\ h &\longmapsto \min_{\mu\in\mathfrak{M}(L):\,\rho(\mu)=h} A_L(\mu)\,. \end{aligned} \tag{1.5}$$

Observe that this minimum is actually achieved (see [57]). This function β is what is generally known as *Mather's β-function* and it is also related to the notion of *stable norm* for a metric d (see for instance [49]).

We can now define what we mean by action minimizing measure with a given rotation vector.

Definition 1.2.4. *A measure $\mu \in \mathfrak{M}(L)$ realizing the minimum in (1.5), i.e., such that $A_L(\mu) = \beta(\rho(\mu))$, is called an action minimizing (or minimal or Mather's) measure with rotation vector $\rho(\mu)$.*

Dual to the concept of rotation vector, one can introduce the notion of *cohomology* of an action minimizing measure. Let us try to describe what we mean. Since the β-function is convex, one can consider its *conjugate* function (given by Fenchel's duality):

$$\begin{aligned} \alpha : \mathrm{H}^1(M;\mathbb{R}) &\longrightarrow \mathbb{R} \\ c &\longmapsto \max_{h\in\mathrm{H}_1(M;\mathbb{R})} \left(\langle c,h\rangle - \beta(h)\right). \end{aligned} \tag{1.6}$$

In particular,

$$\begin{aligned} \alpha(c) &:= \max_{h\in\mathrm{H}_1(M;\mathbb{R})} \left(\langle c,h\rangle - \beta(h)\right) = \\ &= -\min_{h\in\mathrm{H}_1(M;\mathbb{R})} \left(\beta(h) - \langle c,h\rangle\right) = \\ &= -\min_{\mu\in\mathfrak{M}(L)} \left(A_L(\mu) - \langle c,\rho(\mu)\rangle\right) = \\ &= -\min_{\mu\in\mathfrak{M}(L)} A_{L-c}(\mu). \end{aligned} \tag{1.7}$$

Note that for a given μ invariant under the Euler-Lagrange flow Φ_t^L, we have $\langle c, \rho(\mu)\rangle = \int_{\mathrm{T}M} \hat{\eta}\, d\mu$, for any closed 1-form η on M with cohomology class c; therefore, we have $A_{L-c}(\mu) = \int_{\mathrm{T}M}(L-\hat{\eta})\, d\mu$. It is interesting to remark that the value of this α - function coincides with what is called *Mañé's critical value*, which will be introduced later in sections 1.3 and 1.4. Analogously to what we have already done before, we want to single out the invariant measures that minimize this c-average action.

Definition 1.2.5. *If $\mu \in \mathfrak{M}(L)$ and μ minimizes A_{L-c}, i.e., $A_L(\mu) = -\alpha(c)$, we will say that μ is a c-action minimizing measure (or c-minimal measure, or Mather's measure with cohomology c).*

An important fact is next lemma, which will help to clearify the relation (and duality) between these two minimizing procedures. To state it, recall that, like any convex function on a finite-dimensional space, the Mather function β admits a subderivative at each point $h \in \mathrm{H}_1(M;\mathbb{R})$, *i.e.,* we can find $c \in \mathrm{H}^1(M;\mathbb{R})$ such that

$$\forall h' \in \mathrm{H}_1(M;\mathbb{R}), \quad \beta(h') - \beta(h) \geq \langle c, h' - h\rangle.$$

As it is usually done, we will denote by $\partial\beta(h)$ the set of $c \in \mathrm{H}^1(M;\mathbb{R})$ that are subderivatives of β at h, *i.e.,* the set of c which satisfy the inequality above. By Fenchel's duality, we have

$$c \in \partial\beta(h) \Longleftrightarrow \langle c, h\rangle = \alpha(c) + \beta(h).$$

Lemma 1.2.6. *If $\mu \in \mathfrak{M}(L)$, then $A_L(\mu) = \beta(\rho(\mu))$ if and only if there exists $c \in \mathrm{H}^1(M;\mathbb{R})$ such that μ minimizes A_{L-c} (i.e., $A_{L-c}(\mu) = -\alpha(c)$). Moreover, if $\mu \in \mathfrak{M}(L)$ satisfies $A_L(\mu) = \beta(\rho(\mu))$, and $c \in \mathrm{H}^1(M;\mathbb{R})$, then μ minimizes A_{L-c} if and only if $c \in \partial\beta(\rho(\mu))$ (or $\langle c, h\rangle = \alpha(c) + \beta(\rho(\mu))$).*

Proof. We will prove both statement at the same time. Assume $A_L(\mu_0) = \beta(\rho(\mu_0))$.

Let $c \in \partial\beta(\rho(\mu_0))$, by Fenchel's duality this is equivalent to

$$\begin{aligned}\alpha(c) &= \langle c, \rho(\mu_0)\rangle - \beta(\rho(\mu_0))\\ &= \langle c, \rho(\mu_0)\rangle - A_L(\mu_0)\\ &= -A_{L-c}(\mu_0).\end{aligned}$$

Therefore by (1.7): $A_{L-c}(\mu_0) = \min_{\mu\in\mathfrak{M}} A_{L-c}(\mu)$.

Assume conversely that $A_{L-c}(\mu_0) = \min_{\mu\in\mathfrak{M}} A_{L-c}(\mu)$, for some given cohomology class c. Therefore if follows from (1.7) that

$$\alpha(c) = -A_{L-c}(\mu_0),$$

which can be written as

$$\langle c, \rho(\mu_0)\rangle = \alpha(c) + A_L(\mu_0).$$

It now suffices to use the Fenchel inequality $\langle c, \rho(\mu_0)\rangle \leq \alpha(c) + \beta(\rho(\mu_0)$, and the inequality $\beta(\rho(\mu_0) \leq A_L(\mu_0)$, given by the definition of β, to obtain the equality

$$\langle c, \rho(\mu_0)\rangle = \alpha(c) + \beta(\rho(\mu_0)).$$

In particular, we have $A_L(\mu_0) = \beta(\rho(\mu_0))$. □

The above discussion leads to two equivalent formulations for the minimality of a measure μ:

- there exists a homology class $h \in \mathrm{H}_1(M;\mathbb{R})$, namely its rotation vector $\rho(\mu)$, such that μ minimizes A_L amongst all measures in $\mathfrak{M}(L)$ with rotation vector h; *i.e.*, $A_L(\mu) = \beta(h)$;
- there exists a cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$, namely any subderivative of β at $\rho(\mu)$ (*i.e.*, the slope of a supporting hyperplane of the epigraph of β at

$\rho(\mu)$), such that μ minimizes A_{L-c} amongst all probability measures in $\mathfrak{M}(L)$; *i.e.*, $A_{L-c}(\mu) = -\alpha(c)$.

For $h \in \mathrm{H}_1(M;\mathbb{R})$ and $c \in \mathrm{H}^1(M;\mathbb{R})$, let us define

$$\begin{aligned}
\mathfrak{M}^h &:= \mathfrak{M}^h(L) = \{\mu \in \mathfrak{M}(L) :\ A_L(\mu) < +\infty,\ \rho(\mu) = h \text{ and } A_L(\mu) = \beta(h)\} \\
\mathfrak{M}_c &:= \mathfrak{M}_c(L) = \{\mu \in \mathfrak{M}(L) :\ A_L(\mu) < +\infty \text{ and } A_{L-c}(\mu) = -\alpha(c)\}.
\end{aligned}$$

Observe that because of the superlinear growth condition in the fiber, $A_L(\mu) < +\infty$ implies $A_{L-c}(\mu) < +\infty$. Moreover, both procedures lead to the same sets of minimal measures:

$$\bigcup_{h\in \mathrm{H}_1(M;\mathbb{R})} \mathfrak{M}^h = \bigcup_{c\in \mathrm{H}^1(M;\mathbb{R})} \mathfrak{M}_c\,.$$

In fact, minimizing over a set of invariant measures with a fixed rotation vector or minimizing - globally - the modified Lagrangian (corresponding to a certain cohomology class) are dual problems, as the ones that often appears in linear programming and optimization.

We can now begin to discuss some dynamical implications of this approach. Let us start by defining a first important family of invariant sets: *Mather sets*. For a cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$, we call *Mather set of cohomology class* c the set:

$$\widetilde{\mathcal{M}}_c := \overline{\bigcup_{\mu\in\mathfrak{M}_c} \operatorname{supp}\mu} \subset \mathrm{T}M \tag{1.8}$$

while the projection on the base manifold $\mathcal{M}_c = \pi\left(\widetilde{\mathcal{M}}_c\right) \subseteq M$ is called *projected Mather set* (with cohomology class c). In [57] Mather proved the celebrated *graph theorem*:

Theorem 1.2.7. *Let $\widetilde{\mathcal{M}}_c$ be defined as in (1.8). The set $\widetilde{\mathcal{M}}_c$ is compact, invariant* *under the Euler-Lagrange flow and* $\pi|\widetilde{\mathcal{M}}_c$ *is an injective mapping of* $\widetilde{\mathcal{M}}_c$ *into* M*, and its inverse* $\pi^{-1} : \mathcal{M}_c \longrightarrow \widetilde{\mathcal{M}}_c$ *is Lipschitz. Moreover this set is contained in the energy level corresponding to the value* $\alpha(c)$*, i.e.,*

$$H \circ \mathcal{L}(x, v) = \alpha(c) \qquad \forall\, (x, v) \in \widetilde{\mathcal{M}}_c\,. \tag{1.9}$$

Remark 1.2.8. The last statement, corresponding to (1.9), is due to Dias-Carneiro [20] and holds only in the autonomous case.

Analogously, one can consider the *Mather set corresponding to a rotation vector* $h \in \mathrm{H}_1(M;\mathbb{R})$ as

$$\widetilde{\mathcal{M}}^h := \overline{\bigcup_{\mu \in \mathfrak{M}^h} \operatorname{supp} \mu} \subset \mathrm{T}M\,, \tag{1.10}$$

and the projected one $\mathcal{M}^h = \pi\left(\widetilde{\mathcal{M}}^h\right) \subseteq M$. Notice that by Lemma 1.2.6, if $c \in \partial\beta(h)$, we have

$$\widetilde{\mathcal{M}}^h \subseteq \widetilde{\mathcal{M}}_c.$$

Therefore, although this was not shown in [57], the set $\widetilde{\mathcal{M}}^h$ also has a Lipschitz graph over the base.

Theorem 1.2.9. *Let* $\widetilde{\mathcal{M}}^h$ *be defined as in (1.10).* $\widetilde{\mathcal{M}}^h$ *is compact, invariant under the Euler-Lagrange flow and* $\pi|\widetilde{\mathcal{M}}^h$ *is an injective mapping of* $\widetilde{\mathcal{M}}^h$ *into* M *and its inverse* $\pi^{-1} : \mathcal{M}^h \longrightarrow \widetilde{\mathcal{M}}^h$ *is Lipschitz.*

Remark 1.2.10. Though the graph property for $\widetilde{\mathcal{M}}^h$ is not proved in [57], it was shown there that the support of an action minimizing measure has the graph property. The graph property for $\widetilde{\mathcal{M}}^h$ can be also deduced from this last property. In fact, since the space of probability measures on $\mathrm{T}M$ is a separable metric space, one can take a countable dense set $\{\mu_n\}_{n=1}^{\infty}$ of Mather's measures with rotation vector h and consider the new measure $\tilde{\mu} = \sum_{n=1}^{\infty} \frac{1}{2^n}\mu_n$. This is still an invariant probability measure with rotation vector h and $\operatorname{supp}\tilde{\mu} = \widetilde{\mathcal{M}}_h$. Therefore, as the support of a single action minimizing measure, $\widetilde{\mathcal{M}}^h$ has the graph property.

Moreover, this remark points out that there always exist Mather's measures μ^h and μ_c of full support, *i.e.*, $\operatorname{supp}\mu^h = \widetilde{\mathcal{M}}^h$ and $\operatorname{supp}\mu_c = \widetilde{\mathcal{M}}_c$. We will say that an action minimizing measure μ with rotation vector h (resp. cohomology c) has maximal support if $\operatorname{supp}\mu = \widetilde{\mathcal{M}}^h$ (resp. $\operatorname{supp}\mu = \widetilde{\mathcal{M}}_c$).

ADDENDUM

Holonomic measures

Before concluding this section, we would like to stress that using the above approach the minimizing measures are obtained through a variational principle over the set of invariant probability measure. Because of the request of "invariance", this set clearly depends on the Lagrangian that one is considering.

An alternative approach, slightly different under this respect, was due to Ricardo Mañé [46] (see also [23]). This deals with the bigger set of *holonomic measures* and prove extremely advantageous when dealing with different Lagrangians at the same time. In this addendum we want to sketch the basic ideas behind it.

Let C^0_ℓ be the set of continuous functions $f : \mathrm{T}M \longrightarrow \mathbb{R}$ growing (fiberwise) at most linearly, *i.e.*,

$$\|f\|_\ell := \sup_{(x,v)\in \mathrm{T}M} \frac{f(x,v)}{1+\|v\|} < +\infty\,,$$

and let $\mathfrak{M}^\ell_M$ be the set of probability measures on the Borel σ-algebra of $\mathrm{T}M$ such that $\int_{\mathrm{T}M} \|v\|\, d\mu < \infty$, endowed with the unique metrizable topology given by:

$$\mu_n \longrightarrow \mu \qquad \Longleftrightarrow \qquad \int_{\mathrm{T}M} f(x,v)\, d\mu_n \longrightarrow \int_{\mathrm{T}M} f(x,v)\, d\mu \quad \forall\, f \in C^0_\ell\,.$$

Let $(C^0_\ell)^*$ be the dual of C^0_ℓ. Then $\mathfrak{M}^\ell_M$ can be naturally embedded in $(C^0_\ell)^*$ and its topology coincides with that induced by the weak* topology on $(C^0_\ell)^*$. One can show that this topology is metrizable and a metric is, for instance:

$$d(\mu_1,\mu_2) = \left|\int_{\mathrm{T}M} \|v\|\,d\mu_1 - \int_{\mathrm{T}M} \|v\|\,d\mu_2\right| + \sum_n \frac{1}{2^n c_n}\left|\int_{\mathrm{T}M} \varphi_n\,d\mu_1 - \int_{\mathrm{T}M} \varphi_n\,d\mu_2\right|,$$

where $\{\varphi_n\}_n$ is a sequence of functions with compact support on C^0_ℓ, which is dense on C^0_ℓ (in the topology of uniform convergence on compact subsets of $\mathrm{T}M$) and $c_n := \sup_{\mathrm{T}M}|\varphi_n(x,v)|$. The space of probability measures that we will be considering is a closed subset of $\mathfrak{M}^\ell_M$ (endowed with the induced topology), which is defined as follows. If $\gamma : [0,T] \to M$ is a closed absolutely continuous curve, let μ_γ be such that

$$\int_{\mathrm{T}M} f(x,v)\,d\mu_\gamma = \frac{1}{T}\int_0^T f(\gamma(t),\dot\gamma(t))\,dt \qquad \forall\, f \in C^0_\ell\,.$$

Observe that $\mu_\gamma \in \mathfrak{M}^\ell_M$ because if γ is absolutely continuous then $\int|\dot\gamma(t)|\,dt < +\infty$. Let $\mathcal{C}(M)$ be the set of such μ_γ's and $\overline{\mathcal{C}(M)}$ its closure in $\mathfrak{M}^\ell_M$. This set is convex and it is called the set of *holonomic measures* on M.

One can check that the following properties are satisfied:

i) $\mathfrak{M}(L) \subseteq \overline{\mathcal{C}(M)} \subseteq \mathfrak{M}^\ell_M$. In particular, for every Tonelli Lagrangian L on M, all probabilities measures μ that are invariant with respect to the Euler-Lagrange flow and such that $\int_{\mathrm{T}M} L\,d\mu < +\infty$, are contained in $\overline{\mathcal{C}(M)}$.

ii) To any given probability $\mu \in \mathcal{C}(M)$, one can associate a rotation vector $\rho(\mu) \in \mathrm{H}_1(M;\mathbb{R})$. This map extends continuously to a map

$$\rho : \overline{\mathcal{C}(M)} \longrightarrow \mathrm{H}_1(M;\mathbb{R})$$

and this extension is surjective.

iii) For each $C \in \mathbb{R}$ the set $\left\{\mu \in \overline{\mathcal{C}(M)} :\ A_L(\mu) \le C\right\}$ is compact.

iv) If a measure $\mu \in \overline{\mathcal{C}(M)}$ satisfies

$$A_L(\mu) = \min\left\{A_L(\nu) :\ \nu \in \overline{\mathcal{C}(M)}\right\},$$

then $\mu \in \mathfrak{M}(L)$ (and in particular it is invariant). Observe that the existence of probabilities attaining the minimum follows from iii).

In view of these properties, it is clear that the corresponding minimizing problem, although on a bigger space of measures, will lead to the same results as before and the same definition of Mather sets.

1.3 Global action minimizing curves: Aubry and Mañé sets

In addition to the Mather sets, one can construct other compact invariant sets, which are also particularly significant from both a dynamical system and a geometric point of view: the *Aubry sets* and the *Mañé sets*. Instead of considering action minimizing invariant probability measures, we now shift our attention to *c-action-minimizing curves*, also called *c-global minimizers*, for any given cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$.

In the light of remark 1.2.1, let us fix a cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$ and choose a smooth 1-form η on M that represents c. As we have already pointed out in Section 1.1, there is a close relation between solutions of the Euler-Lagrange flow and extremals of the action functional A_{L_η} for the fixed end-point problem (which are the same as the extremals of A_L). In general, these extremals are not minima (they are local minima only if the time length is very short). Nevertheless, such minima exist;

this is a classical result of the calculus of variations, known as *Tonelli Theorem*, that has been reproven - in the setting of Tonelli Lagrangians - by John Mather [57].

Theorem 1.3.1 (Tonelli Theorem, [57]). *Let M be a compact manifold and L a Tonelli Lagrangian on $\mathrm{T}M$. For all $a < b \in \mathbb{R}$ and $x, y \in M$, there exists - in the set of absolutely continuous curves $\gamma : [a, b] \longrightarrow M$ such that $\gamma(a) = x$ and $\gamma(b) = y$ - a curve that minimizes the action $A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt$.*

A curve minimizing $A_{L\eta}(\gamma) = \int_a^b L_\eta(\gamma(t), \dot{\gamma}(t))\, dt$ subject to the fixed end-point condition $\gamma(a) = x$ and $\gamma(b) = y$, is called a *c-Tonelli minimizer*. Recall that such minimizers do only depend on c and not on the chosen representative η (see remark 1.2.1). As Mañé pointed out in [48], for these minimizers to exist it is not necessary to assume the compactness of M: the superlinear growth condition with respect to some complete Riemannian metric on M is enough.

Remark 1.3.2. A Tonelli minimizer which is C^1 is C^r (if the Lagrangian L is C^r) and satisfies the Euler-Lagrange equation; this follows from the usual elementary arguments in the calculus of variations, together with Caratheodory's remark on differentiability. In the autonomous case, Tonelli minimizers will be always C^1. In the non-autonomous time-periodic case (Tonelli Theorem holds also in this case [57]), as already remarked in remark 1.1.2, one needs to require that the Euler-Lagrange flow is complete.

Our interest will be for particular Tonelli minimizers that are defined for all times and whose action is minimal with respect to any given time length. We will see that these curves present a very rich structure.

Definition 1.3.3 (c-minimizers). *An absolutely continuous curve $\gamma : \mathbb{R} \longrightarrow M$ is a c-(global) minimizer if for any given $a < b \in \mathbb{R}$*

$$A_{L_\eta}(\gamma\big|[a, b]) = \min A_{L_\eta}(\sigma)$$

where the miminimum is taken over all $\sigma : [a,b] \to M$ *such that* $\sigma(a) = \gamma(a)$ *and* $\sigma(b) = \gamma(b)$.

A first interesting property of such curves is the following [23].

Proposition 1.3.4. *Any c-minimizer is contained in the energy level*

$$\widetilde{E}_c = \{(x,v) \in \mathrm{T}M : \ H \circ \mathcal{L}(x,v) = \alpha(c)\},$$

where $\alpha : \mathrm{H}^1(M;\mathbb{R}) \to \mathbb{R}$ *is Mather's* α*-function defined in (1.6).*

This result can be also deduced from Carneiro result [20] (see also 1.9 in Theorem 1.2.7), observing that the energy is conserved along solutions and each c-minimizer contains c-action minimizing measures in its α and ω-limit sets.

Remark 1.3.5. It is a result by Albert Fathi [32], that these c-minimizers satisfy a stronger property: they are *time-free* minimizer for $L_\eta + \alpha(c)$, *i.e.,* for any $a < b \in \mathbb{R}$ they minimize the action of $L_\eta + \alpha(c)$ over the set of all curves with the same end-points, independently of their time lengths. Namely, if γ is a c-minimizer, then for any $\sigma : [a', b'] \to M$ such that $\sigma(a') = \gamma(a)$ and $\sigma(b') = \gamma(b)$ we have

$$\int_a^b L_\eta(\gamma(t), \dot\gamma(t)) + \alpha(c)\, dt \leq \int_{a'}^{b'} L_\eta(\sigma(t), \dot\sigma(t)) + \alpha(c)\, dt\,.$$

This is not true anymore in the non-autonomous case. See the addendum at the end of this section for more details.

We can now define another family of invariant sets.

Definition 1.3.6 (Mañé set). *The Mañé set (with cohomology class c) is:*

$$\widetilde{\mathcal{N}}_c = \bigcup \{(\gamma(t), \dot\gamma(t)) : \ \gamma \text{ is a c-minimizer and } t \in \mathbb{R}\}\,. \tag{1.11}$$

One can show that $\widetilde{\mathcal{N}}_c$ is non-empty and compact, it is contained in the energy level $\widetilde{E}_c$ corresponding to $\alpha(c)$ and, above all, it contains the Mather set $\widetilde{\mathcal{M}}_c$ (see theorem 1.3.17). Differently from what happens with the Mather set, in general, the Mañé set is not a graph.

Another important family of sets, which also play an extremely significant role in the study of the dynamics of Lagrangian system, consists of the *Aubry sets*. In order to define these sets, we need some extra tools that will allow us to single out particular kinds of c-minimizers. Let us start by introducing a so-called "barrier" function. As done by Mather in [59], it is convenient to define the following quantity for $t > 0$ and $x,\, y \in M$:

$$h_{\eta,t}(x,y) = \min \int_0^t L_\eta(\gamma(s),\dot\gamma(s))\,ds\,, \tag{1.12}$$

where the minimum is taken over all piecewise C^1 paths $\gamma : [0,t] \longrightarrow M$, such that $\gamma(0) = x$ and $\gamma(t) = y$. This minimum is achieved because of Tonelli theorem. We define the *Peierls barrier* as:

$$h_\eta(x,y) = \liminf_{t\to+\infty}(h_{\eta,t}(x,y) + \alpha(c)t)\,. \tag{1.13}$$

Remark 1.3.7. Observe that h_η does not depend only on the cohomology class c, but also on the choice of the representative η; namely, if $\eta' = \eta + df$, then $h_{\eta'}(x,y) = h_\eta(x,y) + f(y) - f(x)$. Anyhow, this dependence will not be harmful for what we are going to do in the following.

This function is finite for all $x, y \in M$ and it can be shown that it is Lipschitz. Moreover, Albert Fathi [32] showed that - in the autonomous case - this $\liminf$ can be replaced with a $\lim$. This is not generally true in the non-autonomous time-periodic case (see for instance [36] for some counterexamples); Tonelli Lagrangians for which this convergence result holds are called *regular*. Patrick Bernard [11] showed that under suitable assumptions on the Mather set it is possible to prove that the Lagrangian is regular. For instance, if the Mather set $\tilde{\mathcal{M}}_c$ is union of 1-periodic orbits, then L_η is regular. This problem turned out to be strictly related to the convergence of the so-called *Lax-Oleinik semigroup* (see [32] for its definition).

Remark 1.3.8. The function h_η is a generalization of Peierls Barrier introduced by Aubry [6] and Mather [52, 54, 58] in their study of twist maps. In some sense we are comparing, in the limit, the action of Tonelli minimizers of time length T with the corresponding average c-minimimal action $-\alpha(c)T$. Remember, in fact, that $-\alpha(c)$ is the "average action" of a c-minimal measure. In particular, the finiteness of such liminf can be used to get a further characterization of $\alpha(c)$:

$$\begin{aligned}\alpha(c) &= \sup\left\{k \in \mathbb{R} : \liminf_{t\to+\infty}(h_{\eta,t}(x,y)+kt) = -\infty \quad \text{for all } x,y \in M\right\} = \\ &= \min\left\{k \in \mathbb{R} : \liminf_{t\to+\infty}(h_{\eta,t}(x,y)+kt) \in \mathbb{R} \quad \text{for all } x,y \in M\right\}.\end{aligned}$$

See [48, 22, 23] for these and other similar characterizations. This quantity is often referred to as *Mañé critical value*) (see also the addendum at the end of this section and remark 1.4.5).

Many properties of this barrier have been studied by several authors, leading to a better understanding of its dynamical meaning and, we will see in section 1.4, its relation with "weak" solutions of Hamilton-Jacobi equations. We can summarize some of them in the following proposition (see for example [11, 23, 32, 59] for a proof).

Proposition 1.3.9. *The values of the map h_η are finite. Moreover, the following properties hold:*

i) h_η is Lipschitz;

ii) for each x, $y \in M$, $h_\eta(x,y) + h_\eta(y,x) \geq 0$.

iii) for each $x \in M$, $h_\eta(x,x) \geq 0$*;*

iv) for each $x \in \mathcal{M}_c$, $h_\eta(x,x) = 0$*;*

v) for each x, y, $z \in M$ *and* $t > 0$, $h_\eta(x,y) \leq h_\eta(x,z) + h_{\eta,t}(z,y) + \alpha(c)t$ *and* $h_\eta(x,y) \leq h_{\eta,t}(x,z) + \alpha(c)t + h_{\eta,t}(z,y)$*;*

vi) for each $x,y,z \in M$, $h_\eta(x,y) \leq h_\eta(x,z) + h_\eta(z,y)$*.*

Inspired by these properties, one can consider its symmetrization:

$$\begin{aligned} \delta_c : \ M \times M \ &\longrightarrow \ \mathbb{R} \\ (x,y) \ &\longmapsto \ h_\eta(x,y) + h_\eta(y,x). \end{aligned} \tag{1.14}$$

Observe that this function does now depend only on the cohomology class c and moreover it is non-negative, symmetric and satisfies the triangle inequality; therefore, it is a pseudometric on the set

$$\mathcal{A}_c = \{x \in M : \ \delta_c(x,x) = 0\}. \tag{1.15}$$

$\mathcal{A}_c$ is called the *projected Aubry set* associated to L and c, and δ_c is *Mather's pseudometric.*

Remark 1.3.10. One can easily construct a metric space out of $(\mathcal{A}_c, \delta_c)$. We call *quotient Aubry set* the metric space $(\bar{\mathcal{A}}_c, \bar{\delta}_c)$ obtained by identifying two points in $\mathcal{A}_c$, if their δ_c-pseudodistance is zero. This set will the main object of study in chapter 2.

An equivalent definition of the projected Aubry set is the following (it follows immediately from the definition of δ_c):

Proposition 1.3.11. *$x \in \mathcal{A}_c$ if and only if there exists a sequence of absolutely continuous curves $\gamma_n : [0, t_n] \to M$ such that:*

- *for each n, we have $\gamma_n(0) = \gamma_n(t_n) = x$;*

- *the sequence $t_n \to +\infty$, as $n \to +\infty$;*

- *as $n \to +\infty$, $\int_0^{t_n} L_\eta(\gamma_n(s), \dot{\gamma}_n(s))\, ds + \alpha(c)t_n \to 0$.*

Therefore, it consists of points that are contained in loops with period as long as we want and action as close as we want to the minimal average one.

Remark 1.3.12. An interesting property of the pseudometric δ_c is the following (see [59]). If d denotes the distance induced on M by the Riemannian metric g, then there exists $C > 0$ such that for each $x, y \in M$ we have

$$\delta_c(x, y) \leq Cd(x, y)^2.$$

The same estimate continues to be true for the non-autonomous time-periodic case. In this case we have that

$$\delta_c((x, \tau_0), (y, \tau_1)) \leq C[d(x, y) + \|\tau_1 - \tau_0\|]^2$$

for each $(x, \tau_0), (y, \tau_1) \in \mathcal{A}_c$, where

$$\|\tau_1 - \tau_0\| = \inf\left\{|t_1 - t_0| : \ t_i \in \mathbb{R}, \ t_i \equiv \tau_i \,(\text{mod. } 1), \ i = 0, 1\right\}.$$

This estimate will be particularly useful in section 2.2 (addendum) to prove the total disconnectedness of the quotient Aubry set in low dimension.

We want to show now that $\mathcal{A}_c$ is the projection of an invariant compact subset of $\mathrm{T}M$, which will be called *Aubry set*. Let start with the following observation.

Remark 1.3.13. Let $\gamma : \mathbb{R} \longrightarrow M$ be a c-minimizer and consider x_α, x'_α in the

α-limit set[1] of γ and x_ω, x'_ω in the ω-limit set[2] of γ. John Mather in [59] proved that $\delta_c(x_\alpha, x'_\alpha) = \delta_c(x_\omega, x'_\omega) = 0$. In general, it is not true that $\delta_c(x_\alpha, x_\omega)$=0; what one can prove is that this value does not depend on the particular x_α and x_ω, *i.e.*, $\delta_c(x_\alpha, x_\omega) = \delta_c(x'_\alpha, x'_\omega)$: it is a property of the limit sets rather than of their elements. Nevertheless, there will exist particular c-minimizers for which this value is equal to 0 and these will be the c-minimizers that we want to single out.

Definition 1.3.14 (c-regular minimizers). *A c-minimizer $\gamma : \mathbb{R} \longrightarrow M$ is called a c-regular minimizer, if $\delta_c(x_\alpha, x_\omega) = 0$ for each x_α in the α-limit set of γ and x_ω in the ω-limit set of γ.*

We can now define the *Aubry set* as the union of the support of all these c-regular minimizers.

Definition 1.3.15 (Aubry set). *The Aubry set (with cohomology class c) is:*

$$\widetilde{\mathcal{A}}_c = \bigcup \left\{(\gamma(t), \dot\gamma(t)) : \ \gamma \textit{ is a c-regular minimizer and } t \in \mathbb{R}\right\}. \tag{1.16}$$

This set $\widetilde{\mathcal{A}}_c$ is clearly contained in the Mañé set $\widetilde{\mathcal{N}}_c$. In particular, each c-minimizer is asymptotic to $\widetilde{\mathcal{A}}_c$ (see also [23]). One of its most important properties is that, similarly to the Mather set, it is also a graph over the base (see [59, 32]):

Theorem 1.3.16. *$\pi|\widetilde{\mathcal{A}}_c$ is a bi-Lipschitz homeomorphism and $\pi\big(\widetilde{\mathcal{A}}_c\big) = \mathcal{A}_c$.*

Furthermore, there is a clear relation between the support of c-minimal measures and these sets.

[1] Recall that a point z is in the α-limit set of γ, if there exists a sequence $t_n \to -\infty$ such that $\gamma(t_n) \to z$.

[2] Recall that a point z is in the ω-limit set of γ, if there exists a sequence $t_n \to +\infty$ such that $\gamma(t_n) \to z$.

Theorem 1.3.17. *For any* $c \in \mathrm{H}^1(M;\mathbb{R})$, $\widetilde{\mathcal{M}}_c \subseteq \widetilde{\mathcal{A}}_c$. *Moreover:*

$$\begin{aligned} \mu \in \mathfrak{M}_c \quad &\Longleftrightarrow \quad \mu \in \mathfrak{M}(L) \text{ and } \operatorname{supp}\mu \subseteq \widetilde{\mathcal{A}}_c \\ &\Longleftrightarrow \quad \mu \in \mathfrak{M}(L) \text{ and } \operatorname{supp}\mu \subseteq \widetilde{\mathcal{N}}_c\,. \end{aligned} \tag{1.17}$$

The fact that an invariant measure whose support is contained in $\widetilde{\mathcal{N}}_c$ (or $\widetilde{\mathcal{A}}_c$) is c-minimizing might be seen, for example, as a consequence of the finiteness of Peierls barrier. The proof of the first equivalence in (1.17) goes back to Mañé [48]. As far as the second equivalence is concerned, it follows from the fact that the *non-wandering set* of $\Phi_t^L\big|\widetilde{\mathcal{N}}_c$ is contained in $\widetilde{\mathcal{A}}_c$ (we will prove it in the addendum in section 1.4).

Moreover, the following relation between these sets holds.

Proposition 1.3.18. *Let* ρ, c *be respectively an arbitrary homology class in* $\mathrm{H}_1(M;\mathbb{R})$ *and an arbitrary cohomology class* $\mathrm{H}^1(M;\mathbb{R})$. *We have*

$$(1)\ \widetilde{\mathcal{M}^\rho} \cap \widetilde{\mathcal{A}_c} \neq \emptyset \quad \Longleftrightarrow \quad (2)\ \widetilde{\mathcal{M}^\rho} \subseteq \widetilde{\mathcal{A}_c} \quad \Longleftrightarrow \quad (3)\ \rho \in \partial\alpha(c)\,.$$

Proof. The implication (2) $\Longrightarrow$ (1) is trivial. Let us prove that (1) $\Longrightarrow$ (3). If $\widetilde{\mathcal{M}^\rho} \cap \widetilde{\mathcal{A}_c} \neq 0$, then there exists a c-minimizing invariant measure μ with rotation vector ρ. Let η be a closed 1-form with $[\eta] = c$; from the definition of α and β:

$$-\alpha(c) = \int_{\mathrm{T}M} (L - \hat{\eta})\, d\mu = \int_{\mathrm{T}M} L\, d\mu - \langle c, \rho\rangle = \beta(\rho) - \langle c, \rho\rangle\,;$$

since β and α are convex conjugated, then ρ is a subderivative of α at c.

Finally, in order to show (3) $\Longrightarrow$ (2), let us prove that any action minimizing measure with rotation vector ρ is c-minimizing. In fact, if $\rho \in \partial\alpha(c)$ then $\alpha(c) = \langle c, \rho\rangle - \beta(\rho)$; therefore for any $\mu \in \mathfrak{M}^\rho$ and η as above:

$$-\alpha(c) = \beta(\rho) - \langle c, \rho\rangle = \int_{\mathrm{T}M} (L - \hat{\eta})\, d\mu.$$

This proves that $\mu \in \mathfrak{M}_c$ and concludes the proof.

□

We can summarize what discussed so far in the following diagram:[3]

$$\begin{array}{ccccccccc}
\widetilde{\mathcal{M}}_c & \subseteq & \widetilde{\mathcal{A}}_c & \subseteq & \widetilde{\mathcal{N}}_c & \subseteq & \widetilde{\mathcal{E}}_c & \subseteq & \mathrm{T}M \\
\downarrow & & \downarrow & & & & & & \downarrow{\scriptstyle \pi} \\
\mathcal{M}_c & \subseteq & \mathcal{A}_c & \subseteq & & & & \subseteq & M
\end{array}$$

We have already observed that the Mañé set $\widetilde{\mathcal{N}}_c$ in general is not a graph over the base. Anyhow, one can prove a sort of "graph" property over the *Aubry set* (see [11]). More precisely, $\widetilde{\mathcal{N}}_c \cap \pi^{-1}(\mathcal{A}_c) = \widetilde{\mathcal{A}}_c$, where $\pi : \mathrm{T}M \to M$ denotes the canonical projection. In other words, there is a Lipschitz section $\mathcal{V}_c : \mathcal{A}_c \longrightarrow \mathrm{T}M$, such that for each $x \in \mathcal{A}$ there exists one and only c-minimizer $\gamma : \mathbb{R} \to M$ satisfying $\gamma(0) = x$; this minimizer is *regular* and is given by $\gamma(t) = \pi\left(\Phi_t^L(x, \mathcal{V}_c(x)\right)$.

Moreover, one can prove the following properties of $\widetilde{\mathcal{A}}_c$ and $\widetilde{\mathcal{N}}_c$.

Proposition 1.3.19 ([22])**.** *$\widetilde{\mathcal{N}}_c$ is chain transitive* [4] *and $\widetilde{\mathcal{A}}_c$ is chain recurrent* [5].

ADDENDA

In these addenda we want to discuss some other definitions of minimizers that appear in the literature and some differences between the autonomous and non-autonomous case.

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[3]The typographical "coincidence", honoring Ricardo Mañé, was noticed by Albert Fathi.

[4]Namely, for each $\varepsilon > 0$ and for all $(x,v), (y,w) \in \widetilde{\mathcal{N}}_c$, there exists an ε-pseudo-orbit for the flow Φ^L connecting them. In other words, there exist $\{(x_n, v_n)\}_{n=0}^{k_\varepsilon} \subset \widetilde{\mathcal{N}}_c$ and positive times $t_1, \ldots, t_{k_\varepsilon} > 0$ such that $(x_0, v_0) = (x,v)$, $(x_{k_\varepsilon}, v_{k_\varepsilon}) = (y,w)$ and $\operatorname{dist}\left(\Phi_{t_{i+1}}^L\left((x_i, v_i)\right), (x_{i+1}, v_{i+1})\right) \leq \varepsilon$ for all $i = 0, \ldots, k_\varepsilon$.

[5]As in the definition of chain transitivity, but in this case the end-points are the same

Semi-static and Static curves

In some of the literature, in particular in the production of Ricardo Mañé and his school (see for instance [48, 22, 23]), c-minimizers and c-regular minimizers are replaced by the notions of *c-semi-static curves* and *c-static curves*. They used these curves for defining the equivalent of Aubry and Mañé sets that we have introduced above. It turns out that - at least in the autonomous case - these different kinds of curves do indeed coincide. Let us try to give an idea of these definitions and their equivalence.

First of all they were not considering Peierls barrier as in our definition, but something different, that is commonly called *Mañé potential*:

$$\phi_\eta(x,y) = \inf_{t>0} \left(h_t(x,y) + \alpha(c)t\right).$$

Clearly $\phi_\eta(x,y) \leq h_\eta(x,y)$ everywhere. Moreover, it shares many properties with Peierls barrier: it is finite, Lipschitz, it satisfies the triangle inequality (vi) in proposition 1.3.9) and, above all, $\phi_\eta(x,x) = 0$ if and only if $h_\eta(x,x) = 0$ (see for instance [11]).

Definition 1.3.20 (Semi-static and static curves). *An absolutely continuous curve $\gamma : \mathbb{R} \longrightarrow M$ is a c-semi-static if for any given $a < b \in \mathbb{R}$*

$$\int_a^b L_\eta(\gamma(t), \dot\gamma(t))\,dt + \alpha(c)(b-a) = \phi_\eta(\gamma(a), \gamma(b)).$$

Moreover, an absolutely continuous curve $\gamma : \mathbb{R} \longrightarrow M$ is c-static if for any given $a < b \in \mathbb{R}$

$$\int_a^b L_\eta(\gamma(t), \dot\gamma(t))\,dt + \alpha(c)(b-a) = -\phi_\eta(\gamma(b), \gamma(a)).$$

It is easy to check that c-static curves are c-semistatic. Moreover, in the autonomous case, c-semi-static curves coincide with c-minimizers, while c-static curves with c-regular minimizers (see for instance [32]).

Remark 1.3.21. One can therefore give the definition of c-semi-static and static curves also in terms of Peierls barrier. Namely, using remark 1.3.5 it is easy to check that $\gamma : \mathbb{R} \to M$ is c-semi-static if for any $a < b$

$$\int_a^b L_\eta(\gamma(t), \dot{\gamma}(t))\, dt + \alpha(c)(b-a) = h_\eta(\gamma(a), \gamma(b)).$$

In particular, a c-semi-static curve is c-static (or equivalently, a c-minimizer is regular) if $h_\eta(\gamma(a), \gamma(b)) = -h_\eta(\gamma(b), \gamma(a))$, for any $a < b$.

Minimizers in the non-autonomous time-periodic case

In the non-autonomous time-periodic case, the notion of c-minimizer that we have given above is a weak notion and the invariant set of curves that one obtains in this way is too big to enjoy many of the properties that the Mañé set does. The main problem is that the property highlighted in remark 1.3.5 fails to be true in the non-autonomous case. For this reason, one needs to introduce a stronger notion of action-minimizing curves and distinguish between *weak c-minimizers* and *c-minimizers* (although it would rather make more sense to call this new class of curves *c-strong minimizers*).

Let M be a compact manifold and $L : \mathrm{T}M \times \mathbb{T} \longrightarrow \mathbb{R}$ a Tonelli Lagrangian, whose Euler-Lagrange flow is complete. If η is a closed 1-form, we let $[\eta] = ([\eta]_M, [\eta]_\mathbb{T}) \in \mathrm{H}^1(M \times \mathbb{T}; \mathbb{R}) = \mathrm{H}^1(M; \mathbb{R}) \times \mathbb{R}$ denote its de Rham cohomology class. We will say that η is Mañé critical if $[\eta]_\mathbb{T} = -\alpha([\eta]_M)$. All the theory developed so far in this chapter, can be extended to the non-autonomous case considering Mañé critical 1-forms.

In the following, let us fix η to be a Mañé critical 1-form with $[\eta]_M = c$.

Definition 1.3.22 (c-weak minimizers). *An absolutely continuous curve* $\gamma : \mathbb{R} \longrightarrow M$ *is a c-weak minimizer if for any given* $a < b \in \mathbb{R}$

$$\int_a^b (L - \hat{\eta})(\gamma(t), \dot{\gamma}(t), t \bmod. 1)\, dt \leq \int_a^b (L - \hat{\eta})(\sigma(t), \dot{\sigma}(t), t \bmod. 1)\, dt$$

for any $\sigma : [a, b] \to M$ *such that* $\sigma(a) = \gamma(a)$ *and* $\sigma(b) = \gamma(b)$.

Definition 1.3.23 (c-minimizers). *An absolutely continuous curve* $\gamma : \mathbb{R} \longrightarrow M$ *is a c-minimizer if for any given* $a < b \in \mathbb{R}$

$$\int_a^b (L - \hat{\eta})(\gamma(t), \dot{\gamma}(t), t \bmod. 1)\, dt \leq \int_{a'}^{b'} (L - \hat{\eta})(\sigma(t), \dot{\sigma}(t), t \bmod. 1)\, dt$$

for any $\sigma : [a', b'] \to M$ *such that* $a' < b'$, $a' - a \in \mathbb{Z}$, $b' - b \in \mathbb{Z}$ *and* $\sigma(a') = \gamma(a)$ *and* $\sigma(b') = \gamma(b)$.

One can define the following invariant (compact) subsets:

- $\widetilde{\mathcal{W}}_c = \bigcup \{(\gamma(t), \dot{\gamma}(t)) :\ \gamma \text{ is a } c\text{-weak minimizer and } t \in \mathbb{R}\}$;
- $\widetilde{\mathcal{N}}_c = \bigcup \{(\gamma(t), \dot{\gamma}(t)) :\ \gamma \text{ is a } c\text{-minimizer and } t \in \mathbb{R}\}$ (*Mañé set*);
- $\widetilde{\mathcal{A}}_c = \bigcup \{(\gamma(t), \dot{\gamma}(t)) :\ \gamma \text{ is a } c\text{-regular minimizer and } t \in \mathbb{R}\}$ (*Aubry set*).

Obviously every c-minimizer is a c-weak minimizer, therefore $\widetilde{\mathcal{N}}_c \subseteq \widetilde{\mathcal{W}}_c$. On the other hand, this inclusion could be strict, since there might exist c-weak minimizers that are not c-minimizers (see for example [11, 36]). As already remarked above, in the autonomous case these two definitions/sets coincide. Patrick Bernard in [11] proved that for *regular* Lagrangians, *i.e.,* when the liminf in the definition of Peierls barrier is a limit, these two sets coincide: $\widetilde{\mathcal{N}}_c = \widetilde{\mathcal{W}}_c$.

Summarizing, also in the non-autonomous case we have a similar diagram:

$$\begin{array}{ccccccccc}
\widetilde{\mathcal{M}}_c & \subseteq & \widetilde{\mathcal{A}}_c & \subseteq & \widetilde{\mathcal{N}}_c & \subseteq & \widetilde{\mathcal{W}}_c & \subseteq & \mathrm{T}M \times \mathbb{T} \\
\downarrow & & \downarrow & & & & & & \downarrow \pi \\
\mathcal{M}_c & \subseteq & \mathcal{A}_c & \subseteq & & & & \subseteq & M \times \mathbb{T}
\end{array}$$

Furthermore, the Aubry and Mather sets still satisfy the graph property and proposition 1.3.19 continues to hold, *i.e.,* $\widetilde{\mathcal{N}}_c$ is chain transitive and $\widetilde{\mathcal{A}}_c$ chain recurrent.

1.4 Weak KAM theory

Another interesting approach to the study of these invariant sets is provided by *weak KAM theory*, which represents the functional analytical counterpart of the variational methods discussed in the previous sections. This is mainly based on the concept of "critical" subsolutions and "weak" solutions of Hamilton-Jacobi equation and, we will see, can be interpreted from a symplectic geometric point of view as the study of particular Lagrangian graphs and their non-removable intersections (see [69] and chapter 3). This latter approach is particularly interesting, since it relates the dynamics of the system to the geometry of the space and might potentially open the way to a "symplectic" definition of Aubry-Mather theory (see also section 3.2 and [15]). In this section we want to provide a brief presentation of this theory, omitting most of the proves, for which we remand the reader to the excellent - and self-contained - presentation [32].

The main object of investigation is represented by Hamilton-Jacobi (H-J) equation:

$$H_\eta(x, d_x u) = H(x, \eta(x) + d_x u) = k\,,$$

where η is a closed 1-form on M with a certain cohomology class c. Observe that considering H-J equations for different 1-forms corresponding to different cohomology classes, is equivalent to Mather's idea of changing Lagrangian (see remark 1.2.1).

From now on, we will consider L to be a Tonelli Lagrangian on a compact manifold M and H its associated Hamiltonian. Let us fix η to be a closed 1-form on M with cohomology class c, and, as before, denote by L_η and H_η the modified Lagrangian and Hamiltonian (see remark 1.2.1). In classical mechanics, one is interested in studying solutions of this equation, *i.e.,* C^1 functions $u : M \to \mathbb{R}$ such that $H_\eta(x, d_x u) = k$. It is immediate to check that for any given cohomology class there exists at most one value of k for which these C^1 solutions may exist. In fact, it is enough to observe that if u and v are two C^1 functions on a compact manifold, there will exist a point x_0 at which their differentials coincide (take any critical point of $u - v$). We will see that this value of k for which solutions might exist, coincides with $\alpha(c)$ (see theorem 1.4.15).

The existence of such solutions has significant implications to the dynamics of the system and it is, consequently, quite rare. In particular, they correspond to Lagrangian graphs, which are invariant under the Hamiltonian flow Φ_t^H (Hamilton-Jacobi theorem). For instance, in the case of $M = \mathbb{T}^d$ and nearly-integrable systems these solutions correspond to KAM tori (this might give an idea of their rareness).

One of the main results of weak KAM theory is that, in the case of optical Hamiltonians, a weaker kind of solutions do always exist. In the following we are going to define these generalized solutions and their relation with the dynamics of the systems. It is important to point out that one of the main ingredient in the proof of all these results is provided by Fenchel inequality (1.3).

Let us start by generalizing the concept of subsolutions. In the C^1-case it is easy to check - using Fenchel inequality - that the following property holds.

Proposition 1.4.1. *Let* $u : M \to \mathbb{R}$ *be* C^1*;* u *satisfies* $H(x, \eta(x) + d_x u) \leq k$ *for all*

$x \in M$ *if and only if for all* $\gamma : [a,b] \to M$

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L_\eta(\gamma(t), \dot\gamma(t))\, dt + k(b-a).$$

This last inequality provides the ground for to the definition of *dominated functions*, which will generalize subsolutions in the C^0-case.

Definition 1.4.2 (Dominated functions). *Let $u : M \to \mathbb{R}$ be a continuous function; u is dominated by $L_\eta + k$, and we will write $u \prec L_\eta + k$, if for all $\gamma : [a,b] \to M$*

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L_\eta(\gamma(t), \dot\gamma(t))\, dt + k(b-a). \tag{1.18}$$

One can check that if $u \prec L_\eta + k$ then u is Lipschitz and its Lipschitz constant can be bounded by a constant $C(k)$ independent of u; in particular, all dominated functions for values of k in a compact set are equiLipschitz. On the other hand, it is easy to check that each Lipschitz function is dominated by $L_\eta + k$, for a suitable k depending on its Lipschitz constant: this shows that dominated functions exist.
Dominated functions generalize subsolutions of H-J to the continuous case. In fact:

Proposition 1.4.3. *If $u \prec L_\eta + k$ and $d_x u$ exists, then $H(x, \eta(x) + d_x u) \leq k$. Moreover, if $u : M \to \mathbb{R}$ is Lipschitz and $H(x, \eta(x) + d_x u) \leq k$ a.e., then $u \prec L_\eta + k$.*

Remark 1.4.4. Using the fact that any Lipschitz function is differentiable almost everywhere (Rademacher theorem), one could equivalently define subsolutions in the following way: *a locally Lipschitz function $u : M \longrightarrow \mathbb{R}$ is a subsolution of $H_\eta(x, d_x u) = k$, with $k \in \mathbb{R}$, if $H_\eta(x, d_x u) \leq k$ for almost every $x \in M$.*

Remark 1.4.5. One interesting question is: for which values of k do there exist functions dominated by $L_\eta + k$ (or equivalently subsolutions of $H(x, \eta(x) + d_x u) = k$)? It is easy to show that there exists a value $k_c \in \mathbb{R}$ such that $H(x, \eta + d_x u) = k$ does not admit any subsolution for $k < k_c$, while it has subsolutions for $k \geq k_c$, see [45, 32].

In particular, if $k > k_c$ there exist C^∞ subsolutions. The constant k_c is called *Mañé's critical value* and coincides with $\alpha(c)$, where $c = [\eta]$ (see [23, 32]).

Functions corresponding to this "critical domination" play an important role, since they encode significant information about the dynamics of the system.

Definition 1.4.6 (Critical subsolutions). *A function $u \prec L_\eta + \alpha(c)$ is said to be critically dominated. Equivalently, we will also call it an η-critical subsolution, since $H(x, \eta(x) + d_x u) \leq \alpha(c)$ for almost every $x \in M$.*

Remark 1.4.7. The above observation provides a further definition of $\alpha(c)$:

$$\alpha(c) = \inf_{u \in C^\infty(M)} \max_{x \in M} H(x, \eta(x) + d_x u)\,.$$

This infimum is not a minimum, but it becomes a minimum over the set of Lipschitz functions on M (also over the smaller set of $C^{1,1}$ functions, see the addendum at the end of this section). This characterization has the following geometric interpretation. If we consider the space T^*M equipped with the canonical symplectic form, the graph of the differential of a C^1 η-critical subsolution (plus the 1-form η) is nothing else than a c-Lagrangian graph (*i.e.,* a Lagrangian graph with cohomology class c). Therefore Mañé c-critical energy level $\mathcal{E}_c^* = \{(x,p) \in \mathrm{T}^*M : \ H(x,p) = \alpha(c)\}$ corresponds to a $(2d-1)$-dimensional hypersurface, such that the region it bounds is convex in each fiber and does not contain in its interior any c-Lagrangian graph, while any of its neighborhoods does. See also chapter 3 for more details.

Next step consists in analyzing curves for which equality in (1.18) holds.

Definition 1.4.8 (Calibrated curves). *Let $u \prec L_\eta + k$. A curve $\gamma : I \to M$ is (u, L_η, k)-calibrated if for any $[a,b] \subseteq I$*

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L_\eta(\gamma(t), \dot\gamma(t))\, dt + k(b-a).$$

These curves are very special curves; in fact:

Proposition 1.4.9. *If $u \prec L_\eta + k$ and $\gamma : [a,b] \to M$ is (u, L_η, k)-calibrated, then γ is a c-Tonelli minimizer, i.e.,*

$$\int_a^b L_\eta(\gamma(t), \dot\gamma(t))\,dt \le \int_a^b L_\eta(\sigma(t), \dot\sigma(t))\,dt$$

for any $\sigma : [a,b] \to M$ such that $\sigma(a) = \gamma(a)$ and $\sigma(b) = \gamma(b)$. Most of all, this implies that γ is a solution of the Euler-Lagrange flow and therefore it is C^r (if L is C^r).

Moreover, the following differentiability result holds.

Proposition 1.4.10. *Let $u \prec L_\eta + k$ and $\gamma : [a,b] \to M$ be (u, L_η, k)-calibrated.*

i) If $d_{\gamma(t)}u$ exists for some $t \in [a,b]$, then $H(\gamma(t), \eta(\gamma(t)) + d_{\gamma(t)}u) = k$ and $d_{\gamma(t)}u = \dfrac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t))$.

ii) If $t \in (a,b)$, then $d_{\gamma(t)}u$ exists.

Remark 1.4.11. Calibrated curves are "Lagrangian gradient lines" of $\mathrm{grad}_L u$ (where $\mathrm{grad}_L u$ is a multivalued vector field given by the equation $d_x u = \dfrac{\partial L}{\partial v}(x, \mathrm{grad}_L u)$. Therefore, there is only one possibility for calibrated curves, at each point of differentiability of u.

In order to define *weak solutions*, let us recall this property of classical solutions.

Proposition 1.4.12. *Let $u : M \to \mathbb{R}$ a C^1 function and $k \in \mathbb{R}$. The following conditions are equivalent:*

1. *u is solution of $H(x, \eta(x) + d_x u) = k$;*

2. *$u \prec L_\eta + k$ and for each $x \in M$ there exists $\gamma_x : (-\infty, +\infty) \to M$ such that $\gamma_x(0) = x$ and γ_x is (u, L_η, k)-calibrated;*

3. $u \prec L_\eta + k$ *and for each* $x \in M$ *there exists* $\gamma_x : (-\infty, 0] \to M$ *such that* $\gamma_x(0) = x$ *and* γ_x *is* (u, L_η, k)*-calibrated;*

4. $u \prec L_\eta + k$ *and for each* $x \in M$ *there exists* $\gamma_x : [0, +\infty) \to M$ *such that* $\gamma_x(0) = x$ *and* γ_x *is* (u, L_η, k)*-calibrated.*

This suggests the following definitions.

Definition 1.4.13 (weak KAM solutions). *Let* $u \prec L_\eta + k$*.*

- *u is a weak KAM solution of negative type (or backward Weak KAM solution) if for each* $x \in M$ *there exists* $\gamma_x : (-\infty, 0] \to M$ *such that* $\gamma_x(0) = x$ *and* γ_x *is* (u, L_η, k)*-calibrated;*

- *u is a weak KAM solution of positive type (or forward Weak KAM solution) if for each* $x \in M$ *there exists* $\gamma_x : [0, +\infty) \to M$ *such that* $\gamma_x(0) = x$ *and* γ_x *is* (u, L_η, k)*-calibrated.*

Remark 1.4.14. Observe that any weak KAM solution of negative type u_- (resp. of positive type u_+) for a given Lagrangian L, can be seen as a weak KAM solution of positive type (resp. of negative type) for the *symmetrical* Lagrangian $\tilde{L}(x, v) := L(x, -v)$.

Let us denote with $\mathcal{S}_\eta^-$ the set of Weak KAM solutions of negative type and $\mathcal{S}_\eta^+$ the ones of positive types. Albert Fathi [28, 32] proved that these sets are always non-empty.

Theorem 1.4.15 (Weak KAM theorem). *There is only one value of k for which weak KAM solutions of positive or negative type of* $H(x, \eta(x) + d_x u) = k$ *exist. This value coincides with* $\alpha(c)$*, where* $\alpha : \mathrm{H}^1(M; \mathbb{R}) \to \mathbb{R}$ *is Mather's* α*-function defined in (1.6). In particular, for any* $u \prec L_\eta + \alpha(c)$ *there exist a weak KAM solution of negative type* u_- *and a weak KAM solution of positive type* u_+*, such that* $u_- = u = u_+$ *on the projected Aubry set* $\mathcal{A}_c$*.*

Therefore, for any given weak KAM solution of negative type u_- (resp. of positive type u_+), there exists a weak KAM solution of positive type of positive type u_+ (resp. of negative type u_-) such that $u_- = u_+$ on the projected Aubry set $\mathcal{A}_c$. Such solutions are said to be *conjugate*. In particular, for any u_- or u_+ there exists a unique conjugate solutions. This follows from the following result.

Proposition 1.4.16. *The projected Mather set $\mathcal{M}_c$ is the uniqueness set for weak KAM solutions of the same type. Namely, if u_-, v_- are weak KAM solutions of negative type (resp. u_+, v_+ are weak KAM solutions of positive type) and $u_- = v_-$ on $\mathcal{M}_c$ (resp. $u_+ = v_+$ on $\mathcal{M}_c$), then they coincide everywhere on M.*

Let us try to understand the dynamical meaning of such solutions. Albert Fathi [32] - using these generalized solutions - proved a *weak* version of Hamilton-Jacobi theorem, showing the relation between these weak solutions and the dynamics of the associated Hamiltonian system. We will state it for weak KAM solution of negative type, but - using remark 1.4.14 - one can deduce an analogous statement for weak KAM solutions of positive type.

Theorem 1.4.17 (Weak Hamilton-Jacobi Theorem). *Let $u_- : M \to \mathbb{R}$ be a weak KAM solution of negative type and consider*

$$\mathrm{Graph}(\eta + du_-) := \{(x, \eta(x) + d_x u_-),\ \text{where } d_x u_- \text{ exists}\}.$$

Then:

i) $\overline{\mathrm{Graph}(\eta + du_-)}$ *is compact and is contained in the energy level* $\mathcal{E}_c^* = \{(x,p) \in \mathrm{T}^*M :\ H(x,p) = \alpha(c)\}$*;*

ii) $\Phi_{-t}^H\left(\overline{\mathrm{Graph}(\eta + du_-)}\right) \subseteq \mathrm{Graph}(\eta + du_-)$ *for each* $t > 0$*;*

iii) $M = \pi\left(\overline{\mathrm{Graph}(\eta + du_-)}\right)$, *where* $\pi : \mathrm{T}^*M \to M$ *is the canonical projection.*

Moreover, let us define:

$$\mathfrak{I}^*(u_-) := \bigcap_{t\geq 0} \Phi^H_{-t}\left(\overline{\mathrm{Graph}(\eta + du_-)}\right).$$

$\mathfrak{I}^*(u_-)$ *is non-empty, compact and invariant under* Φ^H_t*. Furthermore, its "unstable set" contains* $\overline{\mathrm{Graph}(\eta + du_-)}$*; i.e.,*

$$\overline{\mathrm{Graph}(\eta + du_-)} \subseteq W^u\left(\mathfrak{I}^*(u_-)\right) := \left\{(y,p):\ \mathrm{dist}\big(\Phi^H_{-t}(y,p), \mathfrak{I}^*(u_-)\big) \stackrel{t\to+\infty}{\longrightarrow} 0\right\}.$$

There is a relation between these invariant sets $\mathfrak{I}^*(u_-)$ (or $\mathfrak{I}^*(u_+)$) and the Aubry set $\widetilde{\mathcal{A}}_c$ (recall that $\mathfrak{I}^*(u_-), \mathfrak{I}^*(u_+) \subset \mathrm{T}^*M$, while $\widetilde{\mathcal{A}}_c \subset \mathrm{T}M$):

$$\begin{aligned} \mathcal{A}^*_c := \mathcal{L}\left(\widetilde{\mathcal{A}}_c\right) &= \bigcap_{u_-\in\mathcal{S}^-_\eta} \mathfrak{I}^*(u_-) = \bigcap_{u_-\in\mathcal{S}^-_\eta} \mathrm{Graph}(\eta + du_-) = \\ &= \bigcap_{u_+\in\mathcal{S}^+_\eta} \mathfrak{I}^*(u_+) = \bigcap_{u_+\in\mathcal{S}^+_\eta} \mathrm{Graph}(\eta + du_+) \end{aligned} \tag{1.19}$$

where $\mathcal{L} : \mathrm{T}M \to \mathrm{T}^*M$ denotes the Legendre transform (1.4).

Since it is easier to work with subsolutions rather than weak solutions, we want to discuss now how η-critical subsolutions, although they contain less dynamical information than weak KAM solutions, can be used to characterize Aubry and Mañé sets in a similar way.

Consider $u \prec L_\eta + \alpha(c)$. For $t \geq 0$ define

$$\widetilde{\mathfrak{I}}_t(u) := \left\{(x,v) \in \mathrm{T}M:\ \gamma_{(x,v)}(s) := \pi\Phi^L_s((x,v)) \text{ is } (u, L_\eta, \alpha(c)) - \text{calibr. on } (-\infty, t]\right\}.$$

We will call the *Aubry set of* u: $\widetilde{\mathfrak{I}}(u) := \bigcap_{t\geq 0} \widetilde{\mathfrak{I}}_t(u)$, that can be also defined as

$$\widetilde{\mathfrak{I}}(u) := \left\{(x,v) \in \mathrm{T}M:\ \gamma_{(x,v)}(s) := \pi\Phi^L_s((x,v)) \text{ is } (u, L_\eta, \alpha(c))\ -\ \text{calibr. on } \mathbb{R}\right\}.$$

These sets $\widetilde{\mathfrak{I}}(u)$ are non-empty, compact and invariant. Moreover, here there are some properties of these sets (compare with theorem 1.4.17).

Proposition 1.4.18. *Let $u \prec L_\eta + \alpha(c)$.*

1. *$\widetilde{\mathfrak{I}}_t(u)$ is compact;*
2. *$\widetilde{\mathfrak{I}}_{t'}(u) \subseteq \widetilde{\mathfrak{I}}_t(u) \subseteq \widetilde{\mathfrak{I}}_0(u)$ for all $t' \geq t \geq 0$;*
3. *$\mathcal{L}\left(\widetilde{\mathfrak{I}}_0(u)\right)$ is contained in the energy level $\mathcal{E}_c^*$ corresponding to $\alpha(c)$;*
4. *$\mathcal{L}\left(\widetilde{\mathfrak{I}}_t(u)\right) \subseteq \mathrm{Graph}(\eta + du)$ for all $t > 0$;*
5. *$\mathcal{L}\left(\widetilde{\mathfrak{I}}_0(u)\right) \subseteq \overline{\mathrm{Graph}(\eta + du)}$ (observe that for weak solutions these two sets coincide);*
6. *$\Phi^L_{-t}\left(\widetilde{\mathfrak{I}}_0(u)\right) = \widetilde{\mathfrak{I}}_t(u)$ for all $t > 0$;*
7. *$\overline{\bigcup_{t>0} \widetilde{\mathfrak{I}}_t(u)} = \widetilde{\mathfrak{I}}_0(u)$;*
8. *$\widetilde{\mathfrak{I}}(u) = \bigcap_{t\geq 0} \Phi^L_{-t}\left(\widetilde{\mathfrak{I}}_0(u)\right)$;*
9. *$\widetilde{\mathfrak{I}}_0(u) \subseteq W^u\left(\widetilde{\mathfrak{I}}(u)\right)$.*
10. *$\pi : \widetilde{\mathfrak{I}}(u) \longrightarrow \pi(\widetilde{\mathfrak{I}(u)})$ is a bi-Lipschitz homeomorphism [Graph Theorem]. The same is true for $\widetilde{\mathfrak{I}}_t(u)$ for each $t > 0$.*

Theorem 1.4.19 (Fathi). *The Aubry and Mañé sets defined in (1.16) and (1.11) can be equivalently defined in the following ways:*

$$\widetilde{\mathcal{A}}_c = \bigcap_{u \prec L_\eta + \alpha(c)} \widetilde{\mathfrak{I}}(u) = \bigcap_{u \prec L_\eta + \alpha(c)} \mathcal{L}^{-1}\left(\mathrm{Graph}(\eta + du)\right)$$

$$\widetilde{\mathcal{N}}_c = \bigcup_{u \prec L_\eta + \alpha(c)} \widetilde{\mathfrak{I}}(u).$$

Moreover, there exists $u_\infty \prec L_\eta + \alpha(c)$ such that $\widetilde{\mathcal{A}}_c = \widetilde{\mathfrak{I}}(u_\infty) \cap \mathcal{L}^{-1}\left(\mathcal{E}_c^*\right)$.

For the last statement is sufficient to observe that the set of critically dominated functions is a separable subset of $C(M)$. Let $\{u_n\}$ be a countable dense family of such functions and define u_∞ as a convex combination of their normalization (with respect to a fixed point $x_0 \in M$), *e.g.*, $u_\infty(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}\left(u_n(x) - u_n(x_0)\right)$.

Remark 1.4.20. Using this characterization, the graph property of the Aubry set (Theorem 1.3.16) follows easily from property 10 in proposition 1.4.18. Moreover, the non-emptiness of $\widetilde{\mathcal{N}}_c$ is a result of the non-emptiness of $\widetilde{\mathfrak{I}}(u)$. As far as the non-emptiness of $\widetilde{\mathcal{A}}_c$ is concerned, one can deduce it from this characterization and proposition 1.4.27.

From theorem 1.4.19 one can also deduce another interesting property of critically dominated functions: their differentiability on the projected Aubry set (recall that a-priori these functions are only Lipschitz, so they are differentiable almost everywhere).

Proposition 1.4.21. *Let $u \prec L_\eta + \alpha(c)$. For each $x \in \mathcal{A}_c$, u is differentiable at x and $d_x u$ does not depend on u; namely, $d_x u = \dfrac{\partial L}{\partial v}(x, \pi^{-1}_{|_{\widetilde{\mathcal{A}}_c}}(x))$.*

In addition to the Aubry set and Mañé set, one can also recover the definition of Peierls barrier h_η (see (1.13)) in terms of these functions. Let us start by observing that, from definition 1.4.2, if $u \prec L_\eta + \alpha(c)$ then for each $x, y \in M$ and $t > 0$ we have that $u(y) - u(x) \leq h_{\eta,t}(x,y) + \alpha(c)t$, where $h_{\eta,t}$ has been defined in (1.12). This implies that for each $u \prec L_\eta + \alpha(c)$ and for each $x, y \in M$, $u(y) - u(x) \leq h_\eta(x,y)$, *i.e.*,

$$h_\eta(x,y) \geq \sup_{u \prec L_\eta + \alpha(c)} [u(y) - u(x)] \qquad \forall\, x, y \in M\,.$$

Moreover, if $u_- \in \mathcal{S}^-_\eta$ and $u_+ \in \mathcal{S}^+_\eta$ are conjugate solutions, the same result holds: $u_-(y) - u_+(x) \leq h_\eta(x,y)$ and consequently

$$h_\eta(x,y) \geq \sup_{(u_-, u_+)} [u_-(y) - u_+(x)] \qquad \forall\, x, y \in M\,,$$

where (u_-, u_+) denotes conjugate weak KAM solutions. In addition to this, it is possible to show that the above inequality is actually an equality. In fact:

Proposition 1.4.22 (Fathi). *For $x \in M$ let us define the function $h_\eta^x : M \to \mathbb{R}$ (resp. $h_{\eta,x} : M \to \mathbb{R}$) by $h_\eta^x(y) = h_\eta(x, y)$ (resp. $h_{\eta,x}(y) = h_\eta(y, x)$). For each $x \in M$, the function h_η^x (resp. $-h_{\eta,x}$) is a weak KAM solution of negative (resp. positive) type. Moreover, its conjugate function $u_+^x \in \mathcal{S}_\eta^+$ (resp. $u_-^x \in \mathcal{S}_\eta^-$) vanishes at x.*

As a consequence of this:

Corollary 1.4.23. *For each $x, y \in M$, we have the equality*

$$h_\eta(x, y) = \sup_{(u_-, u_+)} [u_-(y) - u_+(x)],$$

where the supremum is taken over pairs of conjugate solutions. Moreover, for any given $x, y \in M$ this supremum is actually attained.

Observe that, since for any $u \prec L_\eta + \alpha(c)$ there exists a weak KAM solution of negative type u_- and a weak KAM solution of positive type u_+, such that $u_- = u = u_+$ on the projected Aubry set $\mathcal{A}_c$ (see Theorem 1.4.15), one can get the following representations for Peierls barrier h_η on the projected Aubry set:

$$h_\eta(x, y) \quad = \quad \sup_{u \prec L_\eta + \alpha(c)} [u(y) - u(x)] \tag{1.20}$$

for all $x, y \in \mathcal{A}_c$. This supremum is actually attained for any fixed $x \in \mathcal{A}_c$.

Moreover, a similar representation also holds for Mather's pseudodistance δ_c (see (1.14)). In fact, from the definition of $\delta_c(x, y)$ we immediately get:

$$\begin{aligned}
\delta_c(x, y) &= h_\eta(x, y) + h_\eta(y, x) = \\
&= \sup_{u \prec L_\eta + \alpha(c)} (u(y) - u(x)) + \sup_{v \prec L_\eta + \alpha(c)} (v(x) - v(y)) = \\
&= \sup_{u, v \prec L_\eta + \alpha(c)} [(u(y) - v(y)) - (u(x) - v(x))] \qquad (1.21)
\end{aligned}$$

for all $x, y \in \mathcal{A}_c$. This supremum is also attained for any fixed $x, y \in \mathcal{A}_c$.

ADDENDA

In these addenda we want to say more about the regularity of critically dominated functions or η-critical subsolutions and prove a result about the recurrent points of the Euler-Lagrange flow restricted to the Mañé set that we mentioned in relation with theorem 1.3.17.

Regularity of critical subsolutions

We have remarked above in this section, that for $k < \alpha(c)$ there do not exist functions dominated by $L_\eta + k$, while for $k \geq \alpha(c)$ they do exist. Moreover, if $k > \alpha(c)$ these functions can be chosen to be C^∞ (see also characterization of $\alpha(c)$ in remark 1.4.7). The critical case has totally different features. As a counterpart of their relation with the dynamics of the system, critical dominated functions have very rigid structural properties, that become an obstacle when someone tries to make them smoother. For instance, as we have recalled in proposition 1.4.21, if $u \prec L_\eta + \alpha(c)$ then its differential $d_x u$ exists on $\mathcal{A}_c$ and it is prescribed over there. This means that although it is quite easy to make these functions smoother (*e.g.*, C^∞) out of the projected Aubry set, it is impossible to modify them on this set.

Nevertheless, Albert Fathi and Antonio Siconolfi [37] managed to prove that C^1 η-critical subsolutions do exist and are dense, in the following sense:

Theorem 1.4.24 (Fathi, Siconolfi). *Let $u \prec L_\eta + \alpha(c)$. For each $\varepsilon > 0$, there exists a C^1 function $\tilde{u} : M \longrightarrow \mathbb{R}$ such that:*

i) $\tilde{u} \prec L_\eta + \alpha(c)$*;*

ii) $\tilde{u}(x) = u(x)$ *on* $\mathcal{A}_c$*;*

iii) $|\tilde{u}(x) - u(x)| < \varepsilon$ *on* $M \setminus \mathcal{A}_c$*.*

Moreover, one can choose $\tilde{u}$ *so that it is a strict* η*-critical subsolution, i.e., we have* $H_\eta(x, d_x\tilde{u}) < \ a(c)$ *on* $M \setminus \mathcal{A}_c$*.*

Remark 1.4.25. This result has been extended by Patrick Bernard [13], who showed that every η-critical subsolution coincides, on the Aubry set, with a $C^{1,1}$ η-critical subsolution. In general, this is the best regularity that one can expect: it is easy in fact to construct examples in which C^2 η-critical subsolutions do not exist. For example, consider the case in which the Aubry set projects over all the manifold M and it is not a C^1 graph (*e.g.*, on $M = \mathbb{T}$ take $L(x, v) = \frac{1}{2}\|v\|^2 + \sin^2(\pi x)$ and $\eta = \frac{2}{\pi}dx$). In this case there is only one critical subsolution (up to constants), that is an actual solution: its differential is Lipschitz but not C^1.
It is therefore clear that the structure of the Aubry set plays a crucial role. Patrick Bernard [14] proved that if the Aubry set is a union of finitely many hyperbolic periodic orbits or hyperbolic fixed points, then smoother subsolutions can be constructed. In particular, if the Hamiltonian is C^k, then these subsolutions will be C^k too. The proof of this result is heavily based on the hyperbolic structure of the Aubry set and the result is deduced from the regularity of its local stable and unstable manifolds.

Using the density of C^1 critically dominated functions, one can emprove some of the results in 1.19, theorem 1.4.19, (1.20) and (1.21). Let us denote by $\mathcal{S}^1_\eta$ the set of C^1 η-critical subsolutions and $\mathcal{S}^{1,1}_\eta$ the set of $C^{1,1}$ η-critical subsolutions. Then:

$$\begin{aligned}\widetilde{\mathcal{A}}_c &:= \bigcap_{u \in \mathcal{S}^1_\eta} \widetilde{\mathfrak{I}}(u) = \bigcap_{u \in \mathcal{S}^{1,1}_\eta} \widetilde{\mathfrak{I}}(u) = \\ &= \bigcap_{u \in \mathcal{S}^1_\eta} \mathcal{L}^{-1}\left(\mathrm{Graph}(\eta + du)\right) = \bigcap_{u \in \mathcal{S}^{1,1}_\eta} \mathcal{L}^{-1}\left(\mathrm{Graph}(\eta + du)\right). \end{aligned} \qquad (1.22)$$

In particular, there exists a $C^{1,1}$ η-critical subsolution $\tilde{u}$ such that:

$$\begin{aligned}\widetilde{\mathcal{A}}_c &= \mathcal{L}^{-1}\left(\mathrm{Graph}(\eta + d\tilde{u}) \cap \mathcal{E}_c^*\right) = \\ &= \widetilde{\mathfrak{I}}(\tilde{u}) \cap \mathcal{L}^{-1}\left(\mathcal{E}_c^*\right). \end{aligned} \tag{1.23}$$

Moreover, for any x, $y \in \mathcal{A}_c$:

$$h_\eta(x,y) = \sup_{u \in \mathcal{S}^1_\eta} \{u(y) - u(x)\} = \sup_{u \in \mathcal{S}^{1,1}_\eta} \{u(y) - u(x)\} \tag{1.24}$$

$$\begin{aligned}\delta_c(x,y) &= \sup_{u,v \in \mathcal{S}^1_\eta} \{(u-v)(y) - (u-v)(x)\} = \\ &= \sup_{u,v \in \mathcal{S}^{1,1}_\eta} \{(u-v)(y) - (u-v)(x)\}, \end{aligned} \tag{1.25}$$

where all the above suprema are maxima for any fixed $x, y \in \mathcal{A}_c$.

Non-wandering points of the Mañé set

We want now to show that the non-wandering set of the Euler-Lagrange flow restricted to the Mañé set is contained in the Aubry set. We have already recalled this result for sketching a proof of the second equivalence in theorem 1.3.17. Let us first recall the definition of *non-wandering point* for a flow $\Phi_t : X \longrightarrow X$.

Definition 1.4.26. *A point $x \in X$ is called non-wandering if for each neighborhood $\mathcal{U}$ and each positive integer n, there exists $t > n$ such that $f^t(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$.*

We will denote the set of *non-wandering* points for Φ_t by $\Omega(\Phi_t)$.

Proposition 1.4.27. *If M is a compact manifold and L a Tonelli Lagrangian on $\mathrm{T}M$, then $\Omega\left(\Phi^L_t \big| \widetilde{\mathcal{N}}_c\right) \subseteq \widetilde{\mathcal{A}}_c$ for each $c \in \mathrm{H}^1(M;\mathbb{R})$.*

Remarks.

1) Proposition 1.4.27 also shows that the Aubry set is non-empty. In fact, any continuous flow on a compact space possesses non-wandering points.

2) Since every point in the support of an invariant measure μ is non-wandering, then this also shows that $\widetilde{\mathcal{M}}_c \subseteq \widetilde{\mathcal{A}}_c$.

Proof. Let $(x,v) \in \Omega\left(\Phi_t^L|\widetilde{\mathcal{N}}_0\right)$. By the definition of non-wandering point, there exist a sequence $(x_k,v_k) \in \widetilde{\mathcal{N}}_c$ and $t_k \to +\infty$, such that $(x_k,v_k) \to (x,v)$ and $\Phi_{t_k}^L(x_k,v_k) \to (x,v)$ as $k \to +\infty$. From (1.22), for each (x_k,v_k) there exists a η critical subsolution u_k, such that the curve $\gamma_k(t) = \pi\left(\Phi_t^L(x_k,v_k)\right)$ is $(u_k, L_\eta, \alpha(c))$ - calibrated. Moreover, up to extracting a subsequence, we can assume that, on any compact interval, γ_k converge in the C^1-topology to $\gamma(t) = \pi\left(\Phi_t^L(x,v)\right)$.

Pick now any critical subsolution u. If we show that γ is $(u, L_\eta, \alpha(c))$ - calibrated, using (1.22) we can conclude that $(x,v) \in \widetilde{\mathcal{A}}_c$. First of all, observe that, by the continuity of u,

$$u(\gamma_k(t_k)) - u(x_k) \stackrel{k\to\infty}{\longrightarrow} 0 \ .$$

Using that η-critical subsolutions are equi-Lipschitz (as remarked after definition 1.4.2), we can also conclude that

$$u_k(\gamma_k(t_k)) - u_k(x_k) \stackrel{k\to\infty}{\longrightarrow} 0$$

and, therefore,

$$\int_0^{t_k} L_\eta(\gamma_k(s), \dot\gamma_k(s)) + \alpha(c)\, ds = u_k(\gamma_k(t_k)) - u_k(x_k) \stackrel{k\to\infty}{\longrightarrow} 0 \ . \qquad (1.26)$$

Let $0 \leq a \leq b$ and choose $t_k \geq b$. Observe now that

$$\begin{aligned}
u(\gamma_k(b)) - u(\gamma_k(a)) &= u(\gamma_k(t_k)) - u(x_k) - [u(\gamma_k(t_k)) - u(\gamma_k(b))] - \\
&\quad - [u(\gamma_k(a)) - u(x_k)] \geq \\
&\geq u(\gamma_k(t_k)) - u(x_k) - \int_b^{t_k} L_\eta(\gamma_k(s), \dot{\gamma}_k(s)) + \alpha(c)\, ds - \\
&\quad - \int_0^a L_\eta(\gamma_k(s), \dot{\gamma}_k(s)) + \alpha(c)\, ds = \\
&= u(\gamma_k(t_k)) - u(x_k) + \int_a^b L_\eta(\gamma_k(s), \dot{\gamma}_k(s)) + \alpha(c)\, ds - \\
&\quad - \int_0^{t_k} L_\eta(\gamma_k(s), \dot{\gamma}_k(s)) + \alpha(c)\, ds\,;
\end{aligned}$$

taking the limit as $k \to \infty$ on both sides, one can conclude:

$$u(\gamma(b)) - u(\gamma(a)) \geq \int_a^b L_\eta(\gamma(s), \dot{\gamma}(s)) + \alpha(c)\, ds$$

and therefore, from the fact that $u \prec L_\eta + \alpha(c)$, it follows the equality. This shows that γ is $(u, L_\eta, \alpha(c))$-calibrated on $[0, \infty)$. To show that it is indeed calibrated on all $\mathbb{R}$, one can make a symmetric argument, letting $(y_k, w_k) = \Phi^L_{t_k}(x_k, v_k)$ play the role of (x_k, v_k) in the previous argument. In fact, one has $(y_k, w_k) \to (x, v)$ and $\Phi^L_{-t_k}(y_k, w_k) \to (x, v)$ as $k \to +\infty$ and the very same argument works.

□

Chapter 2

On the structure of the quotient Aubry set

2.1 The quotient Aubry set

In John Mather's studies on the dynamics of Lagrangian systems and the existence of Arnold diffusion, it turns out to be useful to understand certain aspects of the *Aubry set* and, in particular, what is called the *quotient Aubry set*. In fact, in many variational methods that have been developed for construncting orbits with prescribed behaviors [11, 12, 25, 61, 63, 65, 78], the structure of this set plays a crucial role. In this chapter, we want to discuss some its topological properties, in particular its total disconnectedness [75].

Let us start this section by recalling the definition of the quotient Aubry set (see also remark 1.3.10). The setting will be the same as in chapter 1: let M be a compact and connected smooth manifold without boundary, $L : \mathrm{T}M \longrightarrow \mathbb{R}$ a Tonelli Lagrangian (definition 1.1.1) and $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ the associated optical Hamiltonian (definition 1.1.3). Moreover, for each $c \in \mathrm{H}^1(M;\mathbb{R})$ and η closed 1-form with cohomology class $[\eta] = c$, we will denote L_η and H_η the corresponding modified Lagrangian and Hamiltonian (see remark 1.2.1).

In (1.15) we have defined the *projected Aubry set* (with cohomology c), as

$$\mathcal{A}_c = \{x \in M : \ \delta_c(x,x) = 0\} = \{x \in M : \ h_\eta(x,x) = 0\},$$

where h_η is *Peierls Barrier* (see (1.13)) and δ_c its symmetrization, also called *Mather's pseudometric* (see (1.14)).

Definition 2.1.1 (Quotient Aubry set). *The quotient Aubry set* $(\bar{\mathcal{A}}_c, \bar{\delta}_c)$ *is the metric space obtained by identifying two points in* $\mathcal{A}_c$ *if their* δ_c*-pseudodistance is zero.*

We will denote an element of this quotient by $\bar{x} = \{y \in \mathcal{A}_c : \ \delta_c(x,y) = 0\}$. These elements, that are also called *c-static classes* (see [23]), provide a partition of $\mathcal{A}_c$ into compact subsets, that can be lifted to invariant subsets of $\mathrm{T}M$. They are really interesting from a dynamical systems point of view, since they contain the α and ω limit sets of c-minimizing orbits. Using this definition, one can rephrase definition 1.3.14 in the following way: $\gamma : \mathbb{R} \longrightarrow M$ is a c-regular minimizer if it is a c-minimizer and the α and ω-limit sets are both contained in the same c-static class (*i.e.,* in the same element of this quotient set). In fact, if $\gamma : \mathbb{R} \longrightarrow M$ is a c-minimizer, then its α and ω-limit sets are each contained in a c-static class, which might be a-priori different (see remark 1.3.13).

Remark 2.1.2. In general one could ask if given two different c-static classes $\overline{x_\alpha}$ and $\overline{x_\omega}$, there exists a c-minimizer "connecting" them, *i.e.,* such that its α-limit set is contained in $\overline{x_\alpha}$ and its ω-limit set is contained in $\overline{x_\omega}$. In general, this is not true. Consider, for instance, a mechanical Lagrangian on the 1-dimensional torus: in this case the Aubry set is made of equilibria points (exactly the global minima of the potential) and it is definitely not true that there exist heteroclinic connections among any two of them.

A more correct way to formulate this question might be the following. Define a reflexive partial order $\trianglelefteq$ in $\bar{\mathcal{A}}_c$ by: a) $\trianglelefteq$ is reflexive; b) $\trianglelefteq$ is transitive; c) if there exists a c-minimizer γ connecting $\overline{x_\alpha}$ to $\overline{x_\omega}$, then $\overline{x_\alpha} \trianglelefteq \overline{x_\omega}$. Gonzalo Contreras and Gabriel Paternain [25] proved that if the quotient Aubry set is finite, then for any given $\overline{x}, \overline{y} \in \bar{\mathcal{A}}_c$ we have that $\overline{x} \trianglelefteq \overline{y}$. In other words, there exist $\overline{x_1} = \overline{x}, \ldots, \overline{x_n} = \overline{x}$ and c-minimizers $\gamma_1, \ldots, \gamma_{n-1}$ such that for each $k = 1 \ldots, n$ the α-limit set of γ_k is contained in $\overline{x_k}$ and its ω-limit set in $\overline{x_{k+1}}$. Therefore, between any two given static classes there exists a chain of c-minimizers connecting them.
John Mather in [64] proved a more general statement (it was proved in the the non-autonomous case). If $\overline{x}, \overline{y} \in \bar{\mathcal{A}}_c$ and $\overline{x} \neq \overline{y}$, then there exists a subset $\mathfrak{S} = \mathfrak{S}_{\overline{x},\overline{y}} \subseteq \bar{\mathcal{A}}_c$ and an order $\trianglelefteq$ on $\mathfrak{S}$ such that: i) $\overline{x}$ is the least and $\overline{y}$ the greatest element of $\mathfrak{S}$; ii) the topology of $\mathfrak{S}$ associated to the metric $\bar{\delta}_c$ is the same as the topology associated to the order; iii) $\mathfrak{S}$ is compact with respect to this topology; iv) if $\overline{x_1}, \overline{x_2} \in \mathfrak{S}$ and $\overline{x_1} \trianglelefteq \overline{x_2}$, then there exists a c-minimizer $\gamma_{\overline{x_1},\overline{x_2}}$ such that its α-limit set is contained in $\overline{x_1}$ and its ω-limit set in $\overline{x_2}$.

While in the case of twist maps (see for instance [8, 38] and references therein) there is a detailed structure theory for these sets, in more degrees of freedom quite few is known. In particular, it seems to be useful to understand whether the quotient Aubry set is "small" in some sense of dimension (*e.g.,* vanishing topological or Hausdorff dimension).

In [62] Mather showed that if the state space has dimension ≤ 2 (in the non-autonomous case) or the Lagrangian is the kinetic energy associated to a Riemannian metric and the state space has dimension ≤ 3, then the quotient Aubry set is totally disconnected, *i.e.,* every connected component consists of a single point (in a compact metric space this is equivalent to vanishing topological dimension). In the autonomous case, with $\dim M \leq 3$, the same argument shows that this quotient is totally disconnected as long as the Aubry set does not intersect the zero section of TM (this is the case when the cohomology class is large enough in norm). See also the addendum to section 2.2.

What happens in higher dimension? Unfortunately, this is generally not true. In fact, Burago, Ivanov and Kleiner in [18] provided an example that does not satisfy this property (they did not discuss it in their work, but it follows from the results therein). More strikingly, Mather provided in [64] several examples of quotient Aubry sets that are not only non-totally-disconnected, but even isometric to closed intervals. All these examples come from mechanical Lagrangians on T$\mathbb{T}^d$ (*i.e.,* the sum of the kinetic energy and a potential) with $d \geq 3$. In particular, for every $\varepsilon > 0$, he provided a potential $U \in C^{2d-3,1-\varepsilon}(\mathbb{T}^d)$, whose associated quotient Aubry set is isometric to an interval. As the author himself noticed, it is not possible to improve the differentiability of these examples, due to the construction carried out. See the addendum to section 2.2 for such counterexamples. It is interesting to notice that Mather's construction is quite similar, in spirit, to the one used by Whitney for constructing a counterexample to Sard's lemma [77]. We will see in section 2.2 that this relation is not accidental at all and will represent one of the key ingredients in the proof of our main result.

In the following sections we will discuss the problem of the total disconnectedness of $\bar{\mathcal{A}}_c$ in higher dimension and prove it under some extra assumptions on the Lagrangian. In particular, it will follow from our main result that the above counterexamples are optimal, in the sense that for more regular mechanical Lagrangians the quotient Aubry set - corresponding to the zero cohomology class - is totally disconnected.

2.2 Sard's Lemma and the quotient Aubry set

In this section we want to discuss the relation between Sard's lemma and the total disconnectedness of the quotient Aubry set. In order to make it clear, it is convenient to adopt *weak KAM theory*'s point of view, introduced in section 1.4. In the following, we will denote by $\mathcal{S}_\eta$ the set of η-critical subsolutions (or critically dominated functions, see definition 1.4.2 and proposition 1.4.3), by $\mathcal{S}^1_\eta$ the set of C^1 η-critical subsolutions and by $\mathcal{S}^{1,1}_\eta$ the set of $C^{1,1}$ η-critical subsolutions. We have already remarked in section 1.4 and its addendum, that these sets are non-empty and in particular $\mathcal{S}^1_\eta$ and $\mathcal{S}^{1,1}_\eta$ are both "dense" in $\mathcal{S}_\eta$ (see theorem 1.4.24 and [13, 37] for the exact statements). We also showed how to get an expression for Peierls Barrier h_η and Mather's pseudometric δ_c in terms of these functions (see (1.20), (1.21) and in particular (1.24) and (1.25)).

It turns out that is convenient to characterize the elements of $\bar{\mathcal{A}}_c$ (*i.e.,* the c - quotient classes) in terms of η-critical subsolutions. Let us consider the following set:

$$\mathcal{D}_c = \{u - v : \; u,\, v \in \mathcal{S}_\eta\}; \tag{2.1}$$

observe that it depends only on the cohomology class c and not on η (differently from the set of critical subsolutions). Similarly, denote by $\mathcal{D}^1_c$ and $\mathcal{D}^{1,1}_c$, the sets corresponding, respectively, to the difference of C^1 and $C^{1,1}$ η-critical subsolutions. With this new notation (1.25) becomes:

$$\begin{aligned} \delta_c(x,y) &= \sup_{w \in \mathcal{D}_c} (w(y) - w(x)) = \sup_{w \in \mathcal{D}^1_c} (w(y) - w(x)) = \\ &= \sup_{w \in \mathcal{D}^{1,1}_c} (w(y) - w(x)) \end{aligned} \tag{2.2}$$

for any $x,\, y \in \mathcal{A}_c$. Moreover, this suprema are achieved for any fixed $x,\, y \in \mathcal{A}_c$.

Let us start to study some properties of these functions.

Proposition 2.2.1. *If* $w \in \mathcal{D}_c$*, then* $d_x w = 0$ *on* $\mathcal{A}_c$*. Therefore* $\mathcal{A}_c \subseteq \bigcap_{w\in\mathcal{D}_c^{1,1}} \mathrm{Crit}(w)$ *, where* $\mathrm{Crit}(w)$ *is the set of critical points of* w*.*

Proof. This is an immediate consequence of proposition 1.4.21; namely, if $u,\, v \in \mathcal{S}_\eta$, then they are differentiable on $\mathcal{A}_c$ and $d_x u = d_x v$. □

Remark 2.2.2. It is actually possible to show that $\mathcal{A}_c = \bigcap_{w\in\mathcal{D}_c^{1,1}} \mathrm{Crit}(w)$.

Proposition 2.2.3. *If* $w \in \mathcal{D}_c$*, then it is constant on any quotient class of* $\bar{\mathcal{A}}_c$*; namely, if* $x,\, y \in \mathcal{A}_c$ *and* $\delta_c(x,y) = 0$*, then* $w(x) = w(y)$*.*

Proof. It follows easily from (2.2). In fact:

$$
\begin{aligned}
0 &= \delta_c(x,y) = \sup_{\tilde{w}\in\mathcal{D}_c} (\tilde{w}(y) - \tilde{w}(x)) \geq w(y) - w(x) \\
0 &= \delta_c(y,x) = \sup_{\tilde{w}\in\mathcal{D}_c} (\tilde{w}(x) - \tilde{w}(y)) \geq w(x) - w(y)\,.
\end{aligned}
$$

□

For any $w \in \mathcal{D}_c^1$, let us define the following *evaluation function*:

$$
\begin{aligned}
\varphi_w : (\bar{\mathcal{A}}_c,\, \bar{\delta}_c) &\longrightarrow (\mathbb{R},\, |\cdot|) \\
\bar{x} &\longmapsto w(x)\,.
\end{aligned}
$$

- φ_w is well defined, *i.e.*, it does not depend on the element of the class at which w is evaluated;
- $\varphi_w(\bar{\mathcal{A}}_c) = w(\mathcal{A}_c) \subseteq w(\mathrm{Crit}(w))$;
- φ_w is Lipschitz, with Lipschitz constant 1. In fact:

$$
\begin{aligned}
\varphi_w(\bar{x}) - \varphi_w(\bar{y}) &= w(x) - w(y) \leq \delta_c(x,y) = \bar{\delta}_c(\bar{x},\bar{y}) \\
\varphi_w(\bar{y}) - \varphi_w(\bar{x}) &= w(y) - w(x) \leq \delta_c(y,x) = \bar{\delta}_c(\bar{y},\bar{x})\,.
\end{aligned}
$$

Therefore:

$$|\varphi_w(\bar{x}) - \varphi_w(\bar{y})| \leq \bar{\delta}_c(\bar{x}, \bar{y})\,.$$

Remark 2.2.4. In some sense, (2.2) corresponds to reconstruct a metric knowing its 1-Lipschitz functions (*i.e.,* Lipschitz functions with Lipschitz constant equal to 1).

Now the relation with Sard's lemma can be stated more clearly.

Definition 2.2.5. *A C^1 function $f : M \longrightarrow \mathbb{R}$ is of Morse-Sard type if $|f(\mathrm{Crit}(f))| = 0$, where $\mathrm{Crit}(f)$ is the set of critical points of f and $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.*

Proposition 2.2.6. *Let M be a compact connected manifold of dimension $d \geq 1$, L a Tonelli Lagrangian and $c \in H^1(M;\mathbb{R})$. If each $w \in \mathcal{D}_c^{1,1}$ is of Morse-Sard type, then the quotient Aubry set $(\bar{\mathcal{A}}_c,\, \bar{\delta}_c)$ is totally disconnected.*

Proof. Suppose by contradiction that $\bar{\mathcal{A}}_c$ is not totally disconnected; therefore it must contain a connected component $\overline{\Gamma}$ with at least two points $\bar{x}$ and $\bar{y}$. In particular $\bar{\delta}_c(\bar{x}, \bar{y}) > 0$ for some $x \in \bar{x}$ and $y \in \bar{y}$. From the representation formula for δ_c, it follows that there exists $w \in \mathcal{D}_c^{1,1}$, such that $w(x) \neq w(y)$. This implies that the set $\varphi_w(\overline{\Gamma})$ is a connected set in $\mathbb{R}$ with at least two different points, hence it is a non-degenerate interval and its Lebesgue measure is positive. But

$$\varphi_w(\overline{\Gamma}) \subseteq \varphi_w(\bar{\mathcal{A}}_c) = w(\mathcal{A}_c) \subseteq w(\mathrm{Crit}(w)$$

and we get a contradiction

$$0 < \left|\varphi_w(\overline{\Gamma})\right| \leq |w(\mathcal{A}_c)| \leq |w(\mathrm{Crit}(w))| = 0\,.$$

□

Remark 2.2.7. It is clear from the proof that it would be sufficient that Morse-Sard's property held for a dense subset of $\mathcal{D}_c^{1,1}$. Moreover, one could ask w to satisfy only the weaker condition $|w(\mathcal{A}_c)| = 0$.

This proposition and Sard's lemma easily prove total disconnectedness in dimension $d \leq 2$ (see also the addendum at the end of this section and [62], where the non-autonomous case was considered). In fact, it suffices to notice that Sard's lemma in dimension d holds for $C^{d-1,1}$ functions [10].

Corollary 2.2.8. *Let M be a compact connected manifold of dimension $d \leq 2$. For any L Tonelli Lagrangian and $c \in H^1(M;\mathbb{R})$, the quotient Aubry set $(\bar{\mathcal{A}}_c, \bar{\delta}_c)$ is totally disconnected.*

The main problem becomes now to understand under which conditions on L and c, these differences of subsolutions are of *Morse-Sard type*. Unfortunately, one cannot use the classical Sard's lemma, due to a lack of regularity of critical subsolutions. As we have already discussed in remark 1.4.25, in general the best regularity one can get is $C^{1,1}$. In fact, although it is always possible to smooth them up out of the Aubry set and obtain functions in $C^\infty(M \setminus \mathcal{A}_c) \cap C^{1,1}(M)$, the presence of the Aubry set (where the value of their differential is prescribed) represents an obstacle that it is impossible to overcome. On the other hand, in the proof of Sard's lemma regularity is used to gain a control on the behavior of the function near the critical set, *i.e.,* to control the *complexity* (*à la* Yomdin) of the function near this set. In absence of such a condition, one can try to resort to some other properties of the function that may allow one to get a similar control. In our case, these functions are related to Hamilton-Jacobi equation and we will see how the smoothness of the Hamiltonian, rather than the regularity of the subsolutions, may provide for our needs. There are several difficulties involved in pursuing such an approach in the general case, mostly related to the nature of the Aubry set; at any rate, we will see in the next section that this becomes possible with some extra assumptions on the Lagrangian.

ADDENDA

Total disconnectedness in low dimension

In this addendum we want to recall Mather's result in low dimension [60, 62].

Proposition 2.2.9.

i) If M is a compact manifold of dimension ≤ 2 and $L : \mathrm{T}M \times \mathbb{T} \longrightarrow \mathbb{R}$ is a Tonelli Lagrangian, then $\bar{\mathcal{A}}_c$ is totally disconnected for all $c \in \mathrm{H}^1(M;\mathbb{R})$.

ii) If M is a compact manifold of dimension ≤ 3 and $L : \mathrm{T}M \longrightarrow \mathbb{R}$ is a Tonelli Lagrangian, such that $\widetilde{\mathcal{A}}_c$ does not intersect the zero section of $\mathrm{T}M$, then $\bar{\mathcal{A}}_c$ is totally disconnected.

iii) If L is the kinetic energy associated to a Riemannian metric on M and $\dim M \leq 3$, then $\bar{\mathcal{A}}_c$ is totally disconnected for all $c \in \mathrm{H}^1(M;\mathbb{R})$.

In the following we want to give an idea of the proof of such results. One of the key ingredient in the proof will be the following result by John Mather (we refer to [60] for a proof).

Proposition 2.2.10. *Let M be a smooth finite dimensional manifold with a Riemannian metric g and let X be a compact connected subset of M that admits a Lipschitz lamination*[1] *of codimension $d \geq 2$. Let $x, y \in X$ and $\varepsilon > 0$; then, there exists a sequence $x = x_0, \ldots, x_k = y$ of points in X such that $\sum_{i=0}^{k-1} \delta_M(x_i, x_{i+1})^d < \varepsilon$, where δ_M is the metric associated to the Riemannian metric g.*

[1]By Lipschitz lamination we mean an $\mathcal{S}$-smooth lamination, in the sense of Thurston, where $\mathcal{S}$ is the class of bi-Lipschitz homeomorphisms; see for instance [60] for a more precise defintion.

Proof (Proposition 2.2.9). Let us start recalling that from the graph property of the Aubry set (see theorem 1.3.16 or [59]), it follows that $\mathcal{A}_c$ has a 1-dimensional Lipschitz lamination, which is defined by the flow on it, generated by the vector field $\pi^{-1} : \mathcal{A}_c \longrightarrow \widetilde{\mathcal{A}}_c$. Moreover - see remark 1.3.12 - recall that

$$\delta_c((x,\tau_0),(y,\tau_1)) \leq C[d(x,y) + \|\tau_1 - \tau_0\|]^2 \tag{2.3}$$

for each $(x,\tau_0),(y,\tau_1) \in \mathcal{A}_c$, where

$$\|\tau_1 - \tau_0\| = \inf\left\{|t_1 - t_0| :\ t_i \in \mathbb{R},\ t_i \equiv \tau_i\,(\text{mod. } 1),\ i = 0,1\right\}.$$

i) If $\dim M \leq 2$, the 1-dimensional lamination of $\mathcal{A}_c$ is of codimension ≤ 2. Hence, applying proposition 2.2.10 with $d = 2$, we get that if $\overline{(x,\tau)}$ and $\overline{(x',\tau')}$ are two points in the same connected component of $\bar{\mathcal{A}}_c$ and $\varepsilon > 0$, then there is a finite sequence of points $(x_0,\tau_0) = (x,\tau),\ldots,(x_k,\tau_k) = (x',\tau')$ in that connected component such that

$$\sum_{i=0}^{k-1}[d(x_i,x_{i+1}) + \|\tau_{i+1} - \tau_i\|]^2 < \varepsilon\,.$$

Therefore, using (2.3) and the triangle inequality for δ_c, we get:

$$\begin{aligned}
\delta_c((x,\tau),(x',\tau')) &\leq \sum_{i=0}^{k-1}\delta_c((x_i,\tau_i),(x_{i+1},\tau_{i+1})) \leq \\
&\leq C\sum_{i=0}^{k-1}[d(x_i,x_{i+1}) + \|\tau_{i+1} - \tau_i\|]^2 < C\varepsilon\,.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\delta_c((x,\tau),(x',\tau')) = 0$; this means that each connected component of $\bar{\mathcal{A}}_c$ is reduced to a single point.

ii) In the autonomous case, the argument used in i) to show that $\mathcal{A}_c$ admits a

1-dimensional Lipschitz lamination still applies, provided that this section vanishes nowhere. This means that $\widetilde{\mathcal{A}}_c$ does not intersect the zero section of TM.

iii) In the case that L is the kinetic energy associated to a Riemannian metric on M and $c \neq 0$, it is obvious that $\widetilde{\mathcal{A}}_c$ does not intersect the zero section of TM. In this case $\mathcal{A}_c$ admits a 1-dimensional Lipschitz lamination and, since $\dim M \leq 3$, this lamination is of codimension 2. Hence, we can proceed just as in the proof of i).

In the case $c = 0$, the result is trivial since $\mathcal{A}_c = M$ and $\bar{\mathcal{A}}_c$ is one point.

□

Counterexamples in high dimension

As we have recalled in section 2.1, in [64] John Mather has constructed several examples of quotient Aubry sets that are isometric to closed intervals, therefore they are not totally disconnected. All these examples consist of mechanical Lagrangians on T$\mathbb{T}^d$ (*i.e.,* the sum of the kinetic energy and a potential) with $d \geq 3$. In particular, for every $\varepsilon > 0$, he provided a potential $U \in C^{2d-3,1-\varepsilon}(\mathbb{T}^d)$, whose associated quotient Aubry set is isometric to an interval. The basic strategy consists first in defining an appropriate connected compact subset $Z \subset \mathbb{R}^d$ on which the potential will attain its global minimum (it will then coincide with the Aubry set $\mathcal{A}_0$), and then constructing a function U that satisfies some suitable conditions. This construction is very technical and we skip all details, for which we refer to [64]. It is interesting to notice that Mather's construction is quite similar, in spirit, to the one used by Whitney for constructing a counterexample to Sard's lemma [77].

Using this relation with Whitney's counterexample, Albert Fathi explained in [31] how to construct similar counterexamples, but less technically involved. We think that it might be useful to reproduce his unpublished construction here.

Let us start by recalling a result by Patrice Assouad [4] about the bi-Lipschitz embedding of compact metric spaces into an Euclidean space (see also [39] where this result is applied to construct a counterexample to Sard's lemma).

Definition 2.2.11. *Let X be a topological space and δ a metric on it. (X, δ) is said to admit a bi-Lipschitz embedding to $\mathbb{R}^N$, if there exists a mapping $\Psi : X \to \mathbb{R}^N$ such that*

$$C_1\delta(x, y) \leq \|\Psi(x) - \Psi(y)\| \leq C_2\delta(x, y)$$

for all $x, y \in X$ with some fixed constants $C_1, C_2 > 0$.

For instance, any subset of an Euclidean space admits a bi-Lipschitz embedding; moreover, this property is clearly invariant under bi-Lipschitz mappings. A difficult open problem consists in characterizing which metric spaces admit such an embedding.

Definition 2.2.12 (Doubling metric spaces). *Let X be a topological space and δ a metric on it. (X, δ) is called doubling if there exists $K \in \mathbb{N}$ such that any ball $B(x, \varepsilon)$ is covered by at most K balls of radius $\frac{\varepsilon}{2}$, for all $\varepsilon > 0$.*

Every subset of the Euclidean space is doubling and also the doubling condition is invariant under bi-Lipschitz mappings. From this, it follows immediately that a necessary condition for (X, δ) to be bi-Lipschitz embeddable to an Euclidean space, is that (X, δ) is doubling. Unfortunately the doubling condition is not sufficient [41]. However, the following result by Patrice Assouad shows that every doubling metric space admits an "almost" bi-Lipschitz embedding.

Theorem 2.2.13 (Assouad, [4]). *Let (X, δ) be a doubling metric space. Then, for every $0 < s < 1$ there exist N and a bi-Lipschitz embedding $\Psi : (X, \delta^s) \to \mathbb{R}^N$, i.e., a* *mapping such that*

$$C_1\delta(x,y)^s \leq \|\Psi(x)-\Psi(y)\| \leq C_2\delta(x,y)^s$$

for some constants $C_1, C_2 > 0$*, depending on* s*, and for all* $x, y \in X$*.*

Let us see now how to relate this result to the counterexamples mentioned above.

Theorem 2.2.14 (Fathi). *Let* $r \geq 2$ *and* (X,δ) *a doubling metric space. Then, there exist* $N \in \mathbb{N}$ *sufficiently large and a* C^r *mechanical Lagrangian* $L : \mathrm{T}\mathbb{T}^N \to \mathbb{R}$ *such that the quotient Aubry set* $(\bar{\mathcal{A}}_0, \bar{\delta}_0)$ *is bi-Lipschitz equivalent to* (X,δ)*.*

Remark 2.2.15. This construction, contrary to the one in [64], does not provide a clear relation between r and N. It will be only clear that N goes to $+\infty$, as r increases, as it must be because of theorem 2.3.1 (see section 2.3).

Let us try to give a sketch of the proof of such a result.

Proof. From theorem 2.2.13, it follows that for each $p > 1$ there exist N and a bi-Lipschitz embedding $\Psi : (X,\delta) \to (\mathbb{R}^N, \|\cdot\|^p)$, *i.e.,* a mapping such that

$$C^{-1}\|\Psi(x)-\Psi(y)\|^p \leq \delta(x,y) \leq C\|\Psi(x)-\Psi(y)\|^p \tag{2.4}$$

for a positive constant C and for all $x, y \in X$. For the sake of simplicity, we can identify X with $\Psi(X)$, *i.e.,* assume that Ψ is the identity.
Let choose $p > 1$; we will see in the following how to choose p in terms of r. Observe that - as in all metric spaces - we can recover the metric in terms of the Lipschitz functions with Lipschitz constant 1:

$$\delta(x,y) = \sup\{u(y)-u(x) : \ u \text{ is Lipschitz with Lipschitz constant } 1\}.$$

Moreover it follows from (2.4) that for any 1-Lipschitz function u we have:

$$|u(x) - u(y)| \leq C\|x - y\|^p \qquad \text{for all } x, y \in X.$$

Therefore, these functions are $(p-1)$-flat on X and using Whitney extension theorem one can extend them to C^{p-1} functions defined over all $\mathbb{R}^N$, that are flat on X (see section 1.10 for a similar construction). To be more precise, if $u : X \to \mathbb{R}$ is 1-Lipschitz function (with respect to δ), then one can define a C^{p-1} function $\tilde{u} : \mathbb{R}^N \to \mathbb{R}$ such that

i) $\tilde{u} = u$ on X;

ii) $d_x\tilde{u}$ exists on X;

iii) $d_x\tilde{u} = 0$ exactly on X;

iv) $\|d_x\tilde{u}\| \leq \tilde{C}\text{dist}(x, X)^{p-1}$ for each point x in a sufficiently large ball $B(0, R)$.

Next step consists in constructing a mechanical Lagrangian $L(x, v) = \frac{1}{2}\|v\|^2 + U(x)$ on $\mathbb{T}^N$, such that these functions $\tilde{u}$ are 0-critical subsolutions of the associated Hamilton-Jacobi equation. For this purpose, we just need to define a potential U that satisfies suitable growth condition in a neighborhood of X. Up to multiplying U by a constant to obtain the right constant on the left-hand side, one can get:

$$\tilde{C}^2\text{dist}(x, X)^{2(p-1)} \leq U(x) \leq \tilde{C}_1\text{dist}(x, X)^{2(p-1)}, \tag{2.5}$$

where $\tilde{C}$ is the same constant as in iv) and $\tilde{C}_1$ some other positive constant. The usual Whitney argument (see for instance section 2.5 for a similar method) allows one to construct such a function and to show that it is C^{2p-3}. Observe that the set on which U achieves its global minimum is exactly X and therefore the associated Aubry set $\mathcal{A}_0 = X$. It is easy, in fact, to show that for mechanical systems, the projected Aubry set coincides with the set on which the potential U achieves its global minimum (see for instance Lemma 2.3.8 for a similar proof). It remains to show that $(\mathcal{A}_0, \delta_0)$ is bi-Lipschitz equivalent to (X, δ); if this is the case, each static class will consist of just one point and $\mathcal{A}_0 = \bar{\mathcal{A}}_0$, and this will conclude the proof of the theorem (choosing p such that $2p - 3 \geq r$).

From property iv) and (2.5), it follows that on $B(0, R)$:

$$\frac{1}{2}\|d_x u\|^2 \leq C^2 \mathrm{dist}(x, X)^{2p-2} \leq U(x)\,;$$

this shows that $\tilde{u}$ can be modified to be a subsolution of the system, without changing it near the Aubry set. Therefore (see (1.21)):

$$\delta_0(x, y) \geq \sup\{\tilde{u}(y) - \tilde{u}(x) :\ u \text{ is Lipschitz with Lipschitz constant } 1\} = \delta(x, y).$$

Let us prove the other inequality. If v is a 0-critical subsolution, *i.e.*, $H(x, d_x v) \leq \alpha(0) = 0$, then from (2.5) we get: $\frac{1}{2}\|d_x u\|^2 \leq \tilde{C}_1 \mathrm{dist}(x, X)^{2(p-1)}$. Let $x, y \in \mathcal{A}_0$ and $\gamma : [0, \|y - x\|] \to \mathbb{R}^N$ be the unit-speed geodesic joining x to y; integrating, we obtain:

$$\begin{aligned}
v(y) - v(x) &= \int_0^{\|y-x\|} d_{\gamma(s)} v(\dot{\gamma}(s))\, ds \leq \int_0^{\|y-x\|} \|d_{\gamma(s)} v\|\, ds \leq \\
&\leq \int_0^{\|y-x\|} \sqrt{2\tilde{C}_1} \mathrm{dist}(\gamma(s), X)^{p-1}\, ds \leq \\
&\leq \sqrt{2\tilde{C}_1} \int_0^{\|y-x\|} \mathrm{dist}(\gamma(s), x)^{p-1}\, ds \leq \\
&\leq \sqrt{2\tilde{C}_1} \int_0^{\|y-x\|} \|y - x\|^{p-1}\, ds \ \leq \\
&\leq \sqrt{2\tilde{C}_1} \|y - x\|^p \leq C\sqrt{2\tilde{C}_1}\, \delta(x, y);
\end{aligned}$$

this implies $\delta_0(x, y) \leq C\sqrt{2\tilde{C}_1}\, \delta(x, y)$ and concludes the proof of their bi-Lipschitz equivalence. □

2.3 On the total disconnectedness in high dimension

Our main goal in this section is to show that, under suitable hypotheses on L, there is a well specified cohomology class c_L, for which $(\bar{\mathcal{A}}_{c_L}, \bar{\delta}_{c_L})$ is totally disconnected, *i.e.,* every connected component consists of a single point.

As we have already recalled at the end of section 1.1, the cotangent bundle T^*M can be naturally equipped with a canonical symplectic structure $\omega = \sum_{i=1}^d dx_i \wedge dp_i = -d\lambda$, where $\lambda = \sum_{i=1}^d p_i \, dx_i$ is the *Liouville form* (or *tautological form*). Let us denote by $\mathcal{L}$ the Legendre transform introduced in (1.4). Consider now the section of T^*M given by

$$\Lambda_L = \mathcal{L}(M \times \{0\}) = \left\{ \left(x, \frac{\partial L}{\partial v}(x,0) \right) : \quad x \in M \right\},$$

corresponding to the 1-form

$$\eta_L(x) = \frac{\partial L}{\partial v}(x,0) \cdot dx = \sum_{i=1}^d \frac{\partial L}{\partial v_i}(x,0) \, dx_i \, .$$

We would like this 1-form to be closed, that is equivalent to ask Λ_L to be a Lagrangian submanifold (see proposition 3.1.2), in order to consider its cohomology class $c_L = [\eta_L] \in \mathrm{H}^1(M;\mathbb{R})$. Observe that this cohomology class can be defined in a more intrinsic way; in fact the projection

$$\pi_{|\Lambda_L} : \Lambda_L \subset \mathrm{T}^*M \longrightarrow M$$

induces an isomorphism between the cohomology groups $\mathrm{H}^1(M;\mathbb{R})$ and $\mathrm{H}^1(\Lambda_L;\mathbb{R})$. The preimage of $[\lambda_{|\Lambda_L}]$ under this isomorphism is called the *Liouville class* of Λ_L and one can easily show that it coincides with c_L.

We can now define this set:

$$\mathbb{L}(M) = \{L : \mathrm{T}M \longrightarrow \mathbb{R} : \ L \text{ is a Tonelli Lagrangian and } \Lambda_L \text{ is Lagrangian}\} \ .$$

This set is non-empty and consists of Lagrangians of the form

$$L(x,v) = f(x) + \langle \eta(x),\, v\rangle_x + O(\|v\|^2),$$

with $f \in C^2(M)$ and η a C^2 closed 1-form on M.

Examples.

- **Mechanical and Symmetrical Lagrangians.** The set $\mathbb{L}(M)$ includes mechanical Lagrangians, *i.e.,* Lagrangians of the form

  $$L(x,v) = \frac{1}{2}\|v\|_x^2 + U(x)\,,$$

  namely the sum of the kinetic energy and a potential $U : M \longrightarrow \mathbb{R}$, and more generally *symmetrical* (or *reversible*) *Lagrangians*, *i.e.,* Lagrangians $L : \mathrm{T}M \longrightarrow \mathbb{R}$ such that

  $$L(x,v) = L(x,-v)\,,$$

  for every $(x,v) \in \mathrm{T}M$. In fact in both cases, $\frac{\partial L}{\partial v}(x,0) \equiv 0$, therefore $\Lambda_L = M \times \{0\}$ coincides with the zero section of the cotangent space, that is clearly Lagrangian with $c_L = 0$.

- **Mañé's Lagrangians**. Let X be a C^2 vector field on M. We discussed in section 1.1 how to embed its flow into the Euler-Lagrange flow of a Tonelli Lagrangian, namely $L_X(x,v) = \frac{1}{2}\|v - X(x)\|_x^2$.
  It is easy to check that $L_X \in \mathbb{L}(M)$ if and only if X is *irrotational*, *i.e.,* the asso-

ciated 1-form $X^\flat := g(X,\cdot)$ is closed. In particular, $c_{L_X} = [X^\flat] \in \mathrm{H}^1(M;\mathbb{R})$. As a special case, one can consider the Mañé's Lagrangians associated to gradient fields $X = \nabla f$, where $f \in C^3(M;\mathbb{R})$; observe that in this case $c_{L_{\nabla f}} = 0$.

We can now state our main result:[2]

Theorem 2.3.1 (**[75]**). *Let M be a compact connected manifold of dimension $d \geq 1$ and let $L \in \mathbb{L}(M)$ be a Lagrangian such that $L(x,0) \in C^r(M)$, with $r \geq 2d-2$ and $\frac{\partial L}{\partial v}(x,0) \in C^2(M)$. If c_L denotes the Liouville class of Λ_L, then the quotient Aubry set $(\bar{\mathcal{A}}_{c_L},\, \bar{\delta}_{c_L})$ is totally disconnected, i.e., every connected component consists of a single point.*

Remark 2.3.2. Using the same ideas as in [10], the proof of this result can be extended to the case $L(x,0) \in C^{2d-3,1}(M)$.

Remark 2.3.3. Observe that theorem 2.3.1 can be also stated in this form:

Let M be a compact connected manifold of dimension $d \geq 1$ and let L a Tonelli Lagrangian such that $L(x,0) \in C^r(M)$, with $r \geq 2d-2$ (or $L(x,0) \in C^{2d-3,1}$) and $\min_{v \in \mathrm{T}_x M} L(x,v) = L(x,0)$ for all $x \in M$. Then, the quotient Aubry set $(\bar{\mathcal{A}}_0,\, \bar{\delta}_0)$ is totally disconnected.

Lagrangians satisfying this condition are called *unimodal*.

Theorem 2.3.1 easily implies:

Corollary 2.3.4 (**Symmetrical Lagrangians**). *Let M be a compact connected manifold of dimension $d \geq 1$ and let $L(x,v)$ be a symmetrical Tonelli Lagrangian on $\mathrm{T}M$, such that $L(x,0) \in C^r(M)$, with $r \geq 2d-2$. Then, the quotient Aubry set $(\bar{\mathcal{A}}_0,\, \bar{\delta}_0)$ is totally disconnected.*

More specifically,

[2] It is worthy mentioning that a similar result has been also proven independently by Albert Fathi, Alessio Figalli and Ludovic Rifford ([33], in preparation).

Corollary 2.3.5 (Mechanical Lagrangians). *Let M be a compact connected manifold of dimension $d \geq 1$ and let $L(x,v) = \frac{1}{2}\|v\|_x^2 + U(x)$ be a mechanical Lagrangian on $\mathrm{T}M$, such that the potential $U \in C^r(M)$, with $r \geq 2d-2$. Then, the quotient Aubry set $(\bar{\mathcal{A}}_0, \bar{\delta}_0)$ is totally disconnected.*

Remark 2.3.6. This result is optimal in terms of the regularity asked to the potential U. In fact, as we have recalled in section 2.2, Mather [64] constructed examples of quotient Aubry sets isometric to the unit interval, corresponding to mechanical Lagrangians $L \in C^{2d-3,1-\varepsilon}(\mathrm{T}\mathbb{T}^d)$, for any $0 < \varepsilon < 1$.

Moreover, theorem 2.3.1 implies the following result for Mañé's Lagrangians associated to *irrotational* vector fields.

Corollary 2.3.7 (Mañé's Lagrangians). *Let M be a compact connected manifold of dimension $d \geq 1$, equipped with a C^∞ Riemannian metric g. Let X be a C^r vector field on M and consider the associated Mañé's Lagragian $L_X(x,v) = \frac{1}{2}\|v - X(x)\|_x^2$. If $r \geq 2d-2$ then the quotient Aubry set $(\bar{\mathcal{A}}_{c_{L_X}}, \bar{\delta}_{c_{L_X}})$ is totally disconnected, where $X^\flat = g(X, \cdot)$ is the associated 1-form and $c_{L_X} = [X^\flat] \in \mathrm{H}^1(M;\mathbb{R})$.*

Before proving the theorem 2.3.1, it will be useful to show some useful properties. In particular, we will show how the condition $L \in \mathbb{L}(M)$ implies many features of the Aubry set that do not appear in the general case. They will play a key role in the proof of our result.

Lemma 2.3.8. *Let us consider $L \in \mathbb{L}(M)$, such that $\frac{\partial L}{\partial v}(x,0) \in C^2(M)$, and let H be the associated Hamiltonian.*

1. *Every constant function $u \equiv const$ is a η_L-critical subsolution. In particular, all η_L-critical subsolutions are such that $d_x u \equiv 0$ on $\mathcal{A}_{c_L}$.*

2. *For every $x \in M$,*

$$\frac{\partial H_{\eta_L}}{\partial p}(x,0) = \frac{\partial H}{\partial p}(x,\eta_L(x)) = 0\,.$$

Proof.

1. The second part follows immediately from the fact that, if u, $v \in \mathcal{S}_{\eta_L}$, then they are differentiable on $\mathcal{A}_{c_L}$ and $d_x u = d_x v$ (see proposition 1.4.21 and [32]).

   Let us show that $u \equiv const$ is a η_L-critical subsolution; namely, that

$$H_{\eta_L}(x,0) \leq \alpha(c_L)$$

   for every $x \in M$. It is sufficient to observe:

   - $H_{\eta_L}(x,0) = -L(x,0)$; in fact:

$$\begin{aligned} H_{\eta_L}(x,0) &= H(x,\eta_L(x)) = H\left(x, \frac{\partial L}{\partial v}(x,0)\right) = \\ &= \left\langle \frac{\partial L}{\partial v}(x,0),\, 0 \right\rangle_x - L(x,0) = \\ &= -L(x,0)\,. \end{aligned}$$

   - Let v be *dominated* by $L_{\eta_L} + \alpha(c_L)$ (see definition 1.4.2), *i.e.*, for each continuous piecewise C^1 curve $\gamma : [a,b] \longrightarrow M$ we have

$$v(\gamma(b)) - v(\gamma(a)) \leq \int_a^b L_{\eta_L}(\gamma(t),\dot\gamma(t))\,dt + \alpha(c_L)(b-a)\,.$$

     Then, considering the constant path $\gamma(t) \equiv x$, one can easily deduce that

$$\alpha(c_L) \geq \sup_{x\in M}(-L_{\eta_L}(x,0)) = -\inf_{x\in M} L_{\eta_L}(x,0)\,;$$

therefore, for every $x \in M$:

$$\alpha(c_L) \geq -L_{\eta_L}(x,0) = -L(x,0) = H_{\eta_L}(x,0).$$

2. The inverse of the Legendre transform can be written in coordinates

$$\begin{array}{rccc} \mathcal{L}^{-1}: & \mathrm{T}^*M & \longrightarrow & \mathrm{T}M \\ & (x,p) & \longmapsto & \left(x,\, \frac{\partial H}{\partial p}(x,p)\right). \end{array}$$

Therefore,

$$\begin{aligned} (x,0) &= \mathcal{L}^{-1}\left(\mathcal{L}(x,0)\right) = \mathcal{L}^{-1}\left(x,\, \frac{\partial L}{\partial v}(x,0)\right) = \\ &= \mathcal{L}^{-1}((x,\eta_L(x))) = \left(x,\, \frac{\partial H}{\partial p}(x,\eta_L(x))\right). \end{aligned}$$

□

In particular, using that for any η_L-critical subsolution u we have that $H_{\eta_L}(x, d_x u) = \alpha(c_L)$ on $\mathcal{A}_{c_L}$, we can easily deduce that:

$$\mathcal{A}_{c_L} \subseteq \{L(x,0) = -\alpha(c_L)\} = \{H(x,\eta_L(x)) = \alpha(c_L)\}$$

and

$$\alpha(c_L) = \sup_{x\in M}(-L(x,0)) = -\inf_{x\in M} L(x,0) =: e_0;$$

this quantity e_0 has already been introduced in [48, 22], where it is referred to as *strict critical value*. Observe that in general it satisfies

$$e_0 \leq \min_{c\in \mathrm{H}^1(M;\mathbb{R})} \alpha(c) = -\beta(0)\,,$$

where $\beta : \mathrm{H}_1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ is Mather's β-function (see (1.5)); therefore, we are considering an extremal case in which $e_0 = \alpha(c_L) = \min \alpha(c)$; it follows also quite easily that $c_L \in \partial\beta(0)$, namely, it is a subgradient of β at 0.

A crucial step in the proof of our result will be the following lemma, that can be read as a sort of relaxed version of Sard's Lemma.

Lemma 2.3.9. *Let $U \in C^r(M)$, with $r \geq 2d - 2$, be a non-negative function, vanishing somewhere and denote $\mathcal{A} = \{U(x) = 0\}$. If $u : M \longrightarrow \mathbb{R}$ is C^1 and satisfies $\|d_x u\|_x^2 \leq U(x)$ in an open neighborhood of $\mathcal{A}$, then $|u(\mathcal{A})| = 0$ (where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$).*

See section 2.4 for its proof. In particular, when we apply this lemma to our specific case, it implies this essential property, which can be also read as a Sard's lemma for critical subsolutions of Hamilton-Jacobi equation.

Proposition 2.3.10 (Sard's lemma for critical subsolutions). *Under the hypotheses of theorem 2.3.1, if $u \in \mathcal{S}_{\eta_L}$, then $|u(\mathcal{A}_{c_L})| = 0$. In particular, for any $w \in \mathcal{D}_{c_L}$ we have that $|w(\mathcal{A}_{c_L})| = 0$.*

Proof. First of all, we can assume that $u \in \mathcal{S}^1_{\eta_L}$, because of Fathi and Siconolfi's theorem 1.4.24. By Taylor's formula, it follows that there exists an open neighborhood W of $\mathcal{A}_{c_L}$, such that for all $x \in W$:

$$\begin{aligned}\alpha(c_L) \quad &\geq \quad H_{\eta_L}(x, d_x u) = H_{\eta_L}(x, 0) + \frac{\partial H_{\eta_L}}{\partial p}(x,0) \cdot d_x u + \\ &\quad + \int_0^1 (1-t) \frac{\partial^2 H_{\eta_L}}{\partial p^2}(x, t\, d_x u)(d_x u)^2 \, dt\,.\end{aligned}$$

Let us observe the following.

- From the previous lemma, one has that

$$\frac{\partial H_{\eta_L}}{\partial p}(x,0) = 0 \qquad \text{for every } x \in M.$$

- From the strict convexity hypothesis, it follows that there exists $\gamma > 0$ such that:

$$\frac{\partial^2 H}{\partial p^2}(x, t\, d_x u)(d_x u)^2 \geq \gamma \|d_x u\|_x^2$$

for all $x \in M$ and $0 \leq t \leq 1$.

Therefore, for $x \in W$:

$$\begin{aligned} \alpha(c_L) &\geq H_{\eta_L}(x, d_x u) \geq H_{\eta_L}(x, 0) + \frac{\gamma}{2}\|d_x u\|_x^2 = \\ &= -L(x, 0) + \frac{\gamma}{2}\|d_x u\|_x^2 \,. \end{aligned}$$

The assertion will follow from the previous lemma, choosing

$$U(x) = \frac{2}{\gamma}(\alpha(c_L) + L(x, 0))\,.$$

In fact by hypothesis, $U \in C^r$ with $r \geq 2d - 2$; moreover, it satisfies all other conditions since

$$\alpha(c_L) = -\inf_{x \in M} L(x, 0)$$

and

$$\mathcal{A}_{c_L} \subseteq \{x \in W : \ L(x, 0) = -\alpha(c_L)\} = \{x \in W : \ U(x) = 0\} =: \mathcal{A}\,.$$

Therefore, the previous lemma allows us to conclude that $|u(\mathcal{A}_{c_L})| = 0$.

The same proof works for $w \in \mathcal{D}_c$, observing that $\|d_x(u-v)\|^2 \leq 2\,(\|d_x u\|^2 + \|d_x v\|^2)$. It will be sufficient in this case to take $\widetilde{U}(x) = \frac{8}{\gamma}(\alpha(c_L) + L(x, 0))$.

□

Proof (Theorem 2.3.1). The proof follows immediately from proposition 2.2.6 and proposition 2.3.10. □

2.4 Proof of Lemma 2.3.9 and Sard's lemma for critical subsolutions

In this section we want to give a proof of Lemma 2.3.9. As we have already remarked, this will imply a version of Sard's lemma for critical subsolutions of certain Hamilton-Jacobi equations (proposition 2.3.10). See also a related work by Ludovic Rifford [70].

Definition 2.4.1. *Consider a function $f \in C^r(\mathbb{R}^d)$. We say that f is s - flat at $x_0 \in \mathbb{R}^d$ (with $s \leq r$), if all its derivatives, up to the order s, vanish at x_0.*

The proof of the Lemma 2.3.9 is based on the following modified version of *Kneser-Glaeser's Rough composition theorem* (see section 2.5).

Proposition 2.4.2. *Let $V, W \subset \mathbb{R}^d$ be open sets, $A \subset V$, $A^* \subset W$ closed sets. Consider $U \in C^r(V)$, with $r \geq 2$, a non-negative function which is s-flat on $A \subset \{U(x) = 0\}$, with $s \leq r-1$, and $g : W \longrightarrow V$ a C^{r-s} homeomorphism such that $g(A^*) \subset A$.*

Then, for every open pre-compact set $W_1 \supset A^*$ properly contained in W, there exists

$$F : \mathbb{R}^d \longrightarrow \mathbb{R}$$

satisfying the following properties:

i) $F \in C^{r-1}(\mathbb{R}^d)$;

ii) $F \geq 0$;

iii) $F(x) = U(g(x)) = 0$ *on* A^*;

iv) F *is s-flat on* A^*;

v) $\{F(x) = 0\} \cap W_1 = A^*$;

vi) *there exists a constant $K > 0$, such that $U(g(x)) \leq KF(x)$ on W_1.*

To prove Lemma 2.3.9, it will be enough to show that for every $x_0 \in M$, there exists a neighborhood Ω such that the conclusion of the lemma holds. For such a local result, we can assume that $M = \mathcal{U}$ is an open subset of $\mathbb{R}^d$, with $x_0 \in \mathcal{U}$. In the sequel, we will identify $\mathrm{T}^*\mathcal{U}$ with $\mathcal{U} \times \mathbb{R}^d$ and for $x \in \mathcal{U}$ we identify $\mathrm{T}^*_x\mathcal{U} = \{x\} \times \mathbb{R}^d$. Let us equip $\mathcal{U} \times \mathbb{R}^d$ with the natural coordinates $(x_1, \ldots, x_d, p_1, \ldots, p_d)$.
Before proceeding in the proof, let us point out that it is locally possible to replace the norm obtained by the Riemannian metric, by a constant norm on $\mathbb{R}^d$.

Lemma 2.4.3. *For each $0 < \alpha < 1$ and $x_0 \in M$, there exists an open neighborhood Ω of x_0, with $\overline{\Omega} \subset \mathcal{U}$ and such that*

$$(1-\alpha)\|p\|_{x_0} \leq \|p\|_x \leq (1+\alpha)\|p\|_{x_0} \,,$$

for every $p \in \mathrm{T}^*_x\mathcal{U} \cong \mathbb{R}^d$ and each $x \in \overline{\Omega}$.

Proof. By continuity of the Riemannian metric, the norm $\|p\|_x$ tends uniformly to 1 on $\{p : \|p\|_{x_0} = 1\}$, as x tends to x_0. Therefore, for x near to x_0 and every $p \in \mathbb{R}^d \setminus \{0\}$, we have:

$$(1-\alpha) \leq \left\| \frac{p}{\|p\|_{x_0}} \right\|_x \leq (1+\alpha).$$

□

We can now prove the main result of this section.

Proof (Lemma 2.3.9). By choosing local charts and by Lemma 2.4.3, we can assume that $U \in C^r(\Omega)$, with Ω open set in $\mathbb{R}^d$, $\mathcal{A} = \{x \in \Omega : \; U(x) = 0\}$ and $u : \Omega \longrightarrow \mathbb{R}$ is such that $\|d_x u\|^2 \leq \beta U(x)$ in Ω, where β is a positive constant.

Define, for $1 \leq s \leq r$:

$$B_s = \{x \in \mathcal{A} : \; U \text{ is } s \text{ - flat at } x\}$$

and observe that

$$\mathcal{A} = B_1 := \{x \in \mathcal{A} : DU(x) = 0\} .$$

We will prove the lemma by induction on the dimension d. Let us start with the following claim.

Claim 2.4.4. *If $s \geq 2d - 2$, then $|u(B_s)| = 0$.*

Proof. Let $C \subset \Omega$ be a closed cube with edges parallel to the coordinate axes. We will show that $|u(B_s \cap C)| = 0$. Since B_s can be covered by countably many such cubes, this will prove that $|u(B_s)| = 0$.

Let us start observing that, by Taylor's theorem, for any $x \in B_s \cap C$ and $y \in C$ we have

$$U(y) = R_s(x; y),$$

where $R_s(x; y)$ is Taylor's remainder. Therefore, for any $y \in C$

$$U(y) = o(\|y - x\|^s) .$$

Let λ be the length of the edge of C. Choose an integer $N > 0$ and subdivide C in N^d cubes C_i with edges $\frac{\lambda}{N}$, and order them so that one has $C_i \cap B_s \neq \emptyset$, for $1 \leq i \leq N_0 \leq N^d$. Hence,

$$B_s \cap C = \bigcup_{i=1}^{N_0} B_s \cap C_i.$$

Observe that for every $\varepsilon > 0$, there exists $\nu_0 = \nu_0(\varepsilon)$ such that, if $N \geq \nu_0$, $x \in B_s \cap C_i$ and $y \in C_i$, for some $0 \leq i \leq N_0$, then

$$U(y) \leq \frac{\varepsilon^2}{4\beta(d\lambda^2)^d} \|y - x\|^s .$$

Fix $\varepsilon > 0$. Choose $x_i \in B_s \cap C_i$ and call $y_i = u(x_i)$. Define, for $N \geq \nu_0$, the following intervals in $\mathbb{R}$:

$$E_i = \left[y_i - \frac{\varepsilon}{2N^d},\ y_i + \frac{\varepsilon}{2N^d} \right].$$

Let us show that if N is sufficiently big, then $u(B_s \cap C) \subset \bigcup_{i=1}^{N_0} E_i$.

In fact, if $x \in B_s \cap C$, then there exists $1 \leq i \leq N_0$, such that $x \in B_s \cap C_i$. Therefore,

$$\begin{aligned}
|u(x) - y_i| &= |u(x) - u(x_i)| = \\
&= \|d_x u(\tilde{x})\| \cdot \|x - x_i\| \leq \\
&\leq \sqrt{\beta U(\tilde{x})} \|x - x_i\| \leq \\
&\leq \sqrt{\beta \frac{\varepsilon^2}{4\beta (d\lambda^2)^d}} \|\tilde{x} - x_i\|^{\frac{s}{2}} \|x - x_i\| \leq \\
&\leq \frac{\varepsilon}{2(d\lambda^2)^{\frac{d}{2}}} \|x - x_i\|^{\frac{s+2}{2}} \leq \\
&\leq \frac{\varepsilon}{2(d\lambda^2)^{\frac{d}{2}}} \left(\sqrt{d} \frac{\lambda}{N} \right)^{\frac{s+2}{2}},
\end{aligned}$$

where $\tilde{x}$ is a point in the segment joining x and x_i. Since by hypothesis $s \geq 2d - 2$, then $\frac{s+2}{2} \geq d$. Hence, assuming that $N > \max\{\lambda\sqrt{d},\ \nu_0\}$, one gets

$$|u(x) - y_i| \leq \frac{\varepsilon}{2N^d}$$

and can deduce the above inclusion.

To prove the claim, it is now enough to observe:

$$\begin{aligned}
|u(B_s \cap C)| &\leq \left| \bigcup_{i=1}^{N_0} E_i \right| \leq \sum_{i=1}^{N_0} |E_i| \leq \\
&\leq \varepsilon N_0 \frac{1}{N^d} \leq \\
&\leq \varepsilon N^d \frac{1}{N^d} = \\
&= \varepsilon.
\end{aligned}$$

From the arbitrariness of ε, the assertion follows easily. □

This claim immediately implies that $u(B_{2d-2})$ has measure zero. In particular, this proves the case $d = 1$ (since in this case $2d - 2 = 0$) and allows us to start the induction.

Let us suppose to have proven the result for $d - 1$ and try to show it for d. Since

$$\mathcal{A} = (B_1 \setminus B_2) \cup (B_2 \setminus B_3) \cup \ldots \cup (B_{2d-3} \setminus B_{2d-2}) \cup B_{2d-2}\,,$$

it remains to show that $|u(B_s \setminus B_{s+1})| = 0$ for $1 \le s \le 2d - 3 \le r - 1$.

Claim 2.4.5. *Every $\tilde{x} \in B_s \setminus B_{s+1}$ has a neighborhood $\tilde{V}$, such that*

$$|u((B_s \setminus B_{s+1}) \cap \tilde{V})| = 0\,.$$

Since $B_s \setminus B_{s+1}$ can be covered by countably many such neighborhoods, this implies that $u(B_s \setminus B_{s+1})$ has measure zero.

Proof. Choose $\tilde{x} \in B_s \setminus B_{s+1}$. By definition of these sets, all partial derivatives of order s of U vanish at this point, but there is one of order $s+1$ that does not. Assume (without any loss of generality) that there exists a function

$$w(x) = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_s}U(x)$$

such that

$$w(\tilde{x}) = 0 \qquad \text{but} \qquad \partial_1 w(\tilde{x}) \neq 0\,.$$

Define

$$\begin{aligned} h : \Omega \;&\longrightarrow\; \mathbb{R}^d \\ x \;&\longmapsto\; (w(x),\, x_2,\, \ldots,\, x_d)\,, \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_d)$. Clearly, $h \in C^{r-s}(\Omega)$ and $Dh(\tilde{x})$ is non-singular; hence, there is an open neighborhood V of $\tilde{x}$ such that

$$h : V \longrightarrow W$$

is a C^{r-s} isomorphism (with $W = h(V)$).

Let V_1 be an open precompact set, containing $\tilde{x}$ and properly contained in V, and define $A = B_s \cap \overline{V_1}$, $A^* = h(A)$ and $g = h^{-1}$. If we consider W_1, any open set containing A^* and properly contained in W, we can apply proposition 2.4.2 and deduce the existence of $F : \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfying properties i)-vi).

Define

$$\hat{W} = \{(x_2, \ldots, x_d) \in \mathbb{R}^{d-1} : \ (0, x_2, \ldots, x_d) \in W_1\}$$

and

$$\hat{U}(x_2, \ldots, x_d) = C\, F(0, x_2, \ldots, x_d),$$

where C is a positive constant to be chosen sufficiently big. Observe that $\hat{U} \in C^{r-1}(\mathbb{R}^{d-1})$. Moreover, property v) of F and the fact that $A^* = h(A) \subseteq \{0\} \times \hat{W}$ imply that:

$$A^* = \{0\} \times \hat{B}_1\,,$$

where $\hat{B}_1 = \{(x_2, \ldots x_d) \in \hat{W} : \ F(0, x_2, \ldots, x_d) = 0\}$. Denote

$$\hat{\mathcal{A}} := \{(x_2, \ldots, x_d) \in \hat{W} : \ \hat{U} = 0\} = \hat{B}_1$$

and define the following function on $\hat{W}$:

$$\hat{u}(x_2, \ldots, x_d) = u(g(0, x_2, \ldots, x_d)).$$

We want to show that these functions satisfy the hypotheses for the $(d-1)$-dimensional case. In fact:

- $\hat{U} \in C^{r-1}(\mathbb{R}^{d-1})$, with $r-1 \geq 2d-3 > 2(d-1)-2$;
- $\hat{u} \in C^1(\hat{W})$ (since g is in $C^{r-s}(W)$, where $1 \leq s \leq r-1$);
- if we denote by $\mu = \sup_{W_1} \|d_x g\| < +\infty$ (since g is C^1 on $\overline{W_1}$), then we have that for every point in $\hat{W}$:

$$\begin{aligned}
\|d\hat{u}(x_2,\dots,x_d)\|^2 &\leq \|d_x u(g(0,x_2,\dots,x_d))\|^2 \|d_x g(0,x_2,\dots,x_d)\|^2 \leq \\
&\leq \mu^2 \|d_x u(g(0,x_2,\dots,x_d))\|^2 \leq \\
&\leq \beta\mu^2 U(g(0,x_2,\dots,x_d)) \leq \\
&\leq \beta\mu^2 K F(0,x_2,\dots,x_d) \leq \\
&\leq \hat{U}(x_2,\dots,x_d),
\end{aligned}$$

  if we choose $C > \beta\mu^2 K$, where K is the positive constant appearing in proposition 2.4.2, property vi).

Therefore, it follows from the inductive hypothesis that: $|\hat{u}(\hat{\mathcal{A}})| = 0$.

Since

$$\begin{aligned}
u(B_s \cap V_1) &\subseteq u(A) = u(g(A^*)) = u(g(\{0\} \times \hat{B}_1)) = \\
&= \hat{u}(\hat{B}_1) = \hat{u}(\hat{\mathcal{A}}),
\end{aligned}$$

defining $\tilde{V} = V_1$, we may conclude that

$$|u(B_s \cap \tilde{V})| \leq |\hat{u}(\hat{\mathcal{A}})| = 0.$$

□

This completes the proof of Lemma 2.3.9. □

2.5 Proof of a modified version of Kneser-Glaeser's Rough composition theorem.

Now, let us prove proposition 2.4.2. We will prove a modified version of the result presented in [1], adapted to our different needs and setting.

Proof (Proposition 2.4.2). Let us start by defining a family of polynomials. Supposing that g is C^r and using the s-flatness hypothesis, we have, for $x \in A^*$ and $k = 0, 1, \dots, r$:

$$f_k(x) = D^k(U \circ g)(x) = \sum_{s<q\leq k} \sum \sigma_k D^q U(g(x))\, D^{i_1} g(x) \dots D^{i_q} g(x)\,, \tag{2.6}$$

where the second sum is over all the q-tuples of integers $i_1, \dots, i_q \geq 1$ such that $i_1 + \dots + i_q = k$, and $\sigma_k = \sigma_k(i_1, \dots, i_q)$.
The crucial observation is that (2.6) makes sense on A^*, even when g is C^{r-s} smooth (in fact $i_j \leq k - q + 1 \leq r - s$).

We would like to proceed in the fashion of *Whitney's extension theorem*, in order to find a smooth function F such that $D^k F = f_k$ on A^* and satisfying the stated conditions.

Remark 2.5.1. Note that, without any loss of generality, we can assume that W is contained in an open ball of diameter 1. The general case will then follow from this special one, by a straightforward partition of unity argument.

Let us start with some technical lemmata.

Lemma 2.5.2. *For $x, x', x_0 \in A^*$ and $k = 0, \dots, r$, we have:*

$$f_k(x') = \sum_{i\leq r-k} \frac{f_{k+i}(x)}{i!}(x'-x)^i + R_k(x,x')\,,$$

with

$$\frac{|R_k(x,x')|}{\|x'-x\|^{r-k}} \longrightarrow 0$$

as $x,\ x' \longrightarrow x_0$ *in* A^*.

Proof. Let $y \in A^*$, $y' \in V$ and $q = s+1, \ldots, r$. By Taylor's formula, we may write

$$D^q U(y') = \sum_{\alpha \le r-q} \frac{D^{q+\alpha}U(y)}{\alpha!}(y'-y)^\alpha + I_q(y,y')(y'-y)^{r-q}$$

where

$$I_q(y,y') = \int_0^1 \frac{(1-t)^{r-q}}{(r-q)!}\{D^r U(y+t(y'-y)) - D^r U(y)\}\,dt\,.$$

Similarly, for $i \le r-s$ and $x,\ x' \in W$,

$$D^i g(x') = \sum_{\beta \le r-s-i} \frac{D^{i+\beta}g(x)}{\beta!}(x'-x)^\beta + J_i(x,x')(x'-x)^{r-s-i}$$

where

$$J_i(x,x') = \int_0^1 \frac{(1-t)^{r-s}}{(r-s)!}\{D^{r-s}g(x+t(x'-x)) - D^{r-s}g(x)\}\,dt\,.$$

It follows from the formulae for I_q and J_i, that they are locally uniformly continuous in both variables and zero on the diagonal.

Now, take $x,\ x',\ x_0 \in A^*$ and let $y = g(x)$ and $y' = g(x')$. By (2.6) we have

$$f_k(x') = \sum_{s<q\le k} \sum \sigma_k D^q U(y')\, D^{i_1}g(x') \ldots D^{i_q}g(x')\,.$$

Substituting the expressions above, we get:

$$f_k(x') = \sum_{j=0}^{r-k} a_j^k (x'-x)^j + R_k(x,x')\,, \tag{2.7}$$

where $R_k(x,x')$ is the sum of all terms T of one of the following forms:

a) $a(x'-x)^j$ where $j > r-k$;

b) $I_q(y,y')(y'-y)^{r-q}$;

c) $J_{i_j}(x,x')(x'-x)^{r-s-i_j}$.

We will show that if T is any term in $R_k(x,x')$, then

$$\frac{|T|}{\|x'-x\|^{r-k}} \longrightarrow 0 \qquad \text{as } x,\, x' \to x_0 \text{ in } A^* .$$

In fact, if T is of form a), this is immediate. If T has form b), then it follows easily from the fact that $q \leq k$, g is Lipschitz and I_q locally uniformly continuous and zero on the diagonal. For any term of the form c), $i_j \leq k-q+1$ where $s+1 \leq q$. Hence, $s+i_j \leq k$ and $r-s-i_j \geq r-k$. Thus, it can be deduced by the same reasoning as case b).

To conclude the proof of this lemma, we need to show that for each $k = 0,\, 1,\, \ldots,\, r$ and $j = 0,\, \ldots,\, r-k$ we have:

$$j!a_j^k = f_{k+j}(x)\,. \tag{2.8}$$

To see this, suppose for the moment that g is class C^r. Then so is $U \circ g$ and (2.7) is Taylor's formula for $f_k = D^k(U \circ g)$. Then, by the uniqueness of Taylor's formula, we have

$$j!a_j^k = D^{k+j}(U \circ g)(x) = f_{k+j}(x).$$

Since (2.8) is an identity in the derivatives of U and g, and points x and x', and since one can always find a C^r function with prescribed derivatives at any finite set of points, it follows that (2.8) holds in general. □

Define, for $x \in A^*$ and $y \in \mathbb{R}^d$

$$P(x,y) = \sum_{i=s+1}^{r} \frac{f_i(x)}{i!}(y-x)^i$$

and its k-th derivative

$$P_k(x,y) = \sum_{i \le r-k} \frac{f_{i+k}(x)}{i!}(y-x)^i \,.$$

Lemma 2.5.3. *For $x \in A^*$ and $y \in W_1$,*

$$U(g(y)) = P(x,y) + R(x,y)\,,$$

where $|R(x,y)| \le C\|y-x\|^r$.

Proof. The proof follows the same idea of Lemma 2.5.2. By Taylor's formula for U,

$$U(g(y)) = \sum_{q=s+1}^{r} \frac{D^q U(g(x))}{q!}(g(y)-g(x))^q + I(g(x),g(y))(g(x)-g(y))^r \,.$$

Obviously,

$$|I(g(x),g(y))(g(x)-g(y))^r| \le C_1\|y-x\|^r,$$

therefore it is sufficient to estimate the first term.

Observe that:

$$g(y) = g(x) + \sum_{i=1}^{r-s} D^i g(x)(y-x)^i + J(x,y)(y-x)^{r-s} \,.$$

Hence, the first term in the above sum becomes:

$$\sum_{q=s+1}^{r} \frac{D^q U(g(x))}{q!} \left[\sum_{i=1}^{r-s} D^i g(x)(y-x)^i + J(x,y)(y-x)^{r-s} \right]^q =$$

$$= \sum_{k=s+1}^{r} a_k (y-x)^k + \hat{R}(x,y) =$$

$$= P(x,y) + \hat{R}(x,y)\,,$$

since

$$a_k = \sum_{s+1 \le q \le k} \sum D^q U(g(x)) D^{i_1} g(x) \ldots D^{i_q} g(x) = \frac{f_k(x)}{k!}\,.$$

The remainder terms consist of:

- terms containing $(y-x)^k$, with $k > r$;
- terms of the binomial product, containing $J(x,y)(y-x)^{r-s}$. They are of the form:

$$\ldots\ (y-x)^{(r-s)j + \sum_{i=1}^{r-s} i\alpha_i}$$

  where $\alpha_i \ge 0$ and $\sum \alpha_i = q - j$. Since $q \ge s+1$ and $s \le r-1$, then:

$$\begin{aligned}
(r-s)j + \sum_{i=1}^{r-s} i\alpha_i &\ge (r-s)j + \sum_{i=1}^{r-s} \alpha_i = \\
&= (r-s)j + q - j = \\
&= rj - sj + q - j \ge \\
&\ge rj - (s+1)j + s + 1 = \\
&= r + r(j-1) - (s+1)(j-1) = \\
&= r + (r-s-1)(j-1) \ge r\,.
\end{aligned}$$

Therefore, for $x \in A^*$ and $y \in W_1$

$$\left|\hat{R}(x,y)\right| \leq C_2 \|y-x\|^r$$

and the lemma follows taking $C = C_1 + C_2$. $\square$

Next step will consist in creating a *Whitney's partition*. We will start by covering $W_1 \setminus A^*$ with an infinite collection of cubes K_j, such that the size of each K_j is roughly proportional to its distance from A^*.

First, let us fix some notation. We will write $a \prec b$ instead of "there exists a positive real constant M, such that $a \leq Mb$ " and $a \approx b$ as short for $a \prec b$ and $b \prec a$. Let $\lambda = \frac{1}{4\sqrt{d}}$; this choice will come in handy later. For any closed cube K (with edges parallel to the coordinate axes), K^λ will denote the $(1+\lambda)$ - dilation of K about its center.
Let $\|\cdot\|$ be the euclidean metric on $\mathbb{R}^d$ and

$$d(y) = d(y, A^*) = \inf\{\|y-x\| : \ x \in A^*\}\,.$$

If $\{K_j\}_j$ is the sequence of closed cubes defined below, with edges of length e_j, let d_j be its distance from A^*, *i.e.*,

$$d_j = d(A^*, K_j) = \inf\{\|y-x\| : \ x \in A^*,\, y \in K_j\}\,.$$

One can show the following classical lemma.

Lemma 2.5.4. *There exists a sequence of closed cubes $\{K_j\}_j$ with edges parallel to the coordinate axes, that satisfies the following properties:*

i) *the interiors of the K_j's are disjoint;*

ii) $W_1 \setminus A^* \subset \bigcup_j K_j$*;*

iii) $e_j \approx d_j$;

iv) $e_j \approx d(y)$ *for all* $y \in K_j^\lambda$;

v) $e_j \approx d(z)$ *for all* $z \in W_1 \setminus A^*$, *such that the ball with center* z *and radius* $\frac{1}{8}d(z)$ *intersects* K_j^λ;

vi) *each point of* $W_1 \setminus A^*$ *has a neighborhood intersecting at most* N *of the* K_j^λ, *where* N *is an integer depending only on* d.

Proof. Divide $\mathbb{R}^d$ into cubes of unit edge. Accept all cubes which intersect $W_1 \setminus A^*$ and are at distance $\geq \frac{1}{2}$ from A^*. Bisect each of the rejected cubes into 2^d parts and accept all those cubes which intersect $W_1 \setminus A^*$ and are at distance $\geq \frac{1}{4}$ from A^*. Repeat indefinitely to get the sequence $\{K_j\}_j$.

Properties i) and ii) are immediate. To prove iii) we show that:

$$\frac{e_j}{2} \leq d_j \leq (1 + 2\sqrt{d})e_j \quad \text{for all } j\,.$$

That $\frac{e_j}{2} \leq d_j$ follows easily from the definition. Let $e_j = 2^{-k}$. For $k = 0$, we have $d_j \leq 1$, since the diameter of W is less than 1 (see remark above). For $k > 0$, K_j is one of the 2^d dyadic cubes of a cube K^* and K^* was rejected. Then $e^* = 2e_j = 2^{-k+1}$ and

$$d^* = d(A, K^*) < 2^{-k} = e_j\,.$$

From the triangle inequality and the fact that the diameter of a cube is $\sqrt{d}$ times the length of its edge, it follows that:

$$d_j \leq (1 + 2\sqrt{d})e_j\,.$$

This completes the proof of iii).

To prove iv), let us show that

$$\frac{e_j}{4} \le d(y) \le (4+\lambda)\sqrt{d}e_j,$$

whenever $y \in K_j^\lambda$.

Choose $y \in K_j^\lambda$. Let $p_j \in K_j$ satisfy $d(y, K_j) = \|y - p_j\|$. Then

$$\|y - p_j\| \le \lambda\sqrt{d}e_j = \frac{1}{4}e_j\,.$$

By iii),

$$\frac{e_j}{2} \le d_j \le d(p_j) \le d(y) + \|y - p_j\|\,.$$

Hence, $\frac{1}{4}e_j \le d(y)$. Now, let $q_j \in K_j$ satisfy $d(q_j) = d_j$. Then, by iii) and geometric considerations,

$$d(y) \le \|y - q_j\| + d(q_j) \le \sqrt{d}(1+\lambda)e_j + (1+2\sqrt{d})e_j\,.$$

Hence, $d(y) \le (4+\lambda)\sqrt{d}e_j$. This completes the proof of iv).

Now for v), let $a = \frac{1}{4}$ and $b = (4+\lambda)\sqrt{d}$ and y in the ball of radius $\frac{1}{2}ad(z) = \frac{1}{8}d(z)$ and center z, and $y \in K_j^\lambda$. By iv), $ae_j \le d(y) \le be_j$. Since

$$d(y) \le \|y - z\| + d(z) \quad \text{and} \quad d(z) \le \|y - z\| + d(y)\,,$$

we have

$$\frac{a}{2}e_j \le d(z) \le \left(b + \frac{a}{2}\right)e_j\,.$$

This proves v).

Now, let us prove vi). From v), it follows that there are constants p and q, depending only on d, such that for every $z \in W_1 \setminus A^*$, the K_j^λ's intersecting the ball about z with radius $\frac{1}{8}d(z)$ are contained in a ball of radius $pd(z)$ about z and have edges at most $qd(z)$. Let N be the maximum number of cubes with edges $\geq \frac{q}{p}$ that fit in the unit ball; it depends only on d and at most N of the K_j^λ's intersect the ball of radius $\frac{1}{8}d(z)$ and center z. This completes the proof of vi). $\square$

Now, let us construct a partition of unity on $W_1 \setminus A^*$. Let Q be the unit cube centered at the origin. Let η be a C^∞ bump function defined on $\mathbb{R}^d$ such that

$$\eta(y) = \begin{cases} 1 & \text{for } y \in Q \\ 0 & \text{for } y \notin Q^\lambda \end{cases}$$

and $0 \leq \eta \leq 1$. Define

$$\eta_j(y) = \eta\left(\frac{y - c_j}{e_j}\right),$$

where c_j is the center of K_j and e_j is the length of its edge, and consider

$$\sigma(y) = \sum_j \eta_j(y).$$

Then, $1 \leq \sigma(y) \leq N$ for all $y \in W_1 \setminus A^*$. Clearly, for each $k = 0, 1, 2, \ldots$ we have that $D^k \eta_j(y) \prec e_j^{-k}$, for all $y \in W_1 \setminus A^*$. Hence, by properties iv) and vi) of Lemma 2.5.4, we have that for each $k = 0, 1, \ldots, r$:

$$D^k \eta_j(y) \prec d(y)^{-k} \qquad \text{for all } y \in W_1 \setminus A^*$$

and

$$D^k \sigma(y) \prec d(y)^{-k} \qquad \text{for all } y \in W_1 \setminus A^*.$$

Define

$$\varphi_j(y) = \frac{\eta_j(y)}{\sigma(y)}.$$

These functions satisfy the following properties:

i) each φ_j is C^∞ and supported on K_j^λ;

ii) $0 \le \varphi_j(y) \le 1$ and $\sum_j \varphi_j(y) = 1$, for all $y \in W_1 \setminus A^*$;

iii) every point of $W_1 \setminus A^*$ has a neighborhood on which all but at most N of the φ_j's vanish identically;

iv) for each $k = 0, 1, \ldots, r$, $D^k\varphi_j(y) \prec d(y)^{-k}$ for all $y \in W_1 \setminus A^*$; namely, there are constants M_k such that $D^k\varphi_j(y) \le M_k d(y)^{-k}$;

v) there is a constant α and points $x_j \in A^*$, such that:

$$\|x_j - y\| \le \alpha d(y), \qquad \text{whenever } \varphi_j(y) \neq 0\,.$$

This follows from properties iii) and iv) of Lemma 2.5.4.

We can now construct our function F. Observe that, from lemma 2.5.3:

$$0 \le U(g(y)) = P(x_j, y) + R(x_j, y) \le P(x_j, y) + C\|y - x_j\|^r\,;$$

therefore $P(x_j, y) \ge -C\|y - x_j\|^r$.
First, define

$$\hat{P}_j(y) = P(x_j, y) + 2C\|y - x_j\|^r$$

where C is the same constant as in Lemma 2.5.3; for what said above,

$$\hat{P}_j(y) \ge C\|y - x_j\|^r > 0 \qquad \text{in } W_1 \setminus \{x_j\}\,. \tag{2.9}$$

Hence, construct F in the following way:

$$F(y) = \begin{cases} 0 & y \in A^* \\ \sum_j \varphi_j(y)\hat{P}_j(y) & y \in \mathbb{R}^d \setminus A^*\,. \end{cases}$$

We claim that this satisfies all the stated properties i)-vi). In particular, properties ii), iii) and v) follow immediately from the definition of F and (2.9). Moreover, $F \in C^\infty(\mathbb{R}^d \setminus A^*)$. We need to show that $D^k F = f_k$ (for $k = 0, 1, \ldots, r-1$) on ∂A^* (namely, the boundary of A^*) and that $D^{r-1}F$ is continuous on it. The main difficult in the proof, is that $D^k F$ is expressed as a sum containing terms

$$D^{k-m}\varphi_j(y) P_m(x_j, y),$$

where $\varphi_j(y) \neq 0$. Even if y is close to some $x_0 \in A^*$, it could be closer to A^* and hence the bound given by property iv) of φ_j might become large. One can overcome this problem by choosing a point $x^* \in A^*$, so that $\|x^* - y\|$ is roughly the same as $d(y)$ and hence, x_j is close to x^*.

Lemma 2.5.5. *For every $\eta > 0$, there exists $\delta > 0$ such that for all $y \in W_1 \setminus A^*$, x, $x^* \in A^*$ and $x_0 \in \partial A^*$, we have*

$$\|P_k(x,y) - P_k(x^*,y)\| \leq \eta\, d(y)^{r-k} \leq \eta \|y - x_0\|^{r-k},$$

whenever $k \leq r$ and

$$\begin{cases} \|y - x\| < \alpha d(y) \\ \|y - x^*\| < \alpha d(y) \\ \|y - x_0\| < \delta\,, \end{cases}$$

where α is the same constant as in v) *above.*

Proof. Let $\varepsilon > 0$ to be defined below. From Lemma 2.5.2, it follows that there exists $\delta > 0$ such that

$$\|R_{k+q}(x^*, x)\| \leq \varepsilon \|x^* - x\|^{r-k-q} \leq \varepsilon (2\alpha d(y))^{r-k-q}\,.$$

In fact $d(y) \leq \|y - x_0\|$, hence by making y close to x_0 (*i.e.,* δ small), one can make x and x^* close to x_0.

Observe that, for x, $x^* \in A^*$ and $y \in \mathbb{R}^d$ we have

$$P_k(x,y) = P_k(x^*,y) + \sum_{q \le r-k} \frac{R_{k+q}(x^*,x)}{q!}(y-x)^q\,,$$

as it can be easily shown using the definition of P_k, the fact that the $f_k(x^*)$ are multilinear maps and the analogous of the binomial theorem.

Therefore,

$$\begin{aligned}
\|P_k(x,y) - P_k(x^*,y)\| &\le \sum_{q \le r-k} \|R_{k+q}(x^*,x)(y-x)^q\| \le \\
&\le \sum_{q \le r-k} \varepsilon(2\alpha d(y))^{r-k} = \\
&= \varepsilon\,(r-k)(2\alpha d(y))^{r-k}\,.
\end{aligned}$$

Hence, take $\varepsilon = \frac{\eta}{(r-k)(2\alpha)^{r-k}}$ when $r \ne k$, and $\varepsilon = \eta$ for $r = k$. This concludes the proof of the lemma. □

Lemma 2.5.6. *For every $\eta > 0$, there exist $0 < \delta < 1$ and a constant E, such that for all $y \in W_1 \setminus A^*$, $x^* \in A^*$ and $x_0 \in \partial A^*$, we have*

$$\|D^k F(y) - P_k(x^*,y)\| \le E\,d(y)^{r-k} \le \eta\,d(y)^{r-k-1},$$

whenever $k \le r-1$ and

$$\begin{cases} \|y - x^*\| < \alpha d(y) \\ \|y - x_0\| < \delta\,. \end{cases}$$

Proof. Let $S_{j,k}(x^*,y) = \partial_k \hat{P}_j(y) - P_k(x^*,y)$. From Lemma 2.5.5 (with $\eta = \varepsilon$, to be defined later) and the definition of $\hat{P}_j$, we get:

$$\begin{aligned}\|S_{j,k}(x^*,y)\| &\leq \|\partial_k \hat{P}_j(y) - P_k(x_j,y)\| + \|P_k(x_j,y) - P_k(x^*,y)\| \leq \\ &\leq C_k d(y)^{r-k} + \varepsilon d(y)^{r-k} = \\ &= (C_k+\varepsilon) d(y)^{r-k}\,.\end{aligned}$$

Then,

$$F(y) - P(x^*,y) = \sum_j \varphi_j(y) S_{j,0}(x^*,y)$$

and hence

$$D^k F(y) - P_k(x^*,y) = \sum_j \sum_{i\leq k} \binom{k}{i} D^{k-i}\varphi_j(y) S_{j,i}(x^*,y)\,.$$

Therefore, choosing ε sufficiently small:

$$\begin{aligned}\|D^k F(y) - P_k(x^*,y)\| &\leq \sum_j \sum_{i\leq k} \binom{k}{i} \|D^{k-i}\varphi_j(y)\| \cdot \|S_{j,i}(x^*,y)\| \leq \\ &\leq \sum_j \sum_{i\leq k} \binom{k}{i} M_{k-i} d(y)^{-k+i} (C_k+\varepsilon) d(y)^{r-i} \leq \\ &\leq E\, d(y)^{r-k} \leq \eta\, d(y)^{r-k-1}\,.\end{aligned}$$

□

Lemma 2.5.7. *For every $\eta > 0$, there exists $0 < \delta < 1$ such that, for all $y \in W_1 \setminus A^*$, $x^* \in A^*$ and $x_0 \in \partial A^*$, we have*

$$\|P_k(x^*,y) - P_k(x_0,y)\| \leq \eta \|y - x_0\|^{r-k}\,,$$

whenever $k \leq r$ and

$$\begin{cases} \|y - x^*\| < \alpha d(y) \\ \|y - x_0\| < \delta\,. \end{cases}$$

Proof. The proof goes as the one of Lemma 2.5.5, observing that $\|x^* - x_0\| \leq (1+\alpha)\|y - x_0\|$ and

$$P_k(x_0, y) - P_k(x^*, y) = \sum_{q \leq r-k} \frac{R_{k+q}(x^*, x_0)}{q!}(y - x)^q \,.$$

$\square$

Claim 2.5.8. *For every $x_0 \in \partial A^*$ and $k = 0, 1, \dots, r-1$:*

$$D^k F(x_0) = f_k(x_0) \,.$$

Moreover, $D^{r-1}F$ is continuous at $x_0 \in \partial A^*$.

Proof. This claim follows easily from the lemmata above. We proceed by induction on k. For $k = 0$, it follows immediately from the definition of F (defining $f_0 = U \circ g$). Assume $k < r - 1$ and $D^k F(x_0) = f_k(x_0)$. We will show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|D^k F(y) - f_k(x_0) - f_{k+1}(x_0)(y - x_0)\| \leq \varepsilon \|y - x_0\|, \tag{2.10}$$

whenever $\|y - x_0\| < \delta$.

Choose $\varepsilon > 0$. Take $\eta = \frac{\varepsilon}{3}$ in Lemma 2.5.6 and 2.5.7 and the corresponding δ_1, $\delta_2 > 0$. Since

$$\|P_k(x_0, y) - f_k(x_0) - f_{k+1}(x_0)(y - x_0)\| \leq \sum_{2 \leq i \leq r-k} \frac{\|f_{k+i}(x_0)\|}{i!} \|y - x_0\|^i \,,$$

we may choose $\delta_3 > 0$ so small that

$$\|P_k(x_0, y) - f_k(x_0) - f_{k+1}(x_0)(y - x_0)\| \leq \eta \|y - x_0\|,$$

whenever $\|y - x_0\| < \delta_3$.

By Lemma 2.5.2, we know that we can choose δ_4 so small that (2.10) holds when $\|y - x_0\| < \delta_4$ and $y \in A^*$. Now take $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4, 1\}$. Choose y such that $\|y - x_0\| < \delta$. If $y \in A^*$, we have completed the proof. If $y \in W_1 \setminus A^*$, choose $x^* \in A^*$ with $\|y - x^*\| < \alpha d(y)$. Then, by Lemma 2.5.6 and 2.5.7 and the above estimate, we have:

$$\begin{aligned}
&\|D^k F(y) - f_k(x_0) - f_{k+1}(x_0)(y - x_0)\| \leq \\
&\quad \leq \|D^k F(y) - P_k(x^*, y)\| + \|P_k(x^*, y) - P_k(x_0, y)\| + \\
&\quad + \|P_k(x_0, y) - f_k(x_0) - f_{k+1}(x_0)(y - x_0)\| \leq \\
&\quad \leq \eta\|y - x_0\|^{r-k-1} + \eta\|y - x_0\|^{r-k} + \eta\|y - x_0\|^{r-k} \leq \\
&\quad \leq 3\eta\|y - x_0\| = \\
&\quad = \varepsilon\|y - x_0\| .
\end{aligned}$$

□

This proves iv). Moreover,

Claim 2.5.9. *$D^{r-1}F$ is continuous at $x_0 \in \partial A^*$.*

Proof. We must show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|D^{r-1}F(y) - f_{r-1}(x_0)\| \leq \varepsilon,$$

whenever $\|y - x_0\| < \delta$.

In fact, using Lemma 2.5.6 and 2.5.7 and the same argument as in the previous claim (with $\eta = \frac{\varepsilon}{2}$), one can get:

$$\begin{aligned}
&\|D^{r-1}F(y) - f_{r-1}(x_0)\| \leq \\
&\quad \leq \; \|D^{r-1}F(y) - P_{r-1}(x^*, y)\| + \|P_{r-1}(x^*, y) - f_{r-1}(x_0, y)\| = \\
&\quad \leq \; \|D^{r-1}F(y) - P_{r-1}(x^*, y)\| + \|P_{r-1}(x^*, y) - P_{r-1}(x_0, y)\| \leq \\
&\quad \leq \; \eta + \eta\|y - x_0\| \leq \\
&\quad \leq \; \eta(1 + \delta) \leq 2\eta = \varepsilon \, .
\end{aligned}$$

□

This proves that $F \in C^{r-1}(\mathbb{R}^d)$ and completes the proof of i) and iv). It remains to show that property vi) holds, namely that there exists a constant $K > 0$, such that $U(g(x)) \leq KF(x)$ on W_1. Obviously, this holds at every point in A^*, for every choice of K (since both functions vanish there).

Claim 2.5.10. *There exists a constant $K > 0$, such that $\frac{U \circ g}{F} \leq K$ on $W_1 \setminus A^*$.*

Proof. Since $F > 0$ on $W_1 \setminus A^*$, it is sufficient to show that $\frac{U \circ g}{F}$ is uniformly bounded by a constant, as $d(y)$ goes to zero.

Let us start observing that, for $y \in K_j^\lambda$,

$$\hat{P}_j(y) \geq C\|y - x_j\|^r \geq Cd(y)^r \, ;$$

therefore:

$$\begin{aligned}
F(y) \;&=\; \sum_j \varphi_j(y)\hat{P}_j(y) \geq \\
&\geq\; \sum_j \varphi_j(y) C d(y)^r = \\
&=\; Cd(y)^r \, .
\end{aligned}$$

Moreover, if $x^* \in A^*$ such that $d(y) = \|y - x^*\|$, lemmata 2.5.3 and 2.5.6 imply:

$$\begin{aligned} |U(g(y)) - F(y)| &\leq |U(g(y)) - P(x^*, y)| + |P(x^*, y) - F(y)| \leq \\ &\leq Cd(y)^r + Ed(y)^r = (C + E)d(y)^r \,. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{U(g(y))}{F(y)} &= \frac{U(g(y)) - F(y) + F(y)}{F(y)} \leq \\ &\leq 1 + \frac{|U(g(y)) - F(y)|}{F(y)} \leq \\ &\leq 1 + \frac{(C + E)d(y)^r}{Cd(y)^r} \leq \\ &\leq 2 + \frac{E}{C} =: \tilde{K} \,. \end{aligned}$$

□

This proves property vi) and concludes the proof of the proposition. □

Chapter 3

Dynamics and Symplectic geometry

3.1 Symplectic geometry of the phase space

The main goal of this chapter is to highlight some relations between the dynamics of the system and the underlying symplectic geometry of the phase space. As we have recalled in the addendum at the end of section 1.1, the cotangent bundle T^*M may be equipped with a canonical symplectic structure

$$\omega = \sum_{i=1}^{d} dx_i \wedge dp_i,$$

where $(\mathcal{U},\, x_1, \ldots, x_d)$ is a local coordinate chart for M and $(\mathrm{T}^*\mathcal{U},\, x_1, \ldots, x_d, p_1, \ldots, p_d)$ the associated cotangent coordinates. It is easy to show that ω is a symplectic form (*i.e.,* it is non-degenerate and closed). In particular, one can check that ω is intrinsically defined (*i.e.,* independent of the choice of the coordinate charts), by considering the 1-form on $\mathrm{T}^*\mathcal{U}$

$$\lambda = \sum_{i=1}^{d} p_i \, dx_i \,,$$

which satisfies $\omega = -d\lambda$ and is coordinate-independent; in fact, in terms of the natural projection

$$\begin{aligned} \pi : \mathrm{T}^*M &\longrightarrow M \\ (x,p) &\longmapsto x \end{aligned}$$

the form λ may be equivalently defined pointwise by

$$\lambda_{(x,p)} = (d\pi_{(x,p)})^* p \ \in \mathrm{T}^*_{(x,p)}\mathrm{T}^*M\,. \tag{3.1}$$

The 1-form λ is called the *Liouville form* (or the *tautological form*).

We also remarked that, since ω is non-degenerate and closed, the *Hamiltonian vector field* X_H is uniquely determined by:

$$\omega\left(X_H(x,p),\cdot\right) = d_x H(\cdot)\,.$$

One can easily check that, in local coordinates, the above equation is equivalent to Hamilton's equation (1.2).

A distinguished role in the study of the geometry of a symplectic space is played by the so-called *Lagrangian submanifolds*.

Definition 3.1.1 (Lagrangian submanifolds). *Let Λ be a d-dimensional C^1 submanifold of (T^*M,ω). We say that Λ is Lagrangian if for any $(x,p)\in\Lambda$, $\mathrm{T}_{(x,p)}\Lambda$ is a Lagrangian subspace, i.e., $\omega\big|_{\mathrm{T}_{(x,p)}\Lambda} = 0$.*

We will mainly be concerned with Lagrangian graphs, that is Lagrangian manifolds $\Lambda \subset T^*M$ such that $\Lambda = \{(x,\eta(x))\,,\ x\in M\}$. In (T^*M,ω) there exists an interesting well-known relation between Lagrangian graphs and closed 1-forms (recall that a 1-form can be interpreted as a section of T^*M).

Proposition 3.1.2. *Let* $\Lambda = \{(x, \eta(x)),\ x \in M\}$ *be a smooth section of* T^*M*.* Λ *is Lagrangian if and only if* η *is a closed* 1*-form.*

Proof. Let us consider:

$$\begin{aligned} s_\eta : M &\longrightarrow \mathrm{T}^*M \\ x &\longmapsto (x, \eta(x)) \,. \end{aligned}$$

We want to prove first that $s_\eta^*\lambda = \eta$, where λ is the tautological form introduced above and $s_\eta^*\lambda$ denotes its pull-back. Recalling that $\lambda(x,p) = (d\pi(x,p))^*p$, we get:

$$\begin{aligned} (s_\eta^*\lambda)(x) &= (ds_\eta(x))^*\lambda(x,p) = (ds_\eta(x))^*(d\pi(x,p))^*\eta(x) = \\ &= d(\pi \circ s_\eta(x,p))^*\eta(x) = \eta(x)\,, \end{aligned}$$

where in the last equality we used that $\pi \circ s_\eta$ is the identity map. Using this property, the claim follows immediately. In fact:

$$\begin{aligned} \Lambda \text{ is Lagrangian} &\iff \omega\big|_{\mathrm{T}\Lambda} = 0 \iff s_\eta^*\omega = 0 \iff s_\eta^* d\lambda = 0 \\ &\iff ds_\eta^*\lambda = 0 \iff d\eta = 0 \iff \eta \text{ is closed.} \end{aligned}$$

□

In the light of this relation, one can define the *cohomology class* (or *Liouville class*) of Λ to be the cohomology class of the closed 1-form representing it.

Remark 3.1.3. Observe that this cohomology class can be defined in a more intrinsic way; in fact, the projection

$$\pi_{|\Lambda_L} : \Lambda_L \subset \mathrm{T}^*M \longrightarrow M$$

induces an isomorphism between the cohomology groups $\mathrm{H}^1(M;\mathbb{R})$ and $\mathrm{H}^1(\Lambda_L;\mathbb{R})$.

The preimage of $[\lambda_{|\Lambda_L}]$ under this isomorphism is called the *Liouville class* of Λ_L and one can easily check that these two definitions coincide.

This characterization motivates the following extension of the notion of Lagrangian graph to the continuous case.

Definition 3.1.4 (Continuous Lagrangian graphs). *A continuous section Λ of T^*M is a C^0-Lagrangian graph if it locally coincides with the graph of an exact differential. As above, one can define its cohomology class.*

Let us now consider a Hamiltonian $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ and Φ_t^H the associated Hamiltonian flow. We want to analyze the relation between the property of being Lagrangian and the dynamics on it.

Proposition 3.1.5. *Let Λ be a Lagrangian submanifold. Then Λ is invariant if and only if $H\big|_\Lambda \equiv \mathrm{const}$.*

Proof. $[\Longrightarrow]$ The Hamiltonian vector field X_H is defined by $\omega(X_H, \cdot) = dH$. Since Λ is invariant, $X_H\big|_\Lambda$ is tangent to Λ. But Λ is Lagrangian, therefore $0 = \omega(X_H, V) = dH \cdot V$ for any $V \in T\Lambda$, and this implies that H is constant on Λ. $[\Longleftarrow]$ Since H is constant on Λ, we have that $0 = dH \cdot V = \omega(X_H, V)$, for every $V \in \mathrm{T}\Lambda$. Since Λ is Lagrangian, X_H belongs to $\mathrm{T}\Lambda$ itself and therefore Λ is invariant. □

Remark 3.1.6. It follows immediately from the above property that invariant Lagrangian graphs correspond to η-critical subsolutions of Hamilton-Jacobi equation, for some 1-form η, whose cohomology class coincides with that of Λ. Therefore, if Λ is an invariant Lagrangian graph, then the value of the constant in proposition 3.1.5 is given by $\alpha(c_\Lambda)$, where α is the α-function associated to H and c_Λ the cohomology class of Λ. See also remark 1.4.7.

In the next sections we will study more in depth the minimizing properties of Lagrangian graphs and deduce some uniqueness results within a fixed cohomology or homology class (see section 3.3).

Before concluding this section, let us state and prove a remarkable observation due to Michael Herman [42] for the case $M = \mathbb{T}^d$. We will use it in section 3.4 to discuss some features of *KAM tori*.

Proposition 3.1.7. *Given a Hamiltonian H, let $\mathcal{T} \subset \mathbb{T}^d \times \mathbb{R}^d$ be an invariant graph over $\mathbb{T}^d$ such that the Hamiltonian flow on $\mathcal{T}$ is conjugated to a flow R_t on $\mathbb{T}^d$, which is transitive, i.e., with a dense orbit. Then $\mathcal{T}$ is Lagrangian.*

Proof. Let $\varphi : \mathbb{T}^d \to \mathcal{T}$ be the conjugation: $\varphi^{-1} \circ \Phi_t^H \circ \varphi = R_t$, $\forall t \in \mathbb{R}$. Consider the inclusion $i_{\mathcal{T}}$ of $\mathcal{T}$ into $\mathbb{T}^d \times \mathbb{R}^d$. We want to prove that $\omega\big|_{\mathcal{T}} = i_{\mathcal{T}}^* \omega \equiv 0$. Let us start by proving that the restriction of the symplectic form $i_{\mathcal{T}}^* \omega$ is invariant under the Hamiltonian flow Φ_t^H. In fact,

$$\Phi_t^{H*} \left(i_{\mathcal{T}}^* \omega\right) = \left(i_{\mathcal{T}} \circ \Phi_t^H\right)^* \omega = \left(\Phi_t^H \circ i_{\mathcal{T}}\right)^* \omega = i_{\mathcal{T}}^* \left(\Phi_t^{H*} \omega\right) = i_{\mathcal{T}}^* \omega \,,$$

where we used that $\mathcal{T}$ is invariant ($\Phi_t^H \circ i_{\mathcal{T}} = i_{\mathcal{T}} \circ \Phi_t^H$) and Φ_t^H is a symplectomorphism for any $t \in \mathbb{R}$ (*i.e.*, $\Phi_t^{H*} \omega = \omega$). Consider now the 2-form on $\mathbb{T}^d$ given by $\omega_1 = \varphi^* \left(i_{\mathcal{T}}^* \omega\right)$. Let us show that ω_1 is invariant under R_t; in fact,

$$\begin{aligned}
(R_t)^* \omega_1 &= (R_t)^* \left(\varphi^* \left(i_{\mathcal{T}}^* \omega\right)\right) = \left(\varphi \circ R_t\right)^* i_{\mathcal{T}}^* \omega = \left(\Phi_t^H \circ \varphi\right)^* i_{\mathcal{T}}^* \omega = \\
&= \varphi^* \left(\Phi_t^{H*} \left(i_{\mathcal{T}}^* \omega\right)\right) = \varphi^* \left(i_{\mathcal{T}}^* \omega\right) = \omega_1 \,,
\end{aligned}$$

where we used that $\varphi^{-1} \circ \Phi_t^H \circ \varphi = R_t$. Since R_t is transitive, then ω_1 invariant implies ω_1 constant: $\omega_1 = \sum_{i<j} a_{ij}\, dx_i \wedge dx_j$. But ω_1 is exact (since $\omega = -d\lambda$ is exact), therefore $\omega_1 = \varphi^* \left(i_{\mathcal{T}}^* \omega\right) \equiv 0$ and (using that φ is invertible) the result is proved. □

3.2 Symplectic aspects of the Aubry set and the quotient Aubry set

In this section we want to discuss purely symplectic definitions of the Aubry set and the quotient Aubry set, which follow quite easily from weak KAM theory approach. In fact, reinterpreting the notion of critical subsolutions from a more geometric prospect, one can deduce many symplectic properties of these sets; in particular we will deduce that they are "invariant" under exact symplectomorphism. Similar results have also been discussed in [15, 69].

As we have seen in section 1.4, the Aubry set $\mathcal{A}^*_c$ can be equivalently defined as:

$$\mathcal{A}^*_c = \bigcap_{u \in \mathcal{S}^1_\eta} \mathrm{Graph}(\eta + du) \tag{3.2}$$

where $\mathcal{S}^1_\eta$ denotes the set of η-critical subsolutions (see theorem 1.4.19 and (1.22)).

Let us give the following definition.

Definition 3.2.1 (Subcritical Lagrangian graphs). *Given a Lagrangian graph Λ with Liouville class c, we will say that Λ is c-subcritical, or simply subcritical, if $\Lambda \subset \{(x,p) \in \mathrm{T}^*M : \; H(x,p) \leq \alpha(c)\}$, where $\alpha : \mathrm{H}^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ is Mather's α-function. Given a subcritical Lagrangian graph Λ with Liouville class c, we will call $\Lambda_{crit} = \{(x,p) \in \Lambda : \; H(x,p) = \alpha(c)\}$ its critical part.*

Observe that c-critical Lagrangian graphs correspond to η-critical subsolutions, for any η closed 1-form with cohomology c. Therefore, we can deduce from weak KAM theory (in particular theorem 1.4.24) that they do exist and that Lipschitz graphs are "dense" (in the sense of theorem 1.4.24). In terms of these Lagrangian graphs, (3.2) becomes:

$$\mathcal{A}^*_c = \bigcap_{c-\text{subcritical } \Lambda} \Lambda = \bigcap_{c-\text{subcritical } \Lambda} \Lambda_{crit} \tag{3.3}$$

where the intersections are over the set of all c-critical Lagrangian graphs. In particular, there will exist a c-critical Lagrangian graph $\tilde{\Lambda}$ such that $\mathcal{A}^*_c = \tilde{\Lambda}_{crit} = \tilde{\Lambda} \cap \mathcal{E}^*_c$, where $\mathcal{E}^*_c$ is Mañé critical energy level (see (1.23)). Observe that, as discussed in [69], this property can be also interpreted as a "*non-removable intersection property*" for such particular Lagrangian graphs.

In terms of c-critical Lagrangian graphs one can also get the following geometric characterization for the Mañé critical energy level $\mathcal{E}^*_c$ and, consequently, for $\alpha(c)$: it is the only energy level of H such that the bounded region that it "encloses" does not contain any c-Lagrangian graphs in its interior, while each of its open neighborhoods does. In particular, it follows from the existence of critically dominated functions (see remark 1.4.5) that there exist continuous (even Lipschitz) c-Lagrangian graphs contained in the closure of this region bounded by $\mathcal{E}^*_c$: these are exactly what we called c-critical Lagrangian graphs. Moreover, it is immediate from the above definitions that if any energy level of the Hamiltonian contains a c-Lagrangian graph, then this energy level is the Mañé c-critical one: in fact any higher energy level will bound a region containing in its interior a c-Lagrangian graph.

Remark 3.2.2. Observe that the above characterizations provide definitions for the Aubry set and the Mañé critical energy level, which do not depend on the form of the Hamiltonian, except for its energy level. Therefore, if H' is another Hamiltonian that has $\mathcal{E}^*_c$ as energy level, then $\mathcal{E}^*_c$ will be its Mañé c-critical energy level. Moreover, it follows from (3.3) that H' will also have the same Aubry set $\mathcal{A}^*_c$. These properties are not surprising. In fact, it is easy to show that on a given energy level, the flows of all Hamiltonians that possess it as an energy level, is the same up to time-reparameterization.

Now, we want to show how the above concepts allow one to get also a geometric definition of the quotient Aubry set; this is not discussed in [69]. Let us start with

Mather's pseudodistance δ_c defined in (1.14). For any $x, y \in \mathcal{A}_c$ and Λ_1, Λ_2 c-critical Lagrangian graphs, let us consider a closed path $\lambda_{x,y}^{\Lambda_1,\Lambda_2}$ obtained by joining x to y on Λ_1 and then going back to x staying on Λ_2. Using (1.25) it is immediate to check that:

$$\delta_c(x,y) = \sup_{\substack{\Lambda_1,\,\Lambda_2 \\ c-\text{critical}}} \int_{\lambda_{x,y}^{\Lambda_1,\Lambda_2}} \lambda \tag{3.4}$$

where λ is the Liouville form defined in (3.1) and the integral does not depend on the chosen path $\lambda_{x,y}^{\Lambda_1,\Lambda_2}$. As a consequence of this definition, one gets a characterization of c-static classes and, therefore, of the quotient Aubry set $\bar{\mathcal{A}}_c$. From a geometric point of view, two points in the same c-static class are points for which the integral $\int_{\lambda_{x,y}^{\Lambda_1,\Lambda_2}} \lambda$ is zero for all possible loops constructed as above. It would be interesting to understand if this property has a geometric meaning too; Floer homology might provide the useful ground for tackling this question.

Let us now deduce, from these characterizations, the following result concerning the invariance of the Aubry set, the quotient Aubry set and the Mañé critical energy level under exact symplectomorphisms. Let us first recall that a diffeomorphism $\Psi : \mathrm{T}^*M \longrightarrow \mathrm{T}^*M$ is a *symplectomorphism* if it preserves the symplectic form ω, *i.e.,* $\Psi_*\omega = \omega$; in particular this is equivalent to say that $\Psi_*\lambda - \lambda$ is a closed 1-form. We will say that Ψ is an *exact symplectomorphism* if $\Psi_*\lambda - \lambda$ is exact.

Theorem 3.2.3. *Let H be an optical Hamiltonian on T^*M and let $\Psi : \mathrm{T}^*M \longrightarrow \mathrm{T}^*M$ be an exact symplectomorphism. Consider the new Hamiltonian $H' = H \circ \Psi^{-1}$. Then for all $c \in \mathrm{H}^1(M;\mathbb{R})$:* [1]

- $\mathcal{E}_c^{*\prime} = \Psi(\mathcal{E}_c^*)$ *and therefore* $\alpha'(c) = \alpha(c)$*;*
- $\mathcal{A}_c^{*\prime} = \Psi(\mathcal{A}_c^*)$*;*

[1] We will indicate with a *prime* all quantities associated to H' (*e.g.,* $\mathcal{E}_c^{*\prime}, \mathcal{A}', \delta_c'$, etc...)

- *$\delta_c(x,y) = \delta'_c(\Psi(x),\Psi(y))$; therefore it maps c-static classes of H' into c-static classes of H and the induced map $\overline{\Psi} : \bar{\mathcal{A}}_c \to \bar{\mathcal{A}}'_c$ is an isometry.*

A similar result has also been proven, with different techniques, by Patrick Bernard in [15].

Proof. It is enough to observe that exact symplectomorphisms transform c-Lagrangian graphs into c-Lagrangian graphs. The proof will then easily follow from (3.3), (3.4) and the characterization of Mañé c-critical energy level. □

3.3 Minimizing properties of Lagrangian graphs

In this section we will analyze the minimizing properties of invariant measures supported on Lagrangian graphs and deduce some uniqueness results for invariant Lagrangian graphs within a fixed cohomology or homology class (we will clearify what we mean by homology class later in this section; see definition 3.3.5).

An interesting result is the following characterization of minimizing measures, introduced in section 1.2.

Lemma 3.3.1. *Let μ be an invariant probability measure on $\mathrm{T}M$ and $\mu^* = \mathcal{L}_*\mu$ its push-forward to T^*M, via the Legendre transform $\mathcal{L}$. Then, μ is a Mather's measure if and only if $\operatorname{supp}\mu^*$ is contained in the critical part of a subcritical Lagrangian graph. In particular, any invariant probability measure μ^* on T^*M, whose support is contained in an invariant Lagrangian graph with Liouville class c, is the image, via the Legendre transform, of a c-action minimizing measure.*

Proof. (i) If μ is a Mather's measure with cohomology class c, then the support of μ^* is contained in $\mathcal{L}(\widetilde{\mathcal{M}}_c) \subseteq \mathcal{A}^*_c$. By (1.22), $\mathcal{A}^*_c$ is given by the intersection of all c-subcritical Lagrangian graphs, so $\operatorname{supp}\mu^*$ is contained in at least one c-subcritical Lagrangian graph Λ. In particular $\operatorname{supp}\mu^*$ is contained in the critical part of Λ, simply because $\mathcal{L}(\widetilde{\mathcal{M}}_c)$ is in the energy level $\mathcal{E}_c^* = \{(x,p) \in \mathrm{T}^*M : \; H(x,p) = \alpha(c)\}$, see Theorem 1.2.7.

(*ii*) Let us fix η to be a closed 1-form with $[\eta] = c$. Since we are assuming that Λ is a c-subcritical Lagrangian graph, we can write $\Lambda = \{(x, \eta(x)+du(x))\,,\, x \in M\}$, where $u : M \to \mathbb{R}$ is C^1. By Theorem 1.3.17, in order to show that μ is a c-action minimizing measure, it is enough to show that $\operatorname{supp}\mu \subseteq \widetilde{\mathcal{N}}_c$, *i.e.,* that any orbit in $\operatorname{supp}\mu$ is a c-minimizer. To this purpose, let us consider $(x,v) \in \operatorname{supp}\mu$ and let $\gamma(t) \equiv \pi(\Phi_t(x,v))$, where Φ_t is the Euler-Lagrange flow and π the canonical projection on M. Given any interval $[a,b] \subset \mathbb{R}$, let us consider the difference $u(\gamma(b)) - u(\gamma(a))$ and rewrite it as:

$$\begin{aligned} u(\gamma(b)) - u(\gamma(a)) &= \int_a^b d_{\gamma(s)}u(\gamma(s))\dot{\gamma}(s)\,ds = \\ &= \int_a^b \big[L_\eta(\gamma(s),\dot{\gamma}(s)) + H_\eta(\gamma(s), d_{\gamma(s)}u)\big]\,ds\,, \end{aligned} \tag{3.5}$$

where the second equality follows from the definition of the Hamiltonian as the Legendre-Fenchel transform of the Lagrangian and the fact that $\gamma(s)$ is an orbit of the Euler-Lagrange flow. Note that along the orbit $H_\eta(\gamma(s), d_{\gamma(s)}u) = \alpha(c)$, because $\operatorname{supp}\mu$ is invariant and $\operatorname{supp}\mu^*$ is in the critical part of Λ. Then

$$\int_a^b L_\eta(\gamma(s),\dot{\gamma}(s))ds = u(\gamma(b)) - u(\gamma(a)) - \alpha(c)(b-a)\;. \tag{3.6}$$

On the other hand, any other curve $\gamma_1 : [a,b] \to M$ such that $\gamma_1(a) = \gamma(a)$ and $\gamma_1(b) = \gamma(b)$ satisfies:

$$\begin{aligned} u(\gamma(b)) - u(\gamma(a)) &= \int_a^b d_{\gamma_1(s)}u(\gamma_1(s))\dot{\gamma}(s)\,ds \leq \\ &\leq \int_a^b \big[L_\eta(\gamma_1(s),\dot{\gamma}_1(s)) + H_\eta(\gamma_1(s), d_{\gamma_1(s)}u)\big]ds \end{aligned} \tag{3.7}$$

where the second inequality follows again by the duality between Hamiltonian and

Lagrangian. Note that now $H_\eta(\gamma_1(s), d_{\gamma_1(s)}u) \leq \alpha(c)$, because $\Lambda = \{(x, \eta(x) + du(x))\}$ is subcritical. Then

$$\int_a^b L_\eta(\gamma_1(s), \dot{\gamma}_1(s))ds \geq u(\gamma(b)) - u(\gamma(a)) - \alpha(c)(b-a) \tag{3.8}$$

and this proves that γ is a c-minimizer (see also proposition 1.4.9).

Let us finally observe that the Hamiltonian is constant on any invariant Lagrangian graph $\Lambda = \{(x, \eta + du)\}$, *i.e.*, $H(x, \eta + du) = k$ (see proposition 3.1.5). Then u is a classical solution of the Hamilton-Jacobi equation with cohomology class c. As explained in remark 3.1.6, $k = \alpha(c)$ and this shows that Λ coincides with its critical part. By the result proved in item (*ii*), if μ^* is supported on Λ, then μ is a c-action minimizing measure and this proves the last claim in the statement of the lemma. □

Remark 3.3.2. Note that the fact that the orbits on an invariant Lagrangian graph are action-minimizing can be also deduced from a classical result by Weierstrass, as already pointed out by Jürgen Moser (see remark in [57]). In fact, Weierstrass method or the use of the Hamilton-Jacobi Equation (that we are using in our proof) are essentially two sides of the same coin.

This result easily implies an already-known uniqueness result for Lagrangian graphs supporting invariant measures of full support, in a fixed cohomology class (see also [57], in which a different proof is presented).

Theorem 3.3.3. *If $\Lambda \subset \mathrm{T}^*M$ is a Lagrangian graph on which the Hamiltonian dynamics admits an invariant measure μ^* with full support, then $\Lambda = \mathcal{L}\big(\widetilde{\mathcal{M}}_c\big) = \mathcal{A}^*_c$, where c is the cohomology class of Λ. Therefore, if Λ_1 and Λ_2 are two Lagrangian graphs as above, with the same cohomology class, then $\Lambda_1 = \Lambda_2$. In other words, for any given $c \in \mathrm{H}^1(M;\mathbb{R})$, there exists at most one invariant Lagrangian graph Λ with* *cohomology class c, that carries an invariant measure whose support is the whole of Λ.*

Proof. By Lemma 3.3.1, the measure $\mu = \mathcal{L}_*^{-1}\mu^*$ is c-minimizing. This means that $\mathcal{L}^{-1}(\Lambda) = \operatorname{supp}\mu \subseteq \widetilde{\mathcal{M}}_c \subseteq \widetilde{\mathcal{A}}_c$, where the last inclusion follows from Theorem 1.3.17. Note however that, by Theorems 1.2.7 and 1.3.16, $\widetilde{\mathcal{M}}_c$ and $\widetilde{\mathcal{A}}_c$ are graphs over their bases and, since $\operatorname{supp}\mu$ is a graph over the whole M, it follows that

$$\mathcal{L}^{-1}(\Lambda) = \operatorname{supp}\mu = \widetilde{\mathcal{M}}_c = \widetilde{\mathcal{A}}_c\,. \tag{3.9}$$

□

One can deduce something more from the above proof.

Theorem 3.3.4. *If Λ and μ are as in Theorem 3.3.3 and ρ is the rotation vector of $\mu = \mathcal{L}^{-1}\mu^*$, then $\Lambda = \mathcal{L}\big(\widetilde{\mathcal{M}}^\rho\big)$. Therefore, if Λ_1 and Λ_2 are two Lagrangian graphs supporting measures of full support and the same rotation vector ρ, then $\Lambda_1 = \Lambda_2$. Moreover, Mather's β-function is differentiable at ρ with $\partial\beta(\rho) = c$, where c is the cohomology class of Λ.*

Proof. The first claim follows from the fact that $\widetilde{\mathcal{M}}^\rho$ is a graph over M and that by definition $\widetilde{\mathcal{M}}^\rho \supseteq \operatorname{supp}\mu = \mathcal{L}^{-1}(\Lambda)$. As far as the differentiability of β at ρ is concerned, suppose that $c' \in \mathrm{H}^1(M;\mathbb{R})$ is a subderivative of β at ρ. Hence, $\beta(\rho) = \langle c', \rho\rangle - \alpha(c')$ and this implies that $\mathfrak{M}^\rho \subseteq \mathfrak{M}_{c'}$; in fact, for any $\mu \in \mathfrak{M}^\rho$:

$$\int_{\mathrm{T}M} (L - \hat{\eta}')\, d\mu = \int_{\mathrm{T}M} L\, d\mu - \int_{\mathrm{T}M} \hat{\eta}'\, d\mu = \beta(\rho) - \langle c', \rho\rangle = -\alpha(c')\,,$$

where η' is a closed 1-form of cohomology c'. As a result, $\widetilde{\mathcal{M}}^\rho = \mathcal{L}^{-1}(\Lambda) \subseteq \widetilde{\mathcal{M}}_{c'}$. The graph property of $\widetilde{\mathcal{M}}_{c'}$ and of $\widetilde{\mathcal{A}}_{c'}$ implies that $\widetilde{\mathcal{A}}_{c'} = \widetilde{\mathcal{M}}_{c'} = \mathcal{L}^{-1}(\Lambda)$ and $\mathcal{A}^*_{c'} = \Lambda$. As a consequence, $c' = c$. In fact, by Theorem 1.4.24 and (1.22), there exists a C^1 function $v : M \longrightarrow \mathbb{R}$, such that $\Lambda = \{(x, \eta' + dv) :\ x \in M\}$, where η' is a closed

1-form with $[\eta'] = c'$. This, by definition, means that c' is the cohomology class of Λ and therefore $c' = c$. □

Now, recall that to any invariant probability measure μ on $\mathrm{T}M$ one can associate an element $\rho(\mu)$ of the homology group $\mathrm{H}_1(M;\mathbb{R})$, known as *rotation vector* or *Schwartzman asymptotic cycle* (see sections 1.2 and 3.5). This allows us to define the homology class of certain invariant Lagrangian graphs.

Definition 3.3.5 (Schwartzman uniquely/strictly ergodic Lagrangian graphs). *A Lagrangian graph Λ is called Schwartzman uniquely ergodic if all invariant measures supported on Λ have the same rotation vector ρ, which will be called homology class of Λ. Moreover, if there exists an invariant measure with full support, Λ will be called Schwarztman strictly ergodic.*

See section 3.5 for a more detailed description of such flows. We can now state and prove the following uniqueness result.

Theorem 3.3.6 ([34]). *Let Λ be a Schwartzman strictly ergodic invariant Lagrangian graph with homology class ρ. The following properties are satisfied:*

(i) *if $\Lambda \cap \mathcal{A}^*_c \neq \emptyset$, then $\Lambda = \mathcal{A}^*_c$ and $c = c_\Lambda$, where c_Λ is the cohomology class of Λ.*

(ii) *the Mather function α is differentiable at c_Λ and $\partial\alpha(c_\Lambda) = \rho$.*

Therefore,

(iii) *any invariant Lagrangian graph that carries a measure with rotation vector ρ is equal to the graph Λ;*

(iv) *any invariant Lagrangian graph is either disjoint from Λ or equal to Λ.*

Proof. (i) From Theorem 3.3.3, it follows that $\Lambda = \mathcal{A}^*_{c_\Lambda}$. Let us show that it does not intersect any other Aubry set. Suppose by contradiction that Λ intersects another

Aubry set $\mathcal{A}^*_c$. By Theorem 3.3.4, $\Lambda = \mathcal{L}^{-1}\left(\widetilde{\mathcal{M}}^{\rho}\right)$, then $\widetilde{\mathcal{M}}^{\rho} \cap \widetilde{\mathcal{A}}_c \neq \emptyset$ and, because of the Lemma 1.3.18 and the graph property of $\widetilde{\mathcal{A}}_c$, we can conclude that $\mathcal{A}^*_c = \Lambda$. The same argument used in the proof of Theorem 3.3.4 allows us to conclude that $c = c_\Lambda$.
(*ii*) Suppose that $h \in \partial\alpha(c_\Lambda)$. The previous lemma implies that $\widetilde{\mathcal{M}}^h \subseteq \Lambda$; the Schwartzman unique ergodicity property of Λ implies $h = \rho$. Therefore α is differentiable at c_Λ and $\partial\alpha(c_\Lambda) = \rho$.

To prove (*iv*), let Λ_1 be an invariant Lagrangian graph and call c_1 its cohomology class. If the compact invariant set $\Lambda \cap \Lambda_1$ is not empty, then we can find a probability measure μ^* invariant under the flow and whose support is contained in this intersection. Since μ^* is contained in the Lagrangian graph Λ_1, by Lemma 3.3.1, it is c_1-minimizing. Hence, the support of μ^* is contained in $\mathcal{A}^*_{c_1}$. This shows that the intersection $\Lambda \cap \mathcal{A}^*_{c_1}$ contains the support of μ^* and is therefore not empty. By (*i*), $\Lambda = \mathcal{A}^*_{c_1}$. Moreover, note that $\mathcal{A}^*_{c_1} \subseteq \Lambda_1$, because, on the one hand, $\Lambda_1 = \text{Graph}(\eta_1 + du_1)$, with $[\eta_1] = c_1$ and u_1 a classical solution to the Hamilton-Jacobi equation (see proof of Lemma 3.3.1), and, on the other hand, $\mathcal{A}^*_{c_1} = \bigcap_{u \in \mathcal{S}^1_\eta} \text{Graph}(\eta_1 + du_1)$, see (1.22). Therefore, $\Lambda = \Lambda_1$, since they are both graphs over the base.

To prove (*iii*), consider an invariant Lagrangian graph Λ_1, with cohomology class c_1, which carries an invariant measure μ^* whose rotation vector is ρ. By Lemma 3.3.1, the measure μ^* is c_1-minimizing. Therefore, we have $\widetilde{\mathcal{M}}^{\rho} \cap \widetilde{\mathcal{A}}_{c_1} \neq \emptyset$. By Proposition 1.3.18, it follows that $\mathcal{L}^{-1}(\Lambda) = \widetilde{\mathcal{M}}^{\rho} \subseteq \widetilde{\mathcal{A}}_{c_1} \subseteq \mathcal{L}^{-1}(\Lambda_1)$. Again, this forces the equality $\Lambda = \Lambda_1$ because of the graph property. □

Finally, observe that Lemma 1.2.6, Theorem 1.3.17 and Proposition 1.3.18 imply the following property.

Corollary 3.3.7. *Mather's α function is differentiable at c if and only if the restriction of the Euler-Lagrange flow to $\widetilde{\mathcal{A}}_c$ is Schwartzman uniquely ergodic (see section 3.5 for the definition and a discussion of Schwartzman ergodic flows).*

3.4 Global uniqueness of KAM tori and Herman's tori

In the case $M = \mathbb{T}^d$, it is natural to ask for the implications of the results in section 3.3 for KAM theory. In this section we will use these results to discuss the problem of uniqueness of KAM tori and, more generally, of the invariant tori belonging to the closure of the set of KAM tori (or *Herman's Tori*).

Let us start by recalling what we mean by KAM torus (recall that in this case the homology group $H_1(\mathbb{T}^d;\mathbb{R})$ can be canonically identified with $\mathbb{R}^d$).

Definition 3.4.1 (KAM Torus). *$\mathcal{T} \subset \mathbb{T}^d \times \mathbb{R}^d$ is a (maximal) KAM torus with rotation vector ρ if:*
i) $\mathcal{T} \subset \mathbb{T}^d \times \mathbb{R}^d$ is a continuous graph over $\mathbb{T}^d$;
ii) $\mathcal{T}$ is invariant under the Hamiltonian flow Φ_t^H generated by H;
iii) the Hamiltonian flow on $\mathcal{T}$ is conjugated to a uniform rotation on $\mathbb{T}^d$; i.e., there exists a diffeomorphism $\varphi : \mathbb{T}^d \to \mathcal{T}$ such that $\varphi^{-1} \circ \Phi_t^H \circ \varphi = R_\rho^t$, $\forall t \in \mathbb{R}$, where $R_\rho^t : x \to x + \rho t$.

KAM theory concerns with the existence of KAM tori for quasi-integrable Hamiltonian systems of the form $H(x,p) = H_0(p) + \varepsilon f(x,p)$, where: (x,p) are local coordinates on $\mathbb{T}^d \times \mathbb{R}^d$, ε is a "small" parameter and $f(x,p)$ a smooth function. If $\varepsilon = 0$, the system is integrable, in the sense that the dynamics can be explictly solved: in particular each torus $\mathbb{T}^d \times \{p_0\}$ is invariant and the motion on it corresponds to a rotation with frequency $\rho(p_0) = \frac{\partial H_0}{\partial p}(p_0)$. The question addressed by KAM theory is whether this foliation of the phase space into invariant tori, on which the motion is quasi-periodic, persists when $\varepsilon \neq 0$. In 1954 Kolmogorov [44] proved (and Arnol'd [2] and Moser [66, 67] reproved it later in different contexts and with different techniques) that, in spite of the generic disappearence of the invariant submanifolds filled by periodic orbits, as already pointed out by Poincaré, for small ε it is always possible to find KAM tori corresponding to "*strongly non-resonant*", *i.e., Diophantine*, rotation vectors. Let us recall here the definition and some properties of Diophantine vectors:

Definition 3.4.2. *Given $\gamma, \tau > 0$, we say that $\rho \in \mathbb{R}^d$ is a (γ, τ)-Diophantine vector if and only if $|\langle \rho, \nu \rangle| \geq \gamma |\nu|^{-\tau}$, $\forall \nu \in \mathbb{Z}^d \setminus \{0\}$.*

Remark 3.4.3. The set of (γ, τ)-Diophantine vectors will be denoted by $\mathcal{D}(\gamma, \tau)$. Note that, if $\tau < d-1$, $\mathcal{D}(\gamma, \tau) = \emptyset$, while for $\tau > d-1$, the Diophantine vectors have full measure in $\mathbb{R}^d$, that is $\lim_{R \to \infty} \mu_0 \left(\bigcup_{\gamma > 0} \mathcal{D}(\gamma, \tau) \cap B_R \right) / \mu_0(B_R) = 1$, where μ_0 is the Lesbegue measure and B_R is the ball of radius R centered at 0; for $\tau = d-1$, $\bigcup_{\gamma > 0} \mathcal{D}(\gamma, \tau)$ has measure zero but Hausdorff dimension d.

The celebrated KAM Theorem (in one of its several versions) not only shows the existence of such tori, but also provides an explicit method to construct them.

Theorem 3.4.4 (Kolmogorov–Arnol'd–Moser, [73]). *Let $d \geq 2$, $\tau > d-1$, $\gamma > 0$, $\ell > 2\tau + 2$, $M > 0$ and $r > 0$ be given. Let $B_r \in \mathbb{R}^d$ be the open ball of radius r centered at the origin. Let $H \in C^\ell(\mathbb{T}^d \times B_r)$ be of the form*

$$H(x, p) = H_0(p) + \varepsilon f(x, p) \tag{3.10}$$

with $|H_0|_{C^\ell} \leq M$, $|f|_{C^\ell} \leq M$, $\left| \frac{\partial^2 H_0}{\partial p^2} \right| \geq M^{-1}$ and $\rho = \frac{\partial H_0}{\partial p}(0) \in \mathcal{D}(\gamma, \tau)$. Then, for any $s < \ell - 2\tau - 1$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$ the Hamiltonian (3.10) *admits a $C^{s,s+\tau}$ KAM torus with rotation vector ρ, i.e., a $C^{s+\tau}$ invariant torus such that the Hamiltonian flow on it is C^s-conjugated with a rotation with frequency ρ.*

Remark 3.4.5. 1) If $H \in C^\infty$ then the KAM torus mentioned in the theorem above is C^∞. If H is real analytic then the KAM torus is real analytic.

2) As already mentioned, the proof of this theorem is constructive and it actually

contains much more information than those summarized in the above statement. For instance, in the analytic case, the proof consists in an iterative method allowing one to construct order by order the series defining the conjugation function (and to prove convergence of the formal series). In the differentiable case the proof is based on the idea of iteratively approximating differentiable functions by analytic ones and then using the inductive approximation scheme of the analytic case. However, although these proofs provide an explicit construction of a KAM torus , these construction depends *a priori* on a number of arbitrary choices that one has to make along the proof – *e.g.,* the choice of cutoffs one needs to introduce in the iterative approximation scheme.

3) The invariant torus constructed in the proof of the KAM theorem is locally unique, in the sense that for any prescribed (and admissible) s there is at most one $C^{s,s+\tau}$ KAM torus with rotation vector ρ within a C^s-distance $\delta(d, s, C, \tau)$ to the one constructed in the proof of the KAM Theorem, see [17, 72, 73]. Note that the C^s-distance δ within which one can prove uniqueness of the KAM torus in a prescribed regularity class depends both on the irrationality properties of ρ and on the regularity class s itself. It is then *a priori* possible that even for small ε there exist different KAM tori, within a prescribed C^1-distance from the one constructed in the proof of the theorem, possibly less regular than that torus. Quite surprisingly, even in the analytic case, we are not aware of any proof of "global" uniqueness of the invariant analytic KAM torus with rotation vector ρ; of course in the analytic case the analytic torus one manages to construct is unique within the class of analytic tori – however nothing *a priori* guarantees that less regular invariant tori with the same rotation vector exist.

The question arisen in remark (3) is our main motivation for the study of the problem of global uniqueness of KAM tori. Corollary 3.4.6 below settles the question and shows that, at least in the case of optical Hamiltonians, it is not possible to have two different KAM tori with the same rotation vector. Note that the assumption of strict convexity of the Hamiltonian is necessary to exclude trivial sources of non-uniqueness: for instance, in the context of quasi-integrable Hamiltonians, global uniqueness could be lost simply because the unperturbed Hamiltonian induces a map $p \to \partial_p H_0(p)$ from actions to frequencies that is not one to one. Let us also remark that, apparently, the Hamiltonian considered in KAM theorem is not a Tonelli Hamiltonian, since the latter, by definition, is defined globally on the whole $\mathbb{T}^d \times \mathbb{R}^d$. However any C^ℓ strictly convex Hamiltonian defined on $\mathbb{T}^d \times B_r$ for some $r > 0$ can be extended to a global C^ℓ optical Hamiltonian. Then in the statement of the KAM Theorem above it is actually enough to assume H to be a C^ℓ optical Hamiltonian, locally satisfying the (in)equalities listed after (3.10).

Given the proof of proposition 3.3.6, it follows the following corollary.

Corollary 3.4.6 (Global uniqueness of KAM tori, [34]). *Every optical Hamiltonian H on $\mathrm{T}^*\mathbb{T}^d$ possesses at most one Lagrangian KAM torus for any given rotation vector ρ. In particular, if H and ρ satisfy the assumptions of the KAM Theorem, then there exists one and only one KAM torus with rotation vector ρ.*

Proof. Since the Lagrangian KAM torus $\mathcal{T}$ admits an invariant measure μ^* of full support, which is the image via the conjugation φ of the uniform measure on $\mathbb{T}^d$, then the claims follow from Theorem 3.3.4. Note that for rationally independent rotation vectors, Herman's remark in proposition 3.1.7, implies that $\mathcal{T}$ is automatically Lagrangian. □

An interesting generalization of the result of Corollary 3.4.6 concerns the invariant tori belonging to the C^0-closure $\overline{\Upsilon}$ of the set Υ of all Lagrangian KAM tori. Note that for quasi-integrable systems, Υ is not empty. The set Υ can be seen as a subset of $\mathrm{Lip}(\mathbb{T}^d, \mathbb{R}^d)$. This follows from Theorem 3.3.4 and from Mather's graph theorem, see Theorems 1.2.7, 1.2.9, and the results in [57]. Moreover, any family of invariant Lagrangians graphs on which the function α (or H) is bounded gives rise to a family of functions in $\mathrm{Lip}(\mathbb{T}^d, \mathbb{R}^d)$ with uniformly bounded Lipschitz constant (see section 1.4). By Ascoli-Arzelà theorem, it follows that $\overline{\Upsilon}$ is also a subset of $\mathrm{Lip}(\mathbb{T}^d, \mathbb{R}^d)$, consisting of functions whose graphs are invariant C^0-Lagrangian tori. Michael Herman [43] showed that, for a generic Hamiltonian H close enough to an integrable Hamiltonian H_0, the dynamics on the generic tori in $\overline{\Upsilon}$ is not conjugated to a rotation. These "new" tori therefore represent the majority, in the sense of topology, and hence most invariant tori cannot be obtained by the KAM algorithm. More precisely, Herman showed that in $\overline{\Upsilon}$ there exists a dense G_δ set (*i.e.,* a dense countable intersection of open sets) of invariant Lagrangian graphs on which the dynamics is strictly ergodic and weakly mixing, and for which the rotation vector is not Diophantine. These invariant graphs are therefore not obtained by the KAM theorem, however our uniqueness result do still apply to these graphs since strict ergodicity implies Schwartzman strict ergodicity.

More generally, given any Tonelli Lagrangian on $\mathbb{T}^d$, we consider the set $\tilde{\Upsilon}$ of invariant Lagrangian graphs on which the dynamics of the flow is topologically conjugated to an *ergodic* linear flow on $\mathbb{T}^d$ (of course, far from the canonical integrable Lagrangian the set $\tilde{\Upsilon}$ may be empty). The dynamics on any of the invariant graphs in $\tilde{\Upsilon}$ is strictly ergodic. Since the set of strictly ergodic flows on a compact set is a G_δ in the C^0 topology, see for example [35, Corollaire 4.5], it follows that there exists a dense G_δ subset $\mathcal{G}$ of the C^0 closure of $\tilde{\Upsilon}$ in $\mathrm{Lip}(\mathbb{T}^d, \mathbb{R}^d)$, such that the dynamics on any $\Lambda \in \mathcal{G}$ is strictly ergodic. Therefore we have the following proposition.

Proposition 3.4.7 (Uniqueness of Herman's Tori, [34]). *There exists a dense G_δ set $\mathcal{G}$ in the C^0 closure of $\tilde{\Upsilon}$ consisting of strictly ergodic invariant Lagrangian graphs. Any $\Lambda \in \mathcal{G}$ satisfies the following properties:*

(i) *the invariant graph Λ has a well-defined rotation vector $\rho(\Lambda)$.*

(ii) *Any invariant Lagrangian graph that intersects Λ coincides with Λ.*

(iii) *Any Lagrangian invariant graph that carries an invariant measure whose rotation is* $\rho(\Lambda)$ *coincides with* Λ*.*

3.5 Schwartzman unique and strict ergodicity

In Section 1.2 we have introduced the concept of rotation vector of a measure. This is closely related to the notion of Schwartzman asymptotic cycle of a flow, introduced by Sol Schwartzman in [74], as a first attempt to develop an algebraic topological approach to the study of dynamics. In this section, we would like to provide some examples and investigate some properties of what we have called *Schwartzman uniquely ergodic flows* (see section 3.4).

First, we would like to discuss more in depth the concept of rotation vector and Schwartzman asymptotic cycle. One can provide a different description of the Schwartzman asymptotic cycle of a flow. This is also known as the flux homomorphism in volume preserving and symplectic geometry, see [9][Chapter 3]. We will use the description given in [26][pages 67-70]. This definition has the technical advantadge of not relying on the Krylov-Bogolioubov theory of generic orbits in a dynamical system, although a more geometrical definition showing that "averaged" pieces of long orbits converge almost everywhere in the first homology group for any invariant measure is certainly more heuristic and intuitive.

Let us start with some standard facts. As usual we set $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The space $\mathbb{T}$ is a topological group for the addition. An important feature of $\mathbb{T}$ is that the canonical projection $\pi : \mathbb{R} \to \mathbb{T}$ is a covering map. Therefore given any continuous path $\gamma : [a, b] \to \mathbb{T}$, with $a \leq b$, we can find a continuous lift $\bar{\gamma} : [a, b] \to \mathbb{R}$ such that $\gamma = \pi\bar{\gamma}$. Any two such lifts differ by an integer. It follows that the quantity $\bar{\gamma}(b) - \bar{\gamma}(a)$ does not depend on the lift. We will set

$$\mathcal{V}(\gamma) = \bar{\gamma}(b) - \bar{\gamma}(a) \in \mathbb{R}.$$

This quantity remains constant on the homotopy class, with fixed end-points, of the path γ. Moreover, if γ is a closed path, *i.e.,* we have $\gamma(a) = \gamma(b)$ then $\mathcal{V}(\gamma) \in \mathbb{Z}$, and, if such a closed path is homotopic to 0 (with fixed end-points) then $\mathcal{V}(\gamma) = 0$. It is also clear that for $c \in [a, b]$, we have

$$\mathcal{V}(\gamma|[a, b]) = \mathcal{V}(\gamma|[a, c]) + \mathcal{V}(\gamma|[c, b]).$$

Note also that two continuous paths $\gamma_1, \gamma_2 : [a, b] \to \mathbb{T}$ can be added by the formula

$$(\gamma_1 + \gamma_2)(t) = \gamma_1(t) + \gamma_2(t).$$

For this addition we have

$$\mathcal{V}(\gamma_1 + \gamma_2) = \mathcal{V}(\gamma_1) + \mathcal{V}(\gamma_2).$$

Another important property of the map $\mathcal{V}$ is its continuity on the functional space $C^0([a, b], \mathbb{T})$, endowed with the topology of uniform convergence. Let θ be the closed 1-form on $\mathbb{T}$ whose lift to $\mathbb{R}$ is the usual differential form dt on $\mathbb{R}$, where dt is the differential of the identity map $\mathbb{R} \to \mathbb{R}, t \mapsto t$. It is well-known that when $\gamma : [a, b] \to \mathbb{T}$ is C^1, we have

$$\mathcal{V}(\gamma) = \int_\gamma \theta = \int_a^b \theta_{\gamma(t)}(\dot\gamma(t))\, dt.$$

If X is a topological space and $F : X \times [a, b] \to \mathbb{T}$ is a given map, we will define $\mathcal{V}(F) : X \to \mathbb{R}$ by

$$\forall x \in X,\ \mathcal{V}(F)(x) = \mathcal{V}(F_x),$$

where $F_x : [a, b] \to \mathbb{T}$ is defined by $F_x(t) = F(x, t)$. The continuity of $\mathcal{V}$ on $C^0([a, b], \mathbb{T})$ implies that $\mathcal{V}(F)$ is continuous. Furthermore, the continuity of $\mathcal{V}$ on $C^0([a, b], \mathbb{T})$ also implies that the map $C^0(X \times [a, b], \mathbb{T}) \to C^0(X, \mathbb{T}), F \mapsto \mathcal{V}(F)$ is continuous, when we provide the spaces of continuous maps with the compact open topology.

If F can be lifted to a continuous map $\bar{F} : X \times [a,b] \to \mathbb{R}$ with $F = \pi\bar{F}$, then

$$\mathcal{V}(F)(x) = \bar{F}(x,b) - \bar{F}(x,a).$$

Suppose now that X is a topological space and that $(\phi_t)_{t\in\mathbb{R}}$ is a continuous flow on X. We will define $\Phi : X \times [0,1] \to X$ by $\Phi(x,t) = \phi_t(x)$. If $f : X \to \mathbb{T}$ is continuous, we set

$$\mathcal{V}(f,\phi_t) = \mathcal{V}(f \circ \Phi) : X \to \mathbb{R}.$$

There is another another way to define $\mathcal{V}(f,\phi_t)$ which is used in [26]. The function $F(f,\Phi) : X \times [0,1] \to \mathbb{T}$, defined by

$$F(f,\Phi)(x,t) = f(\phi_t(x)) - f(x)$$

is continuous and identically 0 on $X \times \{0\}$, it is therefore homotopic to a constant and can be lifted to a continuous map $F(f,\Phi) : X \times [0,1] \to \mathbb{R}$, with $F(f,\Phi)|X \times \{0\}$ identically 0. We have

$$\mathcal{V}(f,\phi_t)(x) = F(f,\Phi)(x,1).$$

Note that if f is homotopic to 0 then it can be lifted continuously to $\bar{f} : X \to \mathbb{R}$. In that case $\bar{F}(f,\Phi) = \bar{f}\Phi - \bar{f}$ and

$$\mathcal{V}(f,\phi_t)(x) = \bar{f}(\phi_1(x)) - \bar{f}.$$

If μ is a measure with compact support and invariant under the flow ϕ_t, for a continuous $f : X \to \mathbb{T}$, we can define $\mathcal{S}(\mu,\phi_t)(f)$, or simply $\mathcal{S}(\mu)(f)$ when ϕ_t is fixed, by

$$\mathcal{S}(\mu)(f) = \int_X \mathcal{V}(f,\phi_t)(x)\, d\mu(x).$$

It is not difficult to verify that for $f_1, f_2 : X \to \mathbb{T}$, we have

$$\mathcal{S}(\mu)(f_1 + f_2) = \mathcal{S}(\mu)(f_1) + \mathcal{S}(\mu)(f_2).$$

Moreover, if $f : X \to \mathbb{T}$ is homotopic to 0, it can be lifted to $\bar{f} : X \to \mathbb{R}$ and

$$\mathcal{S}(\mu)(f) = \int_X [\bar{f}(\phi_1(x)) - \bar{f}(x)]\, d\mu(x) = \int_X \bar{f}(\phi_1(x))\, d\mu(x) - \int_X \bar{f}(x)\, d\mu(x) = 0,$$

since μ is invariant by ϕ_1. Therefore, if we denote by $[X, \mathbb{T}]$ the set of homotopy classes of continuous maps from X to $\mathbb{T}$, which is an additive group, the map $\mathcal{S}(\mu)$ is a well-defined additive homomorphism from the additive group $[X, \mathbb{T}]$ to $\mathbb{R}$. When X is a good space (like a manifold or a locally finite polyhedron), it is well-known that $[X, \mathbb{T}]$ is canonically identified with the first cohomology group $\mathrm{H}^1(X; \mathbb{Z})$. In that case $\mathcal{S}(\mu)$ is in $\mathrm{Hom}(\mathrm{H}^1(X; \mathbb{Z}), \mathbb{R})$. Since the first cohomology group with real coefficients $\mathrm{H}^1(X; \mathbb{R})$ is $\mathrm{H}^1(X; \mathbb{Z}) \otimes \mathbb{R}$, we can view $\mathcal{S}(\mu)$ as an element of the dual $\mathrm{H}^1(X; \mathbb{R})^*$ of the $\mathbb{R}$-vector space $\mathrm{H}^1(X; \mathbb{R})$. When $\mathrm{H}^1(X; \mathbb{R})$ is finite-dimensional then $\mathrm{H}^1(X; \mathbb{R})^*$ is in fact equal to the first homology group $\mathrm{H}_1(X; \mathbb{R})$, and therefore $\mathcal{S}(\mu)$ defines an element of $\mathrm{H}_1(X; \mathbb{R})$, *i.e.,* a 1-cycle. This 1-cycle $\mathcal{S}(\mu)$ is called the *Schwartzman asymptotic cycle* of μ. Note that $\mathrm{H}^1(X; \mathbb{R})$ is finite dimensional when X is a finite polyhedron or a compact manifold. It should be also noted that for a manifold M the projection $TM \to M$ is a homotopy equivalence. Therefore $\mathrm{H}^1(TM; \mathbb{R}) = \mathrm{H}^1(M; \mathbb{R})$ is finite dimensional when M is a compact manifold.

We want now to study the behavior of Schwartzman asymptotic cycles under semi-conjugacy.

Proposition 3.5.1. *Suppose that $\phi_t^i : X_i \to X_i, i = 1, 2$ are two continuous flows. Suppose also that $\psi : X_1 \to X_2$ is a continuous semi-conjugation between the flows, i.e., $\psi \circ \phi_t^1 = \phi_t^2 \circ \psi$, for every $t \in \mathbb{R}$. Given a probability measure μ with compact support on X_1 invariant under ϕ_t^1, then, for every continuous map $f : X_2 \to \mathbb{T}$, we* *have*

$$\mathcal{S}(\psi_*\mu, \phi_t^2)([f]) = \mathcal{S}(\mu, \phi_t^1)([f \circ \psi]),$$

where $\psi_*\mu$ is the image of μ under ψ. In particular, if we are in the situation where $\mathrm{Hom}([X_i, \mathbb{T}]) \equiv \mathrm{H}_1(X_i; \mathbb{R}), i = 1, 2,$ *we obtain*

$$\mathcal{S}(\psi_*\mu, \phi_t^2) = \mathrm{H}_1(\psi)(\mathcal{S}(\mu, \phi_t^1)).$$

Proof. Notice that $f\psi\phi_t^1(x) - f\psi(x) = f\phi_t^2(\psi(x)) - f(\psi(x))$. Therefore by uniqueness of the liftings, $\mathcal{V}(f\psi, \phi_t^1)(x) = \mathcal{V}(f, \phi_t^2)(\psi(x))$. An integration with respect to μ finishes the proof. □

We would like now to relate the Schwartzman asymptotic cycles to the rotation vectors $\rho(\mu)$ defined for Lagrangian flows. We first consider the case of a C^1 flow ϕ_t on the manifold N. We call X the continuous vector field on N generating ϕ_t, *i.e.*,

$$\forall x \in N, \quad X(x) = \left.\frac{d\phi_t(x)}{dt}\right|_{t=0}.$$

By the flow property $\phi_{t+t'} = \phi_t \circ \phi_{t'}$, this implies

$$\forall x \in N,\ \forall t \in \mathbb{R}, \quad \frac{d\phi_t(x)}{dt} = X(\phi_t(x)).$$

In the case of a manifold N, the identification of $[N, \mathbb{T}]$ with $\mathrm{H}^1(N; \mathbb{Z})$ is best described with the de Rham cohomology. We consider the natural map $I_N : [X, \mathbb{T}] \to \mathrm{H}^1(X; \mathbb{R})$ defined by

$$I_N([f]) = [f^*\theta],$$

where $[f]$ on the left hand side denotes the homotopy class of the C^∞ map $f : N \to \mathbb{T}$, and $[f^*\theta]$ on the right hand side is the cohomology class of the pullback by f of the closed 1-form on $\mathbb{T}$ whose lift to $\mathbb{R}$ is dt. Note that any homotopy class in $[N, \mathbb{T}]$

contains smooth maps because C^∞ maps are dense in C^0 maps (for the Whitney topology). Therefore the map I_N is indeed defined on the whole of $[N, \mathbb{T}]$. As it is well-known, this map I_N induces an isomorphism of $[N, \mathbb{T}]$ on $\mathrm{H}^1(N; \mathbb{Z}) \subset \mathrm{H}^1(N; \mathbb{R}) = \mathrm{H}^1(N; \mathbb{Z}) \otimes \mathbb{R}$.

Given a C^∞ map $f : N \to \mathbb{T}$, the C^1 flow ϕ_t on N, and $x \in N$, we compute $\mathcal{V}(f, \phi_t)(x)$. If $\gamma_x : [0,1] \to N$ is the path $t \mapsto \phi_t(x)$, by definition, we have $\mathcal{V}(f, \phi_t)(x) = \mathcal{V}(f \circ \gamma_x)$. Since γ_x is C^1, we get

$$\mathcal{V}(f, \phi_t)(x) = \int_{f \circ \gamma_x} \theta = \int_{\gamma_x} f^* \theta.$$

Since $\gamma_x(t) = \phi_t(x)$, we have $\dot{\gamma}_x(t) = X(\phi_t(x))$. It follows that

$$\mathcal{V}(f, \phi_t)(x) = \int_0^1 (f^* \theta)_{\phi_t(x)}(X[\phi_t(x)])\, dt.$$

Recall that the interior product $i_X \omega$ of a differential form ω with X is given by

$$(i_X \omega)_x(\cdots) = \omega_x(X(x), \cdots).$$

When ω is a differential 1-form then $i_X \omega$ is a function. With this notation, we get

$$\mathcal{V}(f, \phi_t)(x) = \int_0^1 (i_X f^* \theta)(\phi_t(x))\, dt.$$

Therefore if μ is an invariant measure for ϕ_t, which we will assume to have a compact support, we obtain

$$\mathcal{S}(\mu) = \int_N \int_0^1 (i_X f^* \theta)(\phi_t(x))\, dt d\mu(x).$$

Since $i_X f^* \theta$ is continuous, and we are assuming that μ has a compact support, we have

$$\mathcal{S}(\mu) = \int_0^1 \int_N (i_X f^* \theta)(\phi_t(x))\, d\mu(x) dt.$$

By invariance of μ under ϕ_t, we get $\int_N (i_X f^*\theta)(\phi_t(x))\, d\mu(x) = \int_N (i_X f^*\theta)(x)\, d\mu(x)$, and therefore

$$\mathcal{S}(\mu) = \int_0^1 \int_N (i_X f^*\theta)(x)\, d\mu(x) dt = \int_N (i_X f^*\theta)\, d\mu.$$

This shows that as an element of $\mathrm{H}^1(M;\mathbb{R})^*$, the Schwarztman asymptotic cycle $\mathcal{S}(\mu)$ is given by

$$\mathcal{S}(\mu)([\omega]) = \int_N i_X \omega\, d\mu.$$

We can now easily compute Schwartzman asypmtotic cycles for linear flows on $\mathbb{T}^d$. Such a flow is determined by a constant vector field $\alpha \in \mathbb{R}^d$ on $\mathbb{T}^d$ (here we use the canonical trivialisation of the tangent bundle of $\mathbb{T}^d$), the associated flow $R_t^\alpha : \mathbb{T}^d \to \mathbb{T}^d$ is defined by $R_t^\alpha(x) = x + [t\alpha]$, where $[t\alpha]$ is the class in $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ of the vector $t\alpha \in \mathbb{R}^d$. If ω is a 1-form with constant coefficients, *i.e.*, $\omega = \sum_{i=1}^d a_i dx_i$, with $a_i \in \mathbb{R}$, the interior product $i_\alpha \omega$ is the constant function $\sum_{i=1}^d \alpha_i a_i$. Therefore, it follows that $\mathcal{S}(\mu) = \alpha \in \mathbb{R}^d \equiv \mathrm{H}_1(\mathbb{T}^d;\mathbb{R})$.

We now compute Schwartzman asymptotic cycles for Euler-Lagrange flows. In this case $N = TM$ and ϕ_t is an Euler-Lagrange flow ϕ_t^L of some Lagrangian L. If we call X_L the vector field generating ϕ_t^L, since this flow is obtained from a second order ODE on M, we get

$$\forall x \in M, \forall v \in T_x M, \quad T\pi(X_L(x,v)) = v,$$

where $T\pi : T(TM) \to TM$ denotes the canonical projection. Since this projection π is a homotopy equivalence, to compute $\mathcal{S}(\mu)$ we only need to consider forms of the type $\pi^*\omega$ where ω is a closed 1-form on the base M. In this case $(i_{X_L}\pi^*\omega)(x,v) = \omega_x(T\pi(X_L(x,v)) = \omega_x(v)$. Therefore, for any probability measure μ on TM with compact support and invariant under ϕ_t^L, we obtain

$$\mathcal{S}(\mu)[\pi^*\omega] = \int_{TM} \omega_x(v)\, d\mu(x,v) = \int_{TM} \hat{\omega}\, d\mu.$$

This is precisely $\rho(\mu)$ as it was defined in section 1.2. Note that the only property we have used is the fact that ϕ_t is the flow of a second order ODE on the base M.

To simplify things, in the remainder of this section, we will assume that X is a compact space, for which we have $[X,\mathbb{T}] = \mathrm{H}^1(X;\mathbb{Z})$, and $\mathrm{H}_1(X;\mathbb{Z})$ is finitely generated. In that case, the dual space $\mathrm{H}^1(X;\mathbb{R})^*$ is $\mathrm{H}_1(X;\mathbb{R})$, and for every flow ϕ_t on X and every probability measure μ on X invariant under ϕ_t, the Schwartzman asymptotic cycle is an element of the finite dimensional-vector space $\mathrm{H}_1(X;\mathbb{R})$.

Suppose that x is a periodic point of ϕ_t or period $T > 0$. One can define an invariant probability measure μ_{x,t_0} for ϕ_t by

$$\int_X g(x)\, d\mu_{x,t_0} = \frac{1}{t_0}\int_0^{t_0} g((\phi_t(x))\, dt,$$

where $g : X \to \mathbb{R}$ is a measurable function. It is easy to verify that $\mathcal{S}(\mu_{x,t_0})$ is equal in $\mathrm{H}_1(X;\mathbb{R})$ to the homology class $[\gamma_{x,t_0}]/t_0$, where γ_{x,t_0} is the loop $t \mapsto \phi_t(x), t \in [0,t_0]$. When x is a fixed point of ϕ_t, then the Dirac mass δ_x at x is invariant under ϕ_t, and in that case $\mathcal{S}(\delta_x) = 0$.

Definition 3.5.2. *For a flow ϕ_t on X, we denote by $\mathcal{S}(\phi_t)$ the set of all Schwartzman asymptotic cycles $\mathcal{S}(\mu)$, where μ is an arbitrary probability measure on X invariant under ϕ_t.*

Since X is compact, note that for the weak topology the set $\mathfrak{M}(X)$ of probability Borel measures on X is compact and convex. It is even metrizable, since we are assuming X metrizable. Furthermore the subset $\mathfrak{M}(X,\phi_t) \subseteq \mathfrak{M}(X)$ of probability measures invariant under ϕ_t is, as it is well-known, compact convex and non-empty.

Therefore $\mathcal{S}(\phi_t)$ is a compact convex non-empty subset of $\mathrm{H}_1(X;\mathbb{R})$.

For the case of a linear flow R^α on $\mathbb{T}^d$, we have shown above that $\mathcal{S}(R^\alpha_t) = \{\alpha\} \subset \mathbb{R}^d \equiv \mathrm{H}_1(\mathbb{T}^d;\mathbb{R})$.

The following corollary is an easy consequence of Proposition 3.5.1.

Corollary 3.5.3. *For $i = 1, 2$, suppose that ϕ^i_t is a continous flow on the compact space X_i, which satisfies $\mathrm{Hom}([X_i, \mathbb{T}], \mathbb{R}) \equiv \mathrm{H}_1(X_i;\mathbb{R})$. If $\psi : X_1 \to X_2$ is a topological conjugacy between ϕ^1_t and ϕ^2_t (i.e., the map ψ is a homeomorphism that satisfies $\psi\phi^1_t = \phi^2_t\psi$, for all $t \in \mathbb{R}$), then we have*

$$\mathcal{S}(\phi^2_t) = \mathrm{H}_1(\psi)[\mathcal{S}(\phi^1_t)].$$

We denote by $\mathfrak{F}(X)$ the set of continuous flows on X. We can embed $\mathfrak{F}(X)$ in $C^0(X \times [0, 1], X)$ by the map $\phi_t \mapsto F^{\phi_t} \in C^0(X \times [0, 1], X)$, where

$$F^{\phi_t}(x, t) = \phi_t(x).$$

The topology on $C^0(X \times [0, 1], X)$ is the compact open (or uniform) topology, and we endow $\mathfrak{F}(X)$ with the topology inherited from the embedding given above.

Lemma 3.5.4. *The map $\phi_t \mapsto \mathcal{S}(\phi_t)$ is upper semi-continuous on $\mathfrak{F}(X)$. This means that for each open subset $U \subseteq \mathrm{H}_1(X;\mathbb{R})$, the set $\{\phi_t \in \mathfrak{F}(X) \mid \mathcal{S}(\phi_t) \subset U\}$ is open in $\mathfrak{F}(X)$.*

Proof. Since the topology on $C^0(X \times [0, 1], X)$ is metrizable, if this were not true we could find an open set $U \subset \mathrm{H}_1(X;\mathbb{R})$ and a sequence ϕ^n_t of continuous flows on X converging uniformly to a flow ϕ_t, with $\mathcal{S}(\phi_t) \subset U$, and $\mathcal{S}(\phi^n_t)$ is not contained in U. This means that for each n we can find a probability measure μ_n on X invariant under ϕ^n_t and such that its Schwartzman asymptotic cycle $\mathcal{S}(\mu_n, \phi^n_t)$ for ϕ^t_n is not in the open set U. Since $\mathfrak{M}(X)$ is compact for the weak topology, extracting a subsequence if necessary, we can assume that $\mu_n \to \mu$. It is not difficult to show that μ is invariant under the flow ϕ_t. We now show that $\mathcal{S}(\mu_n, \phi_t^n) \to \mathcal{S}(\mu, \phi_t)$. This will yield a contradiction and finish the proof because $\mathcal{S}(\mu_n, \phi_t^n)$ is in the closed set $\mathrm{H}_1(X;\mathbb{R}) \setminus U$, for every n, and $\mathcal{S}(\mu, \phi_t) \in U$.

To show that the linear maps $\mathcal{S}(\mu_n, \phi_t^n) \in \mathrm{H}_1(X;\mathbb{R}) = \mathrm{H}^1(X;\mathbb{R})^*$ converge to the linear map $\mathcal{S}(\mu, \phi_t)$, it suffices to show that $\mathcal{S}(\mu_n, \phi_t^n)([f]) \to \mathcal{S}(\mu, \phi_t)([f])$, for every $[f] \in [X, \mathbb{T}] = \mathrm{H}^1(X;\mathbb{Z}) \subset \mathrm{H}^1(X;\mathbb{R}) = \mathrm{H}^1(X;\mathbb{Z}) \otimes \mathbb{R}$. Fix now a continuous map $f : X \to \mathbb{T}$. Denote by $F_n, F : X \times [0,1] \to \mathbb{T}$ the maps defined by

$$F_n(x,t) = f(\phi_t^n(x)) - f(x) \text{ and } F(x,t) = f(\phi_t(x)) - f(x).$$

By the uniform continuity of f on the compact metric space X, the sequence F_n converges uniformly to F. Since $F_n|X \times \{0\} \equiv 0$, if we call $\tilde{F}_n : X \times [0,1] \to \mathbb{R}$ the lift of F_n such that $\tilde{F}_n|X \times \{0\} \equiv 0$, then the sequence $\tilde{F}_n$ also converges uniformly to $\tilde{F}$, that is the lift of F such that $\tilde{F}|X \times \{0\} \equiv 0$. Since the μ_n are probability measures, we have

$$\left| \int_X \tilde{F}_n(x,1)\mu_n(x) - \int_X \tilde{F}(x,1)\mu_n(x) \right| \leq \|\tilde{F}_n - \tilde{F}\|_\infty \longrightarrow 0.$$

Since $\mu_n \to \mu$ weakly, we also have

$$\left| \int_X \tilde{F}(x,1)\mu_n(x) - \int_X \tilde{F}(x,1)\mu(x) \right| \longrightarrow 0.$$

Therefore $\mathcal{S}(\mu_n, \phi_t^n)([f]) = \int_X \tilde{F}_n(x,1)\mu_n(x) \to \mathcal{S}(\mu, \phi_t)([f]) = \int_X \tilde{F}(x,1)\mu_0(x)$. $\square$

Definition 3.5.5. [Schwartzman unique ergodicity] *We say that a flow ϕ_t is Schwartzman uniquely ergodic if $\mathcal{S}(\phi_t)$ is reduced to one point.*

By the computation done above linear flows on the torus $\mathbb{T}^d$ are Schwartzman uniquely ergodic. Of course, all uniquely ergodic flows (*i.e.,* flows having exactly one invariant probability measure) are also Schwartzman uniquely ergodic. Moreover, by Corollary 3.5.3, any flow topologically conjugate to a Schwartzman uniquely ergodic flow is itself Schwartzman uniquely ergodic.

Theorem 3.5.6. *The set $\mathfrak{S}(X)$ of Schwartzman uniquely ergodic flows is a G_δ in $\mathfrak{F}(X)$.*

Proof. Fix some norm on $\mathrm{H}^1(X;\mathbb{R})$. We will measure diameters of subsets of $\mathrm{H}^1(X;\mathbb{R})$ with respect to that norm. Fix $\epsilon > 0$. Call $\mathcal{U}_\epsilon$ the set of flows ϕ_t such that the diameter of $\mathcal{S}(\phi_t) \subset \mathrm{H}_1(X;\mathbb{R})$ is $< \epsilon$. If $\phi_t^0 \in \mathcal{U}_\epsilon$, we can find U an open subset of $\mathrm{H}_1(X;\mathbb{R})$ of diameter $< \epsilon$ and containing $\mathcal{S}(\phi_t^0)$. By the lemma above the set $\{\phi_t \in \mathfrak{F}(X) \mid \mathcal{S}(\phi_t) \subset U\}$ is open in $\mathfrak{F}(X)$, contains ϕ_t^0 and is contained in $\mathcal{U}_\epsilon$. The set of Schwartzman uniquely ergodic flows is then $\bigcap_{n\geq 1} \mathcal{U}_{1/n}$. □

Proposition 3.5.7. *Let $\phi_t : X \longrightarrow X$ be a continuous flow on the compact path connected space X. Suppose that there exist $t_i \uparrow +\infty$ such that $\phi_{t_i} \longrightarrow \phi$ in $C(X, X)$ (with the C^0-topology). Then, ϕ_t is Schwartzman uniquely ergodic. In particular, periodic flows and (uniformly) recurrent flows are Schwartzman uniquely ergodic (in both cases $\phi = \mathrm{Id}$).*

Proof. Fix a continuous map $f : X \to \mathbb{T}$. Consider the function $F : X \times [0, +\infty) \to \mathbb{T}, (x, t) \mapsto f(\phi_t(x)) - f(x)$. We have $F(x, 0) = 0$, for every $x \in X$. Call $\bar{F} : X \times [0, +\infty) \to \mathbb{R}$ the (unique) continuous lift of F such that $\bar{F}(x, 0) = 0$, for every $x \in X$. The definition of the Schwartzman asymptotic cycle gives

$$\mathcal{S}(\mu)([f]) = \int_X \bar{F}(x, 1)\, d\mu(x),$$

for every probability measure invariant under ϕ_t. We claim that we have

$$\forall t, t' \geq 0, \forall x \in X, \quad \bar{F}(x, t + t') = \bar{F}(\phi_t(x), t') + F(x, t).$$

In fact, if we fix t and we consider each side of the equality above as a (continuous) function of (x, t') with values in $\mathbb{R}$, we see that the two sides are equal for $t' = 0$, and that they both lift the function

$$(x, t') \mapsto f(\phi_{t+t'}(x)) - f(x) = f(\phi'_t(\phi_t(x)) - f(\phi_t(x)) + f(\phi_t(x)) - f(x)$$

with values in $\mathbb{T}$. By induction, it follows easily that

$$\forall k \in \mathbb{N}, \quad \bar{F}(x, k) = \sum_{j=0}^{k-1} \bar{F}(\phi_j(x), 1).$$

Therefore, if $t \geq 0$ and $[t]$ is its integer part, we also obtain

$$\bar{F}(x, t) = \bar{F}(\phi_{[t]}(x), t - [t]) + \sum_{j=0}^{[t]-1} \bar{F}(\phi_j(x), 1). \tag{3.11}$$

It follows that

$$\forall t \geq 0, \forall x \in X, \quad |\bar{F}(x, t)| \leq ([t] + 1)\|\bar{F}|X \times [0, 1]\|_\infty. \tag{3.12}$$

By compactness $\|\bar{F}|X \times [0, 1]\|_\infty$ is finite. If we integrate equality (3.11) with respect to a probability measure μ on X invariant under the flow ϕ_t, we obtain

$$\int_X \bar{F}(x, t)\, d\mu(x) = \int_X \bar{F}(x, t - [t])\, d\mu(x) + [t] \int_X \bar{F}(x, 1)\, d\mu(x).$$

Therefore we have

$$\mathcal{S}(\mu)([f]) = \lim_{t \to +\infty} \int_X \frac{\bar{F}(x, t)}{t}\, d\mu(x). \tag{3.13}$$

Suppose now that we set $\gamma_x(s) = \phi_s(x)$; we have $\bar{F}(x, t) = \mathcal{V}(f\gamma_x|[0, t])$. Fix now some point $x_0 \in X$, and consider $t_i \to +\infty$ such that $\phi_{t_i} \to \phi$ in the C^0 topology. Since

$\bar{F}(x_0,t)/t$ is bounded in absolute value by $2\|\bar{F}|X \times [0,1]\|_\infty$, for $t \geq 1$, extracting a subsequence if necessary, we can assume that $\bar{F}(x_0,t_i)/t_i \to c \in \mathbb{R}$. If $x \in X$, we can find a continuous path $\gamma : [0,1] \to M$ with $\gamma(0) = x_0$ and $\gamma(1) = x$. The map $\Gamma : [0,1] \times [0,t] \to \mathbb{T}, (s,s') \to \phi_{s'}(\gamma(s))$ is continuous, therefore we can lift it to a continuous function with values in $\mathbb{R}$, and this implies the equality

$$\mathcal{V}(\Gamma|[0,1] \times \{0\}) + \mathcal{V}(\Gamma|\{1\} \times [0,t]) - \mathcal{V}(\Gamma|[0,1] \times \{1\}) - \mathcal{V}(\Gamma|\{0\} \times [0,t]) = 0.$$

This can be rewritten as

$$\mathcal{V}(f\gamma_x|[0,t]) - \mathcal{V}(f\gamma_{x_0}|[0,t]) = \mathcal{V}(f\phi_t\gamma) - \mathcal{V}(f\gamma),$$

which translates to

$$\bar{F}(x,t) - \bar{F}(x_0,t) = \mathcal{V}(f\phi_t\gamma) - \mathcal{V}(f\gamma).$$

Since $\phi_{t_i} \to \phi$ uniformly, by continuity of $\mathcal{V}$, the left hand-side remains bounded as $t = t_i \to +\infty$. It follows that $(\bar{F}(x,t_i) - \bar{F}(x_0,t_i))/t_i \to 0$. Hence for every $x \in X$, we also have that $\bar{F}(x,t_i)/t_i$ tends to the same limit c as $\bar{F}(x_0,t_i)/t_i$. Since, by (3.12), $\bar{F}(x,t)/t$ is uniformly bounded for $t \geq 1$, by Lebesgue's dominated convergence we obtain from (3.13) that $\mathcal{S}(\mu)([f]) = c$, where c is independent of the invariant measure μ. This is of course true for any $f : X \to \mathbb{T}$. Therefore $\mathcal{S}(\mu)$ does not depend on the invariant measure μ. □

An interesting property of Schwartzman uniquely ergodic flows (which also shows that they have some kind of rigidity) is the following proposition, that follows immediately from the definition of Schwartzman unique ergodicity and what we remarked above about the asymptotic cycles of fixed and periodic points (see also [74]).

Proposition 3.5.8. *Suppose that ϕ_t is a Schwartzman uniquely ergodic flow on X. If there exists either a fixed point or a closed orbit homologous to zero, then all closed orbits are homologous to zero. In the remaining case, if C_1 and C_2 are closed orbits with periods τ_1 and τ_2, then $\frac{C_1}{\tau_1}$ and $\frac{C_2}{\tau_2}$ are homologous. Since $[C_1]$ and $[C_2]$ are in* $\mathrm{H}_1(X;\mathbb{Z})$*, it follows in this case that the ratio of the periods of any two closed orbits must be rational. Consequently, for any continuous family of periodic orbits of ϕ_t, all orbits have the same period.*

Definition 3.5.9. **[Schwartzman strict ergodicity]** *We say that a flow ϕ_t is Schwartzman strictly ergodic if it is Schwartzman uniquely ergodic and it has an invariant measure μ of full support (i.e., $\mu(U) > 0$ for every non-empty open subset U of X).*

Linear flows on the torus $\mathbb{T}^d$ are Schwartzman strictly ergodic (they preserve Lebesgue measure). Of course, all strictly ergodic flows (*i.e.,* flows having exactly one invariant probability measure, and the support of this measure is full) are also Schwartzman strictly ergodic. A minimal flow which is Schwartzman uniquely ergodic is in fact Schwartzman strictly ergodic (because all invariant measures have full support). Moreover, any flow topologically conjugate to a Schwartzman strictly ergodic flow is also Schwartzman strictly ergodic.

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