ON THE MARKED LENGTH SPECTRUM OF GENERIC STRICTLY CONVEX BILLIARD TABLES

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Abstract. In this paper we show that for a generic strictly convex domain, one can recover the eigendata corresponding to Aubry-Mather periodic orbits of the induced billiard map, from the (maximal) marked length spectrum of the domain.

1. Introduction

A mathematical billiard is a system describing the inertial motion of a point mass inside a domain with elastic reflections at the boundary (which is assumed to have infinite mass). This simple model has been first proposed by Birkhoff [4] as a mathematical playground where: "*the formal side, usually so formidable in dynamics, almost completely disappears and only the interesting qualitative questions need to be considered.*"

Since then billiards have become a very popular subject. Not only is their law of motion very physical and intuitive, but billiard-type dynamics is ubiquitous. Mathematically, they offer models in every subclass of dynamical systems (integrable, regular, chaotic, etc). More importantly, techniques initially devised for billiards have often been applied and adapted to other systems, becoming standard tools and having ripple effects beyond the field.

Moreover, despite their apparently simple (local) dynamics, their qualitative dynamical properties are extremely non-local! This global influence on the dynamics translates into several intriguing rigidity phenomena, which are at the basis of many unanswered questions and conjectures. For instance, while the dependence of the dynamics on the geometry of the domain is well perceptible, an intriguing challenge is to understand to which extent dynamical information can be used to reconstruct the shape of the domain. In this article, we will address this inverse problem in the case of periodic orbits in a strictly convex smooth planar domain Ω.

The study of periodic orbits for billiard maps in strictly convex planar domains has been among the first dynamical features of billiards that have been investigated. One of the first results in the theory of billiards, for example, can be considered Birkhoff's application of Poincaré's last geometric theorem to show the existence of infinitely many periodic orbits, which can be topologically distinguished in terms of their *rotation number*[1]. In [4] Birkhoff proved that for every rotation number

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[1]The rotation number of a periodic billiard trajectory is a rational number that can be roughly defined as

$$\frac{p}{q} = \frac{\text{winding number}}{\text{number of reflections}} \in \left(0, \frac{1}{2}\right],$$

where the winding number $p > 1$ is defined as follows. Fix the positive orientation of $\partial\Omega$ and pick any reflection point of the closed geodesic on $\partial\Omega$; then follow the trajectory and count how many times it goes around $\partial\Omega$ in the positive direction until it comes back to the starting point. Notice

$p/q \in (0, 1/2]$ in lowest terms, there are at least two closed orbits of rotation number p/q: one maximizing the total length and the other obtained by min-max methods (see also [23, Theorem 1.2.4]). This result is clearly optimal: in the case of a billiard in an ellipse, for example, there are only two periodic orbits of period 2 (also called *diameters*), which correspond to the two semi-axis of the ellipse. However, it is easy to find cases in which there are more than two periodic orbits for a given rotation number: think, for example, of a billiard in a disk where, due to the existence of a 1-dimensional group of symmetries (rotations), each periodic orbit generates a 1-dimensional family of similar ones (all diameters are periodic orbits with period 2).

A natural question is to understand which information on the geometry of the billiard domain, the set of periodic orbits does encode. More ambitiously, one could wonder whether a complete knowledge of this set allows one to reconstruct the shape of the billiard and hence the whole of the dynamics.

Let us start by introducing the *length spectrum* of a domain Ω.

Definition 1 (**Length Spectrum**). *Given a domain Ω, the length spectrum of Ω is given by the set of lengths of its periodic orbits, counted with multiplicity:*

$$\mathcal{L}_\Omega := \mathbb{N} \cdot \{\text{lengths of closed geodesics in } \Omega\} \cup \mathbb{N} \cdot \ell(\partial\Omega),$$

where $\ell(\partial\Omega)$ denotes the length of the boundary.

Remark 2. A remarkable relation exists between the length spectrum of a billiard in a convex domain Ω and the spectrum of the Laplace operator in Ω with Dirichlet boundary condition (similarly for Neumann boundary one):

$$\begin{cases} \Delta f = \lambda f & \text{in } \Omega \\ f|_{\partial\Omega} = 0. \end{cases} \tag{1}$$

From the physical point of view, the eigenvalues λ are the eigenfrequencies of the membrane Ω with a fixed boundary.

K. Andersson and R. Melrose [1] proved the following relation between the Laplace spectrum and the length spectrum. Call the function

$$w(t) := \sum_{\lambda_i \in spec \Delta} \cos(t\sqrt{-\lambda_i}),$$

the wave trace.

Theorem (Andersson-Melrose). *The wave trace $w(t)$ is a well-defined generalized function (distribution) of t, smooth away from the length spectrum, namely,*

$$\text{sing. supp.}\big(w(t)\big) \subseteq \pm\mathcal{L}_\Omega \cup \{0\}. \tag{2}$$

So if $l > 0$ belongs to the singular support of this distribution, then there exists either a closed billiard trajectory of length l, or a closed geodesic of length l in the boundary of the billiard table.

Generically, equality holds in (2). More precisely, if no two distinct orbits have the same length and the Poincaré map of any periodic orbit is non-degenerate, then

that inverting the direction of motion for every periodic billiard trajectory of rotation number $p/q \in (0, 1/2]$, we obtain an orbit of rotation number $(q-p)/q \in [1/2, 1)$.

the singular support of the wave trace coincides with $\pm\mathcal{L}_\Omega \cup \{0\}$ (see e.g. [20]).

This theorem implies that, at least for generic domains, one can recover the length spectrum from the Laplace one. This relation between periodic orbits and spectral properties of the domain, immediately recalls a more famous spectral problem (probably the most famous): *Can one hear the shape of a drum?*, as formulated in a very suggestive way by Mark Kac [12] (although the problem had been already stated by Hermann Weyl). More precisely: is it possible to infer information about the shape of a drumhead (*i.e.*, a domain) from the sound it makes (*i.e.*, the list of basic harmonics/ eigenvalues of the Laplace operator with Dirichlet or Neumann boundary conditions)? This question has not been completely solved yet: there are several negative answers (for instance by Milnor [19] and Gordon, Webb, and Wolpert [8]), as well as some positive ones.

Hezari–Zelditch, going in the affirmative direction, proved in [11] that, given an ellipse $\mathcal{E}$, any one-parameter C^∞-deformation Ω_ε which preserves the Laplace spectrum (with respect to either Dirichlet or Neumann boundary conditions) and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group of the ellipse has to be *flat* (*i.e.*, all derivatives have to vanish for $\varepsilon = 0$). Popov–Topalov [21] recently extended these results (see also [29]). Further historical remarks on the inverse spectral problem can be also found in [11]. In [22], P. Sarnak conjectures that the set of smooth convex domains isospectral to a given smooth convex domain is finite; for a partial progress on this question, see [6].

One of the difficulties in working with the length spectrum is that all of these information come in a non-formatted way. For example, we lose track of the rotation number corresponding to each length. A way to overcome this difficulty is to "organize" this set of information in a more systematic way, for instance by associating to each length the corresponding rotation number. This new set is called the *marked length spectrum* of Ω and denoted by $\mathcal{ML}_\Omega$.

One could also refine this set of information by considering not the lengths of all orbits, but selecting some of them. More precisely, for each rotation number p/q in lowest terms, one could consider the maximal length among those having rotation number p/q. We call this map $\mathcal{ML}_\Omega^{\max} : \mathbb{Q} \cap \left(0, \frac{1}{2}\right] \longrightarrow \mathbb{R}_+$ the *maximal marked length spectrum*:

$$\mathcal{ML}_\Omega^{\max}(p/q) = \max\Big\{\text{lengths of periodic orbits with rot. number } p/q\Big\}.$$

For convenience, we extend this map to $(0,1) \cap \mathbb{Q}$ by symmetrizing with respect to $\frac{1}{2}$:

$$\mathcal{ML}_\Omega^{\max}(p/q) = \mathcal{ML}_\Omega^{\max}(1 - p/q), \quad p/q \in (\frac{1}{2}, 1) \cap \mathbb{Q}.$$

This map is closely related to Mather's minimal average action (or β-function) and we will explain it in Section 3 (see also [23, 25]).

1.1. **Main result.**

In [9, pp. 677-678] V. Guillemin and R. Melrose ask whether the Length Spectrum and the eigenvalues of the linearizations of the (iterated) billiard map at periodic orbits, constitute a complete set of symplectic invariants for the system.

Our main result shows that for generic domains, the eigendata corresponding to Aubry-Mather periodic orbits (*i.e.*, periodic orbits of maximal perimeter among those with the same rotation number) can be actually recovered from the (Maximal) Marked Length Spectrum. More precisely:

Main Theorem. *For a generic strictly convex $C^{\tau+1}$-billiard table Ω ($\tau \geqslant 2$), we have that for each $p/q \in \mathbb{Q} \cap (0, 1/2]$ in lowest terms: for any sequence $N_n \in \mathbb{N}$ such that $N_n p$ is coprime with $N_n q - 1$, and $N_n \to +\infty$ as $n \to +\infty$,*[2]

(1) *The following limit exists*

$$\lim_{n\to+\infty} \left[\mathcal{ML}_\Omega^{\max}\left(\frac{N_n p}{N_n q - 1}\right) - N_n \cdot \mathcal{ML}_\Omega^{\max}\left(\frac{p}{q}\right)\right] = -B_{p/q},$$

where $B_{p/q}$ denotes the minimum value of Peierls' Barrier function of rotation number p/q (see Section 5).

(2) *Moreover,*

$$\lim_{n\to+\infty} \frac{1}{N_n} \log \left| \mathcal{ML}_\Omega^{\max}(\frac{N_n p}{N_n q - 1}) - N_n \cdot \mathcal{ML}_\Omega^{\max}(\frac{p}{q}) + B_{p/q} \right| = \log \lambda_{p/q},$$

where $\lambda_{p/q}$ is the eigenvalue of the linearization of the Poincaré return map at the Aubry-Mather periodic orbit with rotation number $\frac{p}{q}$.

See Theorem 15 in Section 4 for a rephrasing of item (2) in the previous theorem in terms of Mather's β-function (which will be introduced in Section 3).

The set of generic billiard tables is a (Baire) generic set, *i.e.*, a set that contains a countable intersection of open dense sets. See Section 4 for a precise set of genericity assumptions.

Remark 3. Notice that for exact area-preserving twist maps, all of the above objects (Aubry-Mather periodic orbits, Peierls' barrier and Mather's β-function) are well defined and the argument in the proofs continue to be valid. Hence, our Main Theorem could be rephrased in terms of generic $C^{\tau+1}$ smooth exact area-preserving twist map, for $\tau \geq 2$. However, being our primary interest in this problem motivated by spectral questions in billiard dynamics, we have opted to focus the presentation of our main results in this context.

Remark 4. A natural question is the following: does the limit in item (2) always exist? If yes, does it determine to the eigenvalue $\lambda_{p/q}$?
In [28] the authors show that for a generic domain every hyperbolic periodic orbit admits some homoclinic orbit. This raises the following question: Can one recover the eigenvalue of the linearization of the Poincaré return map at any hyperbolic periodic orbit of a generic domain from its Marked Length Specturm?
See Remark 22 for a more explicit connection between homoclinic orbits and our construction, and a description of the obstacles that one needs to overcome in order to extend our result to a more general setting.

Remark 5. Quite interestingly, our main result could be applied to identify for which irrational rotation number there exists or does not exist an invariant curve (*i.e.,* a caustic) with that rotation number. In [7], J. Greene conjectured a criterion to test the existence of such curves (nowadays called "Greene's residue criterion"), which was tested numerically in the case of the standard map. We recall here a version of this criterion as conjectured in [14].

[2] A simple choice is, for example, $N_n = np$ for $n \in \mathbb{N}$.

Let f be a symplectic twist map of the annulus and let $\rho \in \mathbb{R}$ be an irrational number. Consider a sequence of rational numbers $\frac{p_n}{q_n} \longrightarrow \rho$ as n goes to $+\infty$ and for any minimizing periodic point X_n of rotation number $\frac{p_n}{q_n}$ associates to it its residue, given by $r_n = \frac{1}{4}\left(2 - \mathrm{Tr}(Df^{q_n}(X_n))\right)$. Then, the limit $\lim_{n\to+\infty} |r_n|^{1/q_n} = \mu(\rho)$ exists. Moreover, $\mu(\rho) \leq 1$ if and only if there exists an invariant curve with rotation number ρ.

In [2, Theorem 3], M.-A. Arnaud and P. Berger proved a part of this criterion (the "only if"). More specifically, they proved that if

$$\limsup_{n\to+\infty} |r_n|^{1/q_n} > 1,$$

then there is no homotopically non-trivial invariant curve with rotation number ρ. Our result allows one to obtain a lower-bound for this limsup at all irrational rotation number and hence apply the above result to deduce the non-existence of invariant curves.

Outline of the proof of the Main Theorem.

Let us sketch here the main ideas involved in the proof.

Given a hyperbolic Aubry-Mather periodic orbit (A-M p.o.) of rotation number p/q, in lowest terms, we compare its length (*i.e.*, action) with the lenghts of A-M p.o. of periods $Np/(Nq-1)$, with $N \geq 2$ and such that Np and $Nq-1$ are coprime. Pictorially, as N goes to infinity, these orbits become denser and denser and, in the limit as $N \to +\infty$, they approach the stable and the unstable manifolds of the starting A-M p.o.; in particular, these orbits approximate the homoclinic Aubry-Mather orbit of rotation number $p/q+$ and it is natural to expect that the asymptotic speed of approximation of the homoclinic orbit encodes information on the eigendata of the first return map.

Naively, the length of an A-M p.o. of rotation number $Np/(Nq-1)$ should be of order N times the length of the starting A-M p.o. However, this approximation is not sufficient to serve our needs; hence, a more precise asymptotic, that goes beyond the first order approximation, is required.

This analysis represents the core of this article and it is pursued in two steps, which correspond to the two items of the Main Theorem. More in details:

- As we have already pointed out, in the limit as N goes to infinity, A-M p.o.'s of rotation numbers $Np/(Nq-1)$ approximate the homoclinic Aubry-Mather orbit of rotation number $p/q+$. In particular, the difference between their lenghts and N times the length of the original A-M p.o. has a well-defined limit, which corresponds to a finer invariant of the periodic orbit: the so-called Peierls' barrier, defined by means of Aubry-Mather theory and related to the minimal action of homoclinic configurations (see Section 5). This is the content of item (1) in the Main Theorem and it will be proven in Section 5.
- The above convergence turns out to be exponential and we show that the rate of convergence is related to the eigenvalue of the first return map of the hyperbolic A-M p.o. of rotation number p/q. This is the content of item (2) in the Main Theorem and it will be proven in Section 6. The proof consists of a precise asymptotic analysis of the lengths of approximating A-M p.o., as well as of the construction of a normal form for Peierls' barrier (see Theorem 15), under suitable generic non-degeneracy conditions (Lemma 21). These generic assumptions are explained in Section 4.

For the reader's convenience, in Sections 2 and 3 we provide some background material on billiard maps and Aubry-Mather theory, as well as their mutual relation. Moreover, for the sake of a clearer exposition, we postpone some of the more technical proofs to Appendices A and B.

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2. The billiard map

In this section we would like to recall some properties of the billiard map. We refer to [23, 27] for a more comprehensive introduction to the study of billiards.

Let Ω be a strictly convex domain in $\mathbb{R}^2$ with $C^{\tau+1}$ boundary $\partial\Omega$, with $\tau \geq 2$. The phase space M of the billiard map consists of unit vectors (x, v) whose foot points x are on $\partial\Omega$ and which have inward directions. The billiard ball map $f : M \longrightarrow M$ takes (x, v) to (x', v'), where x' represents the point where the trajectory starting at x with velocity v hits the boundary $\partial\Omega$ again, and v' is the reflected velocity, according to the standard reflection law: angle of incidence is equal to the angle of reflection (figure 1).

Remark 6. Observe that if Ω is not convex, then the billiard map is not continuous. Moreover, as pointed out by Halpern [10], if the boundary is not at least C^3, then the flow might not be complete.

Let us introduce coordinates on M. We suppose that $\partial\Omega$ is parametrized by arclength s and let $\gamma : [0, l] \longrightarrow \mathbb{R}^2$ denote such a parametrization, where $l = l(\partial\Omega)$ denotes the length of $\partial\Omega$. Let φ be the angle between v and the positive tangent to $\partial\Omega$ at x. Hence, M can be identified with the annulus $\mathbb{A} = [0, l] \times (0, \pi)$ and the billiard map f can be described as

$$\begin{aligned} f : [0, l] \times (0, \pi) &\longrightarrow [0, l] \times (0, \pi) \\ (s, \varphi) &\longmapsto (s', \varphi'). \end{aligned}$$

In particular f can be extended to $\bar{\mathbb{A}} = [0, l] \times [0, \pi]$ by fixing $f(s, 0) = f(s, \pi) = \mathrm{Id}$, for all s.

Let us denote by

$$\ell(s, s') := \|\gamma(s) - \gamma(s')\| \tag{3}$$

the Euclidean distance between two points on $\partial\Omega$. It is easy to prove that

$$\begin{cases} \dfrac{\partial \ell}{\partial s}(s, s') = -\cos\varphi \\[2ex] \dfrac{\partial \ell}{\partial s'}(s, s') = \cos\varphi'\,. \end{cases} \tag{4}$$

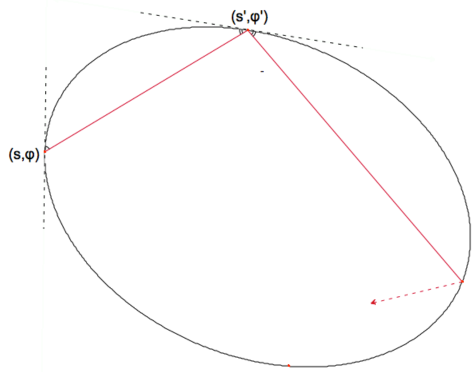

FIGURE 1

Remark 7. If we lift everything to the universal cover and introduce new coordinates $(\tilde{s}, r) = (s, \cos\varphi) \in \mathbb{R} \times (-1, 1)$, then the billiard map is a twist map with ℓ as generating function. See [23, 27].

Particularly interesting billiard orbits are periodic orbits, *i.e.*, billiard orbits $X = \{x_k\}_{k\in\mathbb{Z}} := \{(s_k, \varphi_k)\}_{k\in\mathbb{Z}}$ for which there exists an integer $q \geq 2$ such that $x_k = x_{k+q}$ for all $k \in \mathbb{Z}$. The minimal of such $q's$ represents the *period* of the orbit. However periodic orbits with the same period may be of very different topological types. A useful topological invariant that allows one to distinguish amongst them is the so-called *rotation number*, which can be easily defined as follows. Let X be a periodic orbit of period q and consider the corresponding q-tuple $(s_1, \ldots, s_q) \in \mathbb{R}/l\mathbb{Z}$. For all $1 \leq k \leq q$, there exists $\lambda_k \in (0, l)$ such that $s_{k+1} = s_k + \lambda_k$ (using the periodicity, $s_{q+1} = s_1$). Since the orbit is periodic, then $\lambda_1 + \ldots + \lambda_k \in l\mathbb{Z}$ and takes value between l and $(q-1)l$. The integer $p := \frac{\lambda_1+\ldots+\lambda_k}{l}$ is called the *winding number* of the orbit. The rotation number of X will then be the rational number $\rho(X) := \frac{p}{q}$. Observe that changing the orientation of the orbit replaces the rotation number $\frac{p}{q}$ by $\frac{q-p}{q}$. Since, for the purpose of our result, we do not distinguish between two opposite orientations, then we can assume that $\rho(X) \in (0, \frac{1}{2}] \cap \mathbb{Q}$.

In [4], as an application of Poincaré's last geometric theorem, Birkhoff proved the following result.

Theorem [Birkhoff] *For every $p/q \in (0, 1/2]$ in lowest terms, there are at least two geometrically distinct periodic billiard trajectories with rotation number p/q.*

Remark 8. In [13] V. Lazutkin introduced a very special change of coordinates that reduces the billiard map f to a very simple form.

Let $L_\Omega : [0, l] \times [0, \pi] \to \mathbb{T} \times [0, \delta]$ with small $\delta > 0$ be given by

$$L_\Omega(s, \varphi) = \left(x = C_\Omega^{-1} \int_0^s \rho^{2/3}(s)ds, \quad y = 4C_\Omega^{-1}\rho^{-1/3}(s)\ \sin\varphi/2 \right),$$

where $\rho(s)$ is its radius of curvature at s and $C_\Omega := \int_0^l \rho^{2/3}(s)ds$ is sometimes called the *Lazutkin perimeter* (observe that it is chosen so that period of x is one).

In these new coordinates the billiard map becomes very simple (see [13]):

$$f_L(x,y) = \Big(x + y + O(y^3), y + O(y^4)\Big).$$

In particular, near the boundary $\{\varphi = 0\} = \{y = 0\}$, the billiard map f_L reduces to a small perturbation of the integrable map $(x,y) \longmapsto (x+y, y)$.

Using this result and a version of KAM theorem, Lazutkin proved in [13] that if $\partial\Omega$ is sufficiently smooth (smoothness is determined by KAM theorem), then there exists a positive measure set of invariant curves (corresponding to *caustics*), which accumulates on the boundary and on which the motion is smoothly conjugate to a rigid rotation.

3. Aubry-Mather theory and billiards.

At the beginning of the eighties Serge Aubry and John Mather developed, independently, what nowadays is commonly called *Aubry–Mather theory.* This novel approach to the study of the dynamics of twist diffeomorphisms of the annulus, pointed out the existence of many *action-minimizing orbits* for any given rotation number (for a more detailed introduction, see for example [3, 18, 23, 24]).

More precisely, let $f : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ a monotone twist map, *i.e.*, a C^1 diffeomorphism such that its lift to the universal cover $\tilde{f}$ satisfies the following properties (we denote $(x_1, y_1) = \tilde{f}(x_0, y_0)$):

(i) $\tilde{f}(x_0 + 1, y_0) = \tilde{f}(x_0, y_0) + (1, 0)$,
(ii) $\frac{\partial x_1}{\partial y_0} > 0$ (monotone twist condition),
(iii) $\tilde{f}$ admits a (periodic) generating function h (*i.e.*, it is an exact symplectic map):

$$y_1\, dx_1 - y_0\, dx_0 = dh(x_0, x_1).$$

In particular, it follows from (iii) that:

$$\begin{cases} y_1 = \frac{\partial h}{\partial x_1}(x_0, x_1) \\ y_0 = -\frac{\partial h}{\partial x_0}(x_0, x_1)\,. \end{cases} \tag{5}$$

Remark 9. The billiard map f introduced above is an example of monotone twist map. In particular, its generating function is given by $h(x_0, x_1) = -\ell(x_0, x_1)$, where $\ell(x_0, x_1)$ denotes the euclidean distance between the two points on the boundary of the billiard domain corresponding to $\gamma(x_0)$ and $\gamma(x_1)$.

As it follows from (5), orbits $x = \{x_i\}_{i\in\mathbb{Z}}$ of the monotone twist diffeomorphism f correspond to critical points of the *action functional*

$$\{x_i\}_{i\in\mathbb{Z}} \longmapsto \sum_{i\in\mathbb{Z}} h(x_i, x_{i+1}).$$

Aubry-Mather theory is concerned with the study of orbits that minimize this action-functional amongst all configurations with a prescribed rotation number; recall that the rotation number of an orbit $\{x_i\}_{i\in\mathbb{Z}}$ is given by $\pi\omega = \lim_{i\to\pm\infty} \frac{x_i}{i}$, if this limit exists (in the billiard case, this definition leads to the same notion of rotation number introduced in subsection 1.2). In this context, *minimizing* is meant in the statistical mechanical sense, *i.e.*, every finite segment of the orbit minimizes the action functional with fixed end-points.

Theorem (S. Aubry, J. Mather). *A monotone twist map possesses minimal orbits for every rotation number. For rational numbers there are always at least two periodic minimal orbits. Moreover, every minimal orbit lies on a Lipschitz graph* *over the x-axis.*

Let us denote by $\mathcal{M}_\omega$ the set of minimal trajectories $\underline{x} = \{x_i\}_{i\in\mathbb{Z}}$ with rotation number ω and by $\mathcal{M}_\omega^{\text{rec}}$ the subset of recurrent ones. One can provide a detailed description of the structure of these sets (see [3, 18]):

- If $\omega \in \mathbb{R} \setminus \mathbb{Q}$, then $\mathcal{M}_\omega$ is totally ordered; moreover, there exist a map $f : \mathbb{R} \to \mathbb{R}$, which is the lift of an orientation-preserving circle homeomorphism with rotation number ω, and a closed f-invariant set $A_\omega \subset \mathbb{R}$, such that $\mathcal{M}_\omega$ consists of the orbits of f contained in A_ω. Namely, $\underline{x} \in \mathcal{M}_\omega$ if and only if $x_0 \in A_\omega$ and $x_i = f^i(x_0)$ for all $i \in \mathbb{Z}$. The projection p_0 (which to each $x = \{x_i\}_{i\in\mathbb{Z}}$ associates x_0) maps $\mathcal{M}_\omega$ homeomorphically into A_ω. Furthermore, $\underline{x} \in \mathcal{M}_\omega^{\text{rec}}$ if and only if x_0 is a recurrent point of f.

- If $\omega = \frac{p}{q} \in \mathbb{Q}$ (with p and q relatively prime), then $\mathcal{M}_\omega$ is the union of three disjoint and non-empty[3] sets:
$$\mathcal{M}^{\text{per}}_{\frac{p}{q}} \cup \mathcal{M}^{+}_{\frac{p}{q}} \cup \mathcal{M}^{-}_{\frac{p}{q}}.$$
$\mathcal{M}^{\text{per}}_{\frac{p}{q}}$ denotes the set of periodic minimal ones of rotation number $\frac{p}{q}$. We say that two elements $\underline{x}_- < \underline{x}_+$ of $\mathcal{M}^{\text{per}}_{\frac{p}{q}}$ are neighboring if there is no other element of $\mathcal{M}^{\text{per}}_{\frac{p}{q}}$ between them. We consider the sets $\mathcal{M}^{+}_{\frac{p}{q}}(\underline{x}_-, \underline{x}+)$ of all minimal orbits of rotation number $\frac{p}{q}$ that are asymptotic in the past (*i.e.*, as $i \to -\infty$) to $\underline{x}_-$ and in the future to $\underline{x}_+$. We define
$$\mathcal{M}^{+}_{\frac{p}{q}} = \bigcup_{(\underline{x}_-, \underline{x}_+)} \mathcal{M}^{+}_{\frac{p}{q}}(\underline{x}_-, \underline{x}_+),$$
where $(\underline{x}_-, \underline{x}_+)$ varies among all neighboring elements of $\mathcal{M}^{\text{per}}_{\frac{p}{q}}$.
In a similar way, one defines $\mathcal{M}^{-}_{\frac{p}{q}}$ (just reverse the behaviours in the past and in the future).
Usually orbits in $\mathcal{M}^{\pm}_{\frac{p}{q}}$ are said to have rotation symbol $\frac{p}{q}\pm$.

We can now introduce the *minimal average action* (or *Mather's β-function*).

Definition 10. Let $x^\omega = \{x_i\}_{i\in\mathbb{Z}}$ be any minimal orbit with rotation number ω. Then, the value of the *minimal average action* at ω is given by (this value is well-defined, since it does not depend on the chosen orbit):
$$\beta(\omega) = \lim_{N\to+\infty} \frac{1}{2N} \sum_{i=-N}^{N-1} h(x_i, x_{i+1}). \tag{6}$$

This function $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ enjoys many properties and encodes interesting information on the dynamics. In particular:

i) β is strictly convex and, hence, continuous (see [18]);
ii) β is differentiable at all irrationals (see [17]);
iii) β is differentiable at a rational p/q if and only if there exists an invariant circle consisting of periodic minimal orbits of rotation number p/q (see [17]).

In particular, being β a convex function, one can consider its convex conjugate:
$$\alpha(c) = \sup_{\omega\in\mathbb{R}} [\omega\, c - \beta(\omega)]\,.$$

[3]these sets are non-empty if $\mathcal{M}^{\text{per}}_{\frac{p}{q}}$ does not form an invariant curve

This function – which is generally called *Mather's α-function* – also plays an important role in the study of minimal orbits and in Mather's theory. We refer interested readers to surveys [3, 18, 23, 24].

Observe that for each ω and c one has:

$$\alpha(c) + \beta(\omega) \geq \omega c,$$

where equality is achieved if and only if $c \in \partial\beta(\omega)$ or, equivalently, if and only if $\omega \in \partial\alpha(c)$ (the symbol ∂ denotes in this case the set of 'subderivatives' of the function, which is always non-empty and is a singleton if and only if the function is differentiable).

In the billiard case, since the generating function of the billiard map is the euclidean distance $-\ell$, the action of the orbit coincides – up to a sign – to the length of the trajectory that the ball traces on the table Ω. In particular, these two functions encode many dynamical properties of the billiard (see [23] for more details):

- For each $0 < p/q \leq 1/2$, one has:
$$\beta(p/q) = -\frac{1}{q}\mathcal{M}L_{\Omega}^{\max}(p/q). \tag{7}$$
- β is differentiable at p/q if and only if there exists a caustic of rotation number p/q (*i.e.*, all tangent orbits are periodic of rotation number p/q).
- If Γ_ω is a caustic with rotation number $\omega \in (0, 1/2]$, then β is differentiable at ω and $\beta'(\omega) = -\text{length}(\Gamma_\omega) =: -|\Gamma_\omega|$ (see [23, Theorem 3.2.10]). In particular, β is always differentiable at 0 and $\beta'(0) = -|\partial\Omega|$.
- If Γ_ω is a caustic with rotation number $\omega \in (0, 1/2]$, then one can associate to it another invariant, the so-called *Lazutkin invariant* $Q(\Gamma_\omega)$. More precisely
$$Q(\Gamma_\omega) = |A - P| + |B - P| - |\,\widehat{AB}\,| \tag{8}$$
where $|\cdot|$ denotes the euclidean length and $|\,\widehat{AB}\,|$ the length of the arc on the caustic joining A to B (see figure 2).
This quantity is connected to the value of the α-function. In fact, one can show that (see [23, Theorem 3.2.10]):
$$Q(\Gamma_\omega) = \alpha(\beta'(\omega)) = \alpha(-|\Gamma_\omega|).$$

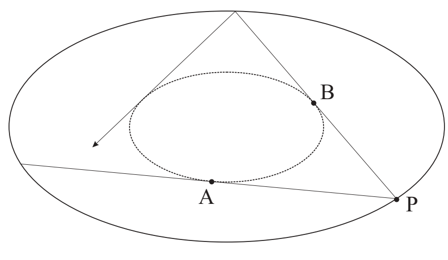

FIGURE 2. Lazutkin invariant

4. The generic assumptions

Let $f:(s,r)\to(s',r')$ denote the billiard map corresponding to a strictly convex domain Ω, parametrized by arclength s, and $h(s,s')=-\ell(s,s')$ (see (3)) denote the corresponding generating function. Then we have

$$\begin{cases} r=-\partial_1 h(s,s') \\ r'=\partial_2 h(s,s'). \end{cases}$$

Moreover

$$Df(s,r)=\begin{pmatrix} -\frac{\partial_{11}h(s,s')}{\partial_{12}h(s,s')} & -\frac{1}{\partial_{12}h(s,s')} \\ \partial_{12}h(s,s')-\partial_{22}h(s,s')\frac{\partial_{11}h(s,s')}{\partial_{12}h(s,s')} & \frac{-\partial_{22}h(s,s')}{\partial_{12}h(s,s')} \end{pmatrix} \tag{9}$$

and

$$Df^{-1}(s',r')=\begin{pmatrix} -\frac{\partial_{22}h(s,s')}{\partial_{12}h(s,s')} & \frac{1}{\partial_{12}h(s,s')} \\ \partial_{11}h(s,s')\frac{\partial_{22}h(s,s')}{\partial_{12}h(s,s')}-\partial_{12}h(s,s') & \frac{-\partial_{11}h(s,s')}{\partial_{12}h(s,s')} \end{pmatrix}. \tag{10}$$

Here and after, we denote

$$\partial_1 h=\partial_s h,\quad \partial_2 h=\partial_{s'}h,\quad \partial_{11}=\partial_s^2 h,\quad \partial_{22}=\partial_{s'}^2 h,\quad \partial_{12}h=\partial_s\partial_{s'}h.$$

Let us describe our main generic assumptions:

Assumptions. For each $0<p/q\in\mathbb{Q}$ in lowest terms,

(1) There exists a unique minimal periodic orbit in $\mathcal{M}^{\rm per}_{\frac{p}{q}}$.
(2) The minimal periodic orbit is hyperbolic.
(3) The stable and unstable manifolds of the minimal periodic orbit intersect transversally.

Under these assumptions, we have the following well known fact due to Aubry-Mather theory (see, e.g. [18]).

Proposition 11. *For every $0<p/q\in\mathbb{Q}$ in lowest terms, there exists a unique minimal orbit in $\mathcal{M}^+_{\frac{p}{q}}$.*

Observe that in Proposition 11, the unique orbit in $\mathcal{M}^+_{p/q}$ connects the unique Aubry-Mather periodic orbit of rotation number p/q to one of its shifts.

Let $\tau\geqslant 2$ and denote $\mathcal{E}^\tau$ the set of all the strictly convex $C^{\tau+1}$-billiard tables, for which the corresponding billiard maps satisfy Assumptions in Section 4. The set $\mathcal{E}^\tau$ is a residual subset of the space formed by strictly convex $C^{\tau+1}$-domains, with $C^{\tau+1}$-topology. See e.g. [5].

Hereafter, we fix $\Omega\in\mathcal{E}^\tau$ and $f:(s,r)\to(s',r')$ is the associated billiard map. Without further specification, all of our discussions are about the billiard map f.

5. Approximation of the Barrier

In this section, we will prove statement (1) in Main Theorem.

For $\frac{p}{q}\in\mathbb{Q}\cap(0,\frac12]$ in lowest terms, let

$$X_{p/q}: x_0,\dots,x_{q-1},$$

be the minimal periodic orbit with rotation number $\frac{p}{q}$ and let $L_{p,q}$ be its perimeter. Denote by $L_{Np,Nq-1}$ be the perimeter of the minimal periodic orbit with rotation number $\frac{Np}{Nq-1}$. Then:

Proposition 12. *For any sequence $N_n \in \mathbb{N}$ such that $N_n p$ is coprime with $N_n q - 1$ and $N_n \to +\infty$ as $n \to +\infty$, we have*

$$\lim_{n\to+\infty} L_{N_n p, N_n q-1} - N_n \cdot L_{p,q} = -p/q\beta'_+(p/q) + \beta(p/q).$$

where $\beta(\cdot)$ is the minimal averaged action of the billiard map f (introduced in Definition 10), and $\beta'_+(\cdot)$ is its one-side derivative.

Proof. Recall relation (7). Since $L_{p,q} = -q\beta(p/q)$ and $L_{N_n p, N_n q-1} = -(N_n q - 1)\beta(\frac{N_n p}{N_n q-1})$, then

$$\begin{aligned} L_{N_n p, N_n q-1} - N_n L_{p,q} &= -[(N_n q-1)\beta(\frac{N_n p}{N_n q - 1}) - N_n q \beta(p/q)] \\ &= -(N_n q - 1)(\beta(\frac{N_n p}{N_n q-1}) - \beta(p/q)) + \beta(p/q) \\ &= -p/q \frac{\beta(\frac{N_n p}{N_n q-1}) - \beta(p/q)}{\frac{N_n p}{N_n q-1} - \frac{p}{q}} + \beta(p/q) \\ &\longrightarrow\ -p/q\beta'_+(p/q) + \beta(p/q) \quad as \quad n \to +\infty. \end{aligned}$$

In the last equality, we have used the convexity of the minimal averaged action $\beta(\cdot)$. This proves the assertion of the Proposition. □

Let now

$$X_{p/q+} : \ldots, z_{-1}, z_0, z_1, \ldots,$$

be the minimal orbit in $\mathcal{M}^+_{\frac{p}{q}}$, and

$$d(f^{Nq}(z_0), f^{Nq}(x_1)) \to 0, \quad d(f^{-Nq}(z_0), f^{-Nq}(x_0)) \to 0, \quad as \quad N \to +\infty, \tag{11}$$

where $d(\cdot,\cdot)$ is the standard Euclidean distance in $\mathbb{R}^2$.

With slight abuse of notation, we will also use the same notation to denote the s-coordinates of the points in the orbits when they are considered as variables of the generating function $h(s,s') = -\ell(s,s')$. It follows from Aubry-Mather theory (see for example [18, Section 13]) that $X_{p/q+}$ minimizes
[18, Section 13]

$$\begin{aligned} B_{p/q}(z'_0) &= \lim_{M,\,K\to+\infty} \sum_{i=-Kq}^{Mq-1} \big(h(z'_i, z'_{i+1}) - h(x_i, x_{i+1})\big) \\ &= \lim_{M,\,K\to+\infty} \sum_{i=-Kq}^{Mq-1} h(z'_i, z'_{i+1}) + (M+K)L_{p,q}, \end{aligned}$$

among all the configurations $\ldots, z'_{-1}, z'_0, z'_1, \ldots$ such that (as $N \to +\infty$)

$$d(z'_{-Nq+i}, x_i) \to 0, \quad d(z'_{Nq+i}, x_{1+i}) \to 0, \quad i = 0, \ldots, q-1. \tag{12}$$

The function $B_{p/q}(\cdot)$ is usually referred as the *Peierls' Barrier function.* In particular, it follows from [18, Section 13]), that $B_{p/q}(z_0)$ is finite and, due to the hyperbolicity of the minimal periodic orbit $X_{p/q}$, one can show that the convergence is exponentially fast (see also Section 6).

Proposition 13. *For any sequence $N_n \in \mathbb{N}$ such that $N_n p$ is coprime with $N_n q - 1$ and $N_n \to +\infty$ as $n \to +\infty$, we have*

$$\lim_{n\to+\infty} L_{N_n p, N_n q-1} - N_n L_{p,q} = -B_{p/q}(z_0).$$

Remark 14. This result proves assertion (1) in Main Theorem.

Proof. For any $\epsilon > 0$ and large enough $N \in \mathbb{N}$ such that Np is coprime with $Nq - 1$, $N/3 < M < 2N/3$, $K = N - M$, let

$$X_{Np,Nq-1} : x'_{-Kq}, \dots, x'_0, \dots, x'_{Mq-2}$$

be the minimal periodic orbit with rotation number $\frac{Np}{Nq-1}$ and $d(x_{-Kq}, x_0) < \epsilon$. Then, clearly the configuration

$$\dots x_{-2}, x_{-1} X_{Np,Nq-1} x_1, x_2 \dots$$

satisfies (12). Therefore, by the minimality of the orbit $X_{p/q+}$, we have

$$-(L_{Np,Nq-1} - NL_{p,q}) \geqslant B_{p/q}(z_0) - C\epsilon,$$

where C is a constant that depends only on the billiard map f.

On the other hand, the configuration $z_{-Kq}, \dots, z_0, \dots, z_{Mq-2}, z_{-Kq}$ is of rotation number $\frac{Np}{Nq-1}$; hence

$$-L_{Np,Nq-1} + NL_{p,q} \leqslant B_{p/q}(z_0) + C\epsilon.$$

Therefore, the assertion of the proposition follows.

□

Using Proposition 12, Proposition 13 and relation (7), observe that item (2) in Main Theorem can be rephrased in terms of Mather's β-function in the following way.

Theorem 15. *For a generic strictly convex $C^{\tau+1}$-billiard table Ω ($\tau \geqslant 2$), we have that for each $p/q \in \mathbb{Q} \cap (0, 1/2]$ in lowest terms: for any sequence $N_n \in \mathbb{N}$ such that $N_n p$ is coprime with $N_n q - 1$ and $N_n \to +\infty$ as $n \to +\infty$,*

$$\lim_{n\to+\infty} \frac{1}{N_n} \log \left| (N_n q - 1)\beta(\frac{N_n p}{N_n q - 1}) - N_n q \beta(\frac{p}{q}) - B_{p/q} \right| = \log \lambda_{p/q},$$

where $\lambda_{p/q}$ is the eigenvalue of the linearization of the Poincaré return map at the Aubry-Mather periodic orbit with rotation number $\frac{p}{q}$ and $B_{p/q} = p/q\beta'_+(p/q) - \beta(p/q)$.

6. Eigenvalues of the Aubry-Mather periodic orbits

In this section, we describe the tools and the estimates which are needed in order prove assertion (2) of Main Theorem (see the end of this section for its proof).

Let $\Lambda_{p/q} = Df^q(x_1)$. Since $X_{p/q}$ is hyperbolic, $\Lambda_{p/q}$ is hyperbolic, *i.e.*, it has two distinguished eigenvalues $0 < \lambda_{\frac{p}{q}} < 1$ and $\lambda_{\frac{p}{q}}^{-1} > 1$. One of the main results of this section is the following theorem, which can be interpreted as a sort of normal form statement for Peierls' barrier.

Theorem 16. *There exists $N_{p,q} > 0$, $C_{p,q} \in \mathbb{R}$ and $C'_{p,q} \in \mathbb{R}$ such that, if $N > N_{p,q}$ and Np is coprime with $Nq - 1$, then there exists a periodic orbit $X_{Np,Nq-1}$ with minimal period $Nq - 1$, rotation number $Np/(Nq - 1)$ and perimeter $L'_{Np,Nq-1}$ satisfying,*

$$L'_{Np,Nq-1} - N \cdot L_{p,q} = -B_{p/q}(z_0) + C_{p,q}\lambda^N_{\frac{p}{q}} + \mathcal{O}(\lambda^{9N/8}_{\frac{p}{q}}), \quad \textit{if } N \textit{ is even},$$

and

$$L'_{Np,Nq-1} - N \cdot L_{p,q} = -B_{p/q}(z_0) + C'_{p,q}\lambda^N_{\frac{p}{q}} + \mathcal{O}(\lambda^{9N/8}_{\frac{p}{q}}), \quad \textit{if } N \textit{ is odd}.$$

Moreover $d(z_0, X_{Np,Nq-1}) = \mathcal{O}(\lambda^N_{\frac{p}{q}})$.

Remark 17. Notice that the constant $C_{p,q}$ for the even case can be different from the constant $C'_{p,q}$ for the odd one (see (35) and (36) respectively). See also Remark 24 in Appendix A.

Remark 18. It seems that Theorem 16 holds true in general, namely, suppose we have a hyperbolic periodic orbit of a billiard map and a transverse homoclinic orbit related to it. Then, the difference of perimeters should satisfy the estimate from Theorem 16.

The proof of this theorem is quite technical, so for the sake of clearer exposition, we postpone it to Appendix A.

Let us now notice the following fact.

Lemma 19. *When N is sufficiently large and Np is coprime with $Nq - 1$, the periodic orbit obtained in Theorem 16, is the one with the maximal perimeter, i.e., an Aubry-Mather periodic orbit.*

Proof. Let $X'_{Np,Nq-1}$ denote the periodic orbit with minimal period $Nq-1$, rotation number $\frac{Np}{Nq-1}$ and the maximal perimeter (minimal action). Then the distance $d(z_0, X'_{Np,Nq-1})$ tends to zero as N tends to $+\infty$. By hyperbolicity, there exists a neighborhood U of z_0 which contains exactly one periodic orbit with minimal period $Nq - 1$ and rotation number $\frac{Np}{Nq-1}$. Therefore $X_{Np,Nq-1}$ and $X'_{Np,Nq-1}$ coincide when N is large enough. □

In particular, combining together Theorem 16 and Lemma 19 we conclude the following.

Lemma 20. *If the constants $C_{p,q}$ and $C'_{p,q}$ in Theorem 16 are not zero, then for any sequence $N_n \in \mathbb{N}$ such that N_nq is coprime with $N_nq - 1$ and $N_n \to +\infty$ as $n \to +\infty$, we have*

$$\lim_{n\to+\infty} \frac{1}{N_n} \log |L_{N_np,N_nq-1} - N_n \cdot L_{p,q} + B_{p/q}(z_0)| = \log \lambda_{p/q}.$$

It turns out that the assumption in Lemma 20 is generic (see Appendix B for the proof).

Lemma 21. *For a generic billiard map f, we have that for each $p/q \in \mathbb{Q} \cap (0, 1/2)$, the constants $C_{p,q}(f)$ and $C'_{p,q}(f)$ in Theorem 16 are not zero.*

We can now conclude this section by proving assertion (2) in Main Theorem.

Proof. [**Main Theorem, item (2)**] Let us denote $\mathcal{E}'$ the set of strictly convex billiard tables, for which the induced billiard maps belong to the residual set $\mathcal{G}'$, as defined in (37). Consider the set

$$\mathcal{E} = \mathcal{E}' \cap \mathcal{E}^{\tau}.$$

Clearly, $\mathcal{E}$ is a residual set. Then the assertion (2) of Main Theorem follows from Lemmas 20 and 21. This concludes the proof. $\square$

Remark 22. In order to extend Main Theorem from Aubry-Mather periodic orbits to arbitrary hyperbolic periodic orbits of a generic domain (namely, determine the eigenvalue of the linearization of the associated Poincaré return map from the Marked Length Spectrum), we face two types of difficulties.

- By a result in [28], for a hyperbolic periodic orbit there is a homoclinic orbit, which is generically transverse. Existence of a transverse homoclinic orbit implies existence of a sequence of hyperbolic periodic orbits accumulating to it. To proceed with our scheme, we need to determine the corresponding sequence in the Marked Length Spectrum. In the light of Remark 18, this should provide Theorem 16.
- In order to prove Lemma 20, we need to know that constant $C_{p,q}$ and $C'_{p,q}$ are non-zero. In Lemma 25 we essentially use the graph property of $p/q+$ orbits, which is however not true in general.

Appendix A. Proof of Theorem 16

In order to prove Theorem 16, let us start by recalling the following lemma, which is well known, see e.g [26, 30].

Lemma 23. *For any $\epsilon > 0$, there exists a $C^{1,\frac{1}{2}}$ diffeomorphism $\Phi : V \to U$, where U, V are neighborhood of x_1 such that*

$$\Phi^{-1} \circ f^q \circ \Phi = \Lambda_{p/q}, \quad \|\Phi - \mathrm{Id}\|_{C^1} \leqslant \epsilon, \quad \textit{and} \quad \|\Phi^{-1} - \mathrm{Id}\|_{C^1} \leqslant \epsilon.$$

Moreover,

$$\Phi(z) - \Phi(z') = z - z' + \mathcal{O}(\max\{|z|^{1/2}, |z'|^{1/2}\}|z - z'|).$$

Let us start now the proof of Theorem 16.

Proof [Theorem 16]. From (11), we have that there exist n_0 and m_0 such that $f^{m_0 q}(z_0) \in U$ and $f^{-n_0 q+1}(z_0) \in U$. Let us denote their images under Φ as

$$A = \Phi(f^{m_0 q}(z_0)) \quad \text{and} \quad B = \Phi(f^{-n_0 q+1}(z_0)).$$

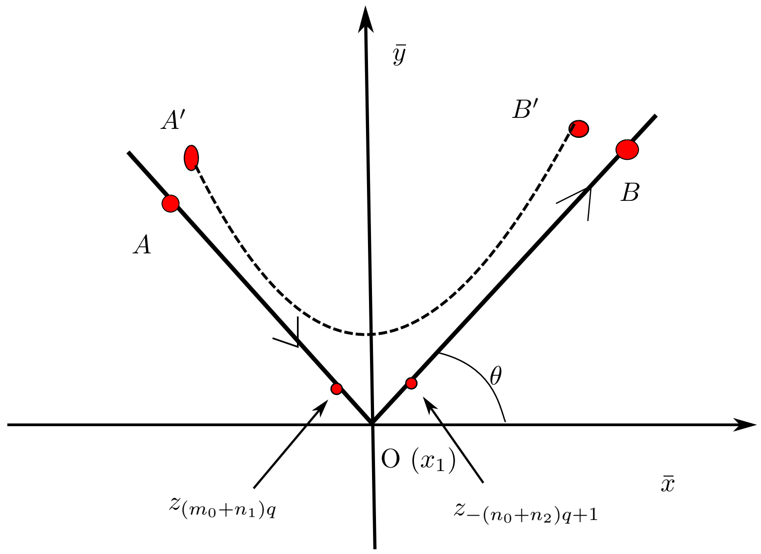

FIGURE 3. Saddle

For the sake of simplicity, hereafter in this proof, we will write $\Lambda_{p/q}$ and $\lambda_{\frac{p}{q}}$ as Λ and λ.

Now we consider the standard $\bar{x}$-$\bar{y}$ plane, where x_1 is located at the origin O. The unit eigenvectors corresponding to the eigenvalues λ and λ^{-1} are respectively,

$$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

See Figure 3. Using the change of coordinates

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \begin{pmatrix} \bar{\xi} \\ \bar{\eta} \end{pmatrix} = R_\theta \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix},$$

we tranform the map

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \mapsto \Lambda \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

into

$$\begin{pmatrix} \bar{\xi} \\ \bar{\eta} \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{-1}\bar{\xi} \\ \lambda\bar{\eta} \end{pmatrix}.$$

In the $\bar{\xi}$-$\bar{\eta}$ coordinate, we denote

$$A = (0, \eta) \quad \text{and} \quad B = (\xi, 0).$$

We choose $n \in \mathbb{N}$ to be sufficiently large and such that $(n + n_0 + m_0)p$ is coprime with $(n + n_0 + m_0)q - 1$. Let $\ldots, y_{-1}, y_0, y_1, \ldots$ be a periodic orbit with minimal period $(n + n_0 + m_0)q - 1$ and rotation number

$$\frac{(n + n_0 + m_0)p}{(n + n_0 + m_0)q - 1},$$

and let U_{z_0} be a small neighborhood of z_0, containing y_0, such that

$$f^{m_0 q}(U_{z_0}) \subset U \quad \text{and} \quad f^{-n_0 q+1}(U_{z_0}) \subset U.$$

Let us denote

$$A' = f^{m_0 q}(y_0), \quad B' = f^{-n_0 q+1}(y_0).$$

Then, in coordinates $\bar{\xi}$-$\bar{\eta}$, they become

$$A' = \begin{pmatrix} \delta_A \\ \eta + \delta'_A \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} \xi + \delta_B \\ \delta'_B \end{pmatrix}.$$

Here the δ's are small numbers to be determined.

Using the periodicity of y_0, we have that

$$\begin{pmatrix} \lambda^{-n}\delta_A \\ \lambda^n(\eta + \delta'_A) \end{pmatrix} = \begin{pmatrix} \xi + \delta_B \\ \delta'_B \end{pmatrix}$$

and

$$\begin{pmatrix} \delta_A \\ \delta'_A \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta_B \\ \delta'_B \end{pmatrix} + \mathcal{O}(\delta_B^{3/2} + (\delta'_B)^{3/2}),$$

where the 2×2 matrix on the right-hand side is the linear part of the global map $R_\theta \circ \Phi^{-1} \circ f^{(n_0+m_0)q-1} \circ \Phi \circ R_{-\theta}$ at the point B (the global map is of $C^{1,1/2}$). Due to the transversal intersections between the stable and unstable manifolds at points A and B, we know that $a \neq 0$. Therefore,

$$\begin{cases} \delta_A = \xi\lambda^n + \mathcal{O}(\lambda^{2n}), & \delta'_A = \frac{c\xi-\eta}{a}\lambda^n + \mathcal{O}(\lambda^{3n/2}), \\ \delta'_B = \eta\lambda^n + \mathcal{O}(\lambda^{2n}), & \delta_B = \frac{\xi-b\eta}{a}\lambda^n + \mathcal{O}(\lambda^{3n/2}). \end{cases}$$

Now, let us denote $n_1 := \lfloor n/2 \rfloor$ and $n_2 := n - n_1$.

In $\bar{\xi}$-$\bar{\eta}$ coordinates, for $i = 0, 1, \ldots, n_1$, the difference between the images of the points $f^{m_0q+iq}(y_0)$ and $f^{m_0q+iq}(z_0)$ is

$$\begin{pmatrix} \lambda^{-i}\delta_A \\ (\eta + \delta'_A)\lambda^i \end{pmatrix} - \begin{pmatrix} 0 \\ \eta\lambda^i \end{pmatrix} = \begin{pmatrix} \xi\lambda^{n-i} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{c\xi-\eta}{a}\lambda^{n+i} \end{pmatrix} + \mathcal{O}(\lambda^{3n/2})$$

and for $j = 0, 1, \ldots, n_2$, the difference between the images of the points $f^{-n_0q+1-jq}(y_0)$ and $f^{-n_0q+1-jq}(z_0)$ is

$$\begin{pmatrix} \lambda^j(\xi + \delta_B) \\ \delta'_B\lambda^{-j} \end{pmatrix} - \begin{pmatrix} \xi\lambda^j \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \eta\lambda^{n-j} \end{pmatrix} + \begin{pmatrix} \frac{\xi-b\eta}{a}\lambda^{n+j} \\ 0 \end{pmatrix} + \mathcal{O}(\lambda^{3n/2}).$$

Let us now switch back to the coordinate $(\bar{x}, \bar{y})$.

For $i = 0, 1, \ldots, n_1$, along the stable direction, the differences between the periodic orbit and the homoclinic orbit is

$$\begin{aligned} &\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \xi\lambda^{n-i} \\ \frac{c\xi-\eta}{a}\lambda^{n+i} \end{pmatrix} + \mathcal{O}(\lambda^{3n/2}) \\ &= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \xi\lambda^{n-i} + \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \frac{c\xi - \eta}{a}\lambda^{n+i} + \mathcal{O}(\lambda^{3n/2}). \end{aligned}$$

For $j = 0, 1, \ldots, n_2$, along the unstable direction

$$\begin{aligned} &\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\xi-b\eta}{a}\lambda^{n+j} \\ \eta\lambda^{n-j} \end{pmatrix} + \mathcal{O}(\lambda^{3n/2}) \\ &= \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \eta\lambda^{n-j} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \frac{\xi - b\eta}{a}\lambda^{n+j} + \mathcal{O}(\lambda^{3n/2}). \end{aligned}$$

For the orbits $\ldots, y_{-1},\ y_0,\ y_1, \ldots$ and $\ldots, z_{-1}, z_0, z_1, \ldots$, by Lemma 23, we have that for $i = 0, 1, \ldots, n_1$,

$$z_{m_0q+iq} - y_{m_0q+iq} = -\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \xi\lambda^{n-i} + \mathcal{O}(\lambda^{i/2}\lambda^{n-i}), \tag{13}$$

and for $j = 0, 1, \dots, n_2$,

$$z_{-n_0q+1-jq} - y_{-n_0q+1-jq} = -\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \eta\lambda^{n-j} + \mathcal{O}(\lambda^{j/2}\lambda^{n-j}). \qquad (14)$$

Now, we want to study the quantity

$$I = \sum_{i=-(n_0+n_2)q+1}^{(m_0+n_1)q-1} h(z_i, z_{i+1}) - h(y_i, y_{i+1}).$$

We split it into three parts:

a) the first part corresponds to the sum "far away" from the minimal periodic orbit $X_{p/q}$:

$$\mathcal{I}_0 = \sum_{i=-n_0q+1}^{m_0q-1} h(z_i, z_{i+1}) - h(y_i, y_{i+1});$$

b) the second part concerns the part along the unstable manifold, in a neighborhood of the minimal periodic orbit $X_{p/q}$:

$$\mathcal{I}_1 = \sum_{i=-(n_0+n_2)q+1}^{-n_0q} h(z_i, z_{i+1}) - h(y_i, y_{i+1});$$

c) the third part concerns the part along the stable manifold, in a neighborhood of the minimal periodic orbit $X_{p/q}$:

$$\mathcal{I}_2 = \sum_{i=m_0q}^{(m_0+n_1)q-1} h(z_i, z_{i+1}) - h(y_i, y_{i+1}).$$

Let us estimate these three contributions independently.

a) Since along the (periodic) orbit y_i, $i \in \mathbb{Z}$,

$$\partial_2 h(y_i, y_{i+1}) + \partial_1 h(y_{i+1}, y_{i+2}) = 0, \quad i \in \mathbb{Z} \qquad (15)$$

using Taylor's expansion we get

$$\begin{aligned}\mathcal{I}_0 &= \sum_{i=-n_0q+1}^{m_0q-1} h(z_i, z_{i+1}) - h(y_i, y_{i+1}) \\ &= \partial_1 h(y_{-n_0q+1}, y_{-n_0q+2})(z_{-n_0q+1} - y_{-n_0q+1}) \\ &\quad + \partial_2 h(y_{m_0q-1}, y_{m_0q})(z_{m_0q} - y_{m_0q}) + \mathcal{O}(\lambda^{2n}),\end{aligned} \qquad (16)$$

where in the last equality we have used that

$$|z_i - y_i| = O(\lambda^n), \quad i = -n_0q+1, \dots, m_0q,$$

as it follows from (13) and (14) and the Lipschitzianity of the map (observe that n_0 and m_0 are fixed).

b) Next, we consider $\mathcal{I}_1$, which is the sum of the terms along the unstable manifold. For $j = 1, \dots, n_2$, let us denote

$$\tilde{z}^j_k = z_{-n_0q-jq+1+k}, \quad \tilde{y}^j_k = y_{-n_0q-jq+1+k}, \quad k = 0, \dots, q,$$

and

$$I_j := \sum_{k=0}^{q-1} h(\tilde{z}_k^j, \tilde{z}_{k+1}^j) - h(\tilde{y}_k^j, \tilde{y}_{k+1}^j).$$

Clearly, $\mathcal{I}_1 = \sum_{j=1}^{n_2} I_j$. We split it into two other sums:

$$\mathcal{I}_1 = \sum_{j=1}^{n_2/2-1} I_j + \sum_{j=n_2/2}^{n_2} I_j.$$

Let us first consider the cases $j = n_2/2, \dots, n_2$. By (14) and (15), we have:

$$I_j = \partial_1 h(\tilde{y}_0^j, \tilde{y}_1^j)(\tilde{z}_0^j - \tilde{y}_0^j) + \partial_2 h(\tilde{y}_{q-1}^j, \tilde{y}_q^j)(\tilde{z}_q^j - \tilde{y}_q^j)$$
$$+ \frac{1}{2} \sum_{k=0}^{q-1} \begin{pmatrix} \tilde{y}_k^j - \tilde{z}_k^j \\ \tilde{y}_{k+1}^j - \tilde{z}_{k+1}^j \end{pmatrix}^T D^2 h(\tilde{y}_k^j, \tilde{y}_{k+1}^j) \begin{pmatrix} \tilde{y}_k^j - \tilde{z}_k^j \\ \tilde{y}_{k+1}^j - \tilde{z}_{k+1}^j \end{pmatrix} + \mathcal{O}(\lambda^{3(n-i)})$$

where

$$D^2 h(\tilde{y}_k^j, \tilde{y}_{k+1}^j) = \begin{pmatrix} \partial_{11} h(\tilde{y}_k^j, \tilde{y}_{k+1}^j) & \partial_{12} h(\tilde{y}_k^j, \tilde{y}_{k+1}^j) \\ \partial_{21} h(\tilde{y}_k^j, \tilde{y}_{k+1}^j) & \partial_{22} h(\tilde{y}_k^j, \tilde{y}_{k+1}^j) \end{pmatrix}$$
$$= \begin{pmatrix} \partial_{11} h(x_{k+1}, x_{k+2}) & \partial_{12} h(x_{k+1}, x_{k+2}) \\ \partial_{21} h(x_{k+1}, x_{k+2}) & \partial_{22} h(x_{k+1}, x_{k+2}) \end{pmatrix} + \mathcal{O}(\lambda^{n/4}).$$

Here we have used that, as it follows from (13) and (14), for $j = n_2/2, \dots, n_2$, $\tilde{y}_k^j$ are at least $\mathcal{O}(\lambda^{n/4})$-close to x_{k+1}, $k = 0, \dots, q$.

From (14) we know that

$$\tilde{z}_k^j - \tilde{y}_k^j = \lambda^{n-j} \prod_{i=0}^{k-1} Df(x_{i+1}) \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \eta + \mathcal{O}(\lambda^{n-j+\frac{n}{8}}), \quad k = 1, \dots, q.$$

Let us denote $Z_0^+ = \sin\theta$, and

$$Z_k^+ = \pi_1 \Big[\prod_{i=0}^{k-1} Df(x_{i+1}) \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \Big], \quad k = 1, \dots, q \tag{17}$$

where π_1 is the projection on the first coordinate. Denote

$$\mathcal{Z}_+ = (Z_0^+, Z_1^+, \dots, Z_q^+). \tag{18}$$

Then we have

$$\sum_{k=1}^{q-1} \begin{pmatrix} \tilde{y}_k^j - \tilde{z}_k^j \\ \tilde{y}_{k+1}^j - \tilde{z}_{k+1}^j \end{pmatrix}^T D^2 h(\tilde{y}_k^i, \tilde{y}_{k+1}^i) \begin{pmatrix} \tilde{y}_k^j - \tilde{z}_k^j \\ \tilde{y}_{k+1}^j - \tilde{z}_{k+1}^j \end{pmatrix}$$
$$= \mathcal{Z}_+ \mathbb{W}(X_{p/q}) \mathcal{Z}_+^T \eta^2 \lambda^{2(n-j)} + \mathcal{O}(\lambda^{2(n-j)+\frac{n}{8}}),$$

where

$$\mathbb{W}(X_{p/q}) = \begin{pmatrix} \eta_1 & \sigma_1 & 0 & \dots & 0 \\ \sigma_1 & \eta_2 & \sigma_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \sigma_{q-1} & \eta_q & \sigma_q \\ 0 & 0 & \dots & \sigma_q & \eta_{q+1} \end{pmatrix}_{(q+1)\times(q+1)} \tag{19}$$

with $\eta_1 = \partial_{11} h(x_1, x_2)$, $\eta_{q+1} = \partial_{22} h(x_0, x_1)$,

$$\eta_i = \partial_{22} h(x_{i-1}, x_i) + \partial_{11} h(x_i, x_{i+1}), \quad i = 2, \dots, q$$

and

$$\sigma_i = \partial_{12} h(x_i, x_{i+1}), \quad i = 1, \dots, q.$$

Then for $j = n_2/2, \dots, n_2$, we have

$$\begin{aligned} I_j &= \partial_1 h(\tilde{y}_0^j, \tilde{y}_1^j)(\tilde{z}_0^j - \tilde{y}_0^j) + \partial_2 h(\tilde{y}_{q-1}^j, \tilde{y}_q^j)(\tilde{z}_q^j - \tilde{y}_q^j) \\ &\quad + C_{q+}\eta^2 \lambda^{2(n-j)} + \mathcal{O}(\lambda^{2(n-j)+\frac{n}{8}}), \end{aligned} \tag{20}$$

where

$$C_{q+} = \frac{1}{2}\mathcal{Z}_+ \mathbb{W}(X_{p/q})\mathcal{Z}_+^T. \tag{21}$$

Moreover, for $j = 1, \dots, n_2/2 - 1$, by (14), we have

$$I_j = \partial_1 h(\tilde{y}_0^j, \tilde{y}_1^j)(\tilde{z}_0^j - \tilde{y}_0^j) + \partial_2 h(\tilde{y}_{q-1}^j, \tilde{y}_q^j)(\tilde{z}_q^j - \tilde{y}_q^j) + \mathcal{O}(\lambda^{2(n-j)}). \tag{22}$$

Hence, using again (15), as well as estimates (20) and (22), we conclude that

$$\begin{aligned} \mathcal{I}_1 &= \sum_{j=1}^{n_2} I_j = \sum_{j=1}^{n_2/2-1} I_j + \sum_{j=n_2/2}^{n_2} I_j \\ &= \partial_1 h(y_{-(n_0+n_2)q+1}, y_{-(n_0+n_2)q+2})(z_{-(n_0+n_2)q+1} - y_{-(n_0+n_2)q+1}) \\ &\quad + \partial_2 h(y_{-n_0 q}, y_{-n_0 q+1})(z_{-n_0 q+1} - y_{-n_0 q+1}) \\ &\quad + C_{q+}\eta^2 \frac{\lambda^{2(n-n_2)}}{1-\lambda^2} + \mathcal{O}(\lambda^{9n/8}). \end{aligned} \tag{23}$$

c) Finally, we deal with $\mathcal{I}_2$, *i.e.*, the sum of the contributions along the stable manifold.

For $i = 1, \dots, n_1$, let us denote

$$\bar{z}_k^i = z_{m_0 q+(i-1)q+k}, \quad \bar{y}_k^i = y_{m_0 q+(i-1)q+k}, \quad k = 0, \dots, q,$$

and

$$\bar{I}_i = \sum_{k=0}^{q-1} h(\bar{z}_k^i, \bar{z}_{k+1}^i) - h(\bar{y}_k^i, \bar{y}_{k+1}^i).$$

Clearly, $\mathcal{I}_2 = \sum_{i=1}^{n_1} \bar{I}_i$. We split it into two parts:

$$\mathcal{I}_2 = \sum_{i=1}^{n_1/2-1} \bar{I}_i + \sum_{i=n_1/2}^{n_1} \bar{I}_i.$$

First, consider the cases $i = n_1/2, \dots, n_1$. By (15), we have that

$$\begin{aligned} \bar{I}_i &= \partial_1 h(\bar{y}_0^i, \bar{y}_1^i)(\bar{z}_0^i - \bar{y}_0^i) + \partial_2 h(\bar{y}_{q-1}^i, \bar{y}_q^i)(\bar{z}_q^i - \bar{y}_q^i) \\ &+ \frac{1}{2}\sum_{k=0}^{q-1} \begin{pmatrix} \bar{y}_k^i - \bar{z}_k^i \\ \bar{y}_{k+1}^i - \bar{z}_{k+1}^i \end{pmatrix}^T D^2 h(\bar{y}_k^i, \bar{y}_{k+1}^i) \begin{pmatrix} \bar{y}_k^i - \bar{z}_k^i \\ \bar{y}_{k+1}^i - \bar{z}_{k+1}^i \end{pmatrix} \\ &+ \mathcal{O}(\lambda^{3(n-i)}). \end{aligned}$$

Using (13), we have

$$\bar{z}_0^i - \bar{y}_0^i = -\lambda^{n-i+1} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \xi + \mathcal{O}(\lambda^{n-i+\frac{n}{8}}),$$

and for $k = 1, \dots, q$,

$$\bar{z}_k^i - \bar{y}_k^i = -\lambda^{n-i+1} \prod_{l=0}^{k-1} Df(x_{l+1}) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \xi + \mathcal{O}(\lambda^{n-i+\frac{n}{8}}).$$

Denote $Z_0^- = -\cos\theta$,

$$Z_k^- = \pi_1\Big[-\prod_{l=0}^{k-1} Df(x_{l+1})\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\Big], k = 1,\dots,q, \tag{24}$$

and

$$\mathcal{Z}_- = (Z_0^-,\dots,Z_q^-). \tag{25}$$

Then:

$$\begin{aligned}\bar{I}_i &= \partial_1 h(\bar{y}_0^i,\bar{y}_1^i)(\bar{z}_0^i-\bar{y}_0^i)+\partial_2 h(\bar{y}_{q-1}^i,\bar{y}_q^i)(\bar{z}_q^i-\bar{y}_q^i)\\ &\quad+\frac{1}{2}\mathcal{Z}_-\mathbb{W}(X_{p/q})\mathcal{Z}_-^T\xi^2\lambda^{2(n-i+1)}+\mathcal{O}(\lambda^{2(n-i)+\frac{n}{8}}),\end{aligned}$$

where $\mathbb{W}(X_{p/q})$ is defined in (19). Moreover, for $i = 1,\dots,n_1/2-1$ we have

$$\partial_1 h(\bar{y}_0^i,\bar{y}_1^i)(\bar{z}_0^i-\bar{y}_0^i)+\partial_2 h(\bar{y}_{q-1}^i,\bar{y}_q^i)(\bar{z}_q^i-\bar{y}_q^i)+\mathcal{O}(\lambda^{2(n-i)}).$$

Therefore:

$$\begin{aligned}\mathcal{I}_2 &= \sum_{i=1}^{n_1/2-1}\bar{I}_i+\sum_{i=n_1/2}^{n_1}\bar{I}_i\\ &= \partial_1 h(y_{m_0q},y_{m_0q+1})(z_{m_0q}-y_{m_0q})\\ &\quad+\partial_2 h(y_{(m_0+n_1)q-1},y_{(m_0+n_1)q})(z_{(m_0+n_1)q}-y_{(m_0+n_1)q})\\ &\quad+C_{q-}\xi^2\frac{\lambda^{2(n-n_1+1)}}{1-\lambda^2}+\mathcal{O}(\lambda^{9n/8}),\end{aligned} \tag{26}$$

where

$$C_{q-} = \frac{1}{2}\mathcal{Z}_-\mathbb{W}(X_{p/q})\mathcal{Z}_-^T. \tag{27}$$

Summing up the contributions (16), (23) and (26), we obtain:

$$\begin{aligned} I &= \mathcal{I}_0+\mathcal{I}_1+\mathcal{I}_2\\ &= \partial_1 h(y_{-(n_0+n_2)q+1},y_{-(n_0+n_2)q+2})(z_{-(n_0+n_2)q+1}-y_{-(n_0+n_2)q+1})\\ &\quad+\partial_2 h(y_{(m_0+n_1)q-1},y_{(m_0+n_1)q})(z_{(m_0+n_1)q}-y_{(m_0+n_1)q})\\ &\quad+C_{q+}\eta^2\frac{\lambda^{2(n-n_2)}}{1-\lambda^2}+C_{q-}\xi^2\frac{\lambda^{2(n-n_1+1)}}{1-\lambda^2}+\mathcal{O}(\lambda^{9n/8}).\end{aligned} \tag{28}$$

Now, we need to consider the tail:

$$I_+ = \sum_{i=m_0q+n_1q}^{+\infty} h(z_i,z_{i+1})-h(x_{i+1},x_{i+2}).$$

Since along the periodic orbit $X_{p/q}$,

$$\partial_2 h(x_i,x_{i+1})+\partial_1 h(x_{i+1},x_i)=0,\quad i\in\mathbb{Z}, \tag{29}$$

we have

$$\begin{aligned} I_+ &= \partial_1 h(x_1,x_2)(z_{m_0q+n_1q}-x_1)\\ &\quad+\frac{1}{2}\sum_{i=m_0q+n_1q}^{+\infty}\begin{pmatrix} z_i-x_{i+1}\\ z_{i+1}-x_{i+2}\end{pmatrix}^T D^2h(x_{i+1},x_{i+2})\begin{pmatrix} z_i-x_{i+1}\\ z_{i+1}-x_{i+2}\end{pmatrix}\\ &\quad+\mathcal{O}(\lambda^{3n/2}).\end{aligned}$$

Since $x_{m_0q+n_1q+1}=x_1$ and

$$z_{m_0q+n_1q}-x_1=\lambda^{n_1}\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\eta+\mathcal{O}(\lambda^{3n_1/2}),$$

by means of the same calculation done for $\mathcal{I}_1$, we obtain

$$\begin{aligned}I_+&=\partial_1 h(x_1,x_2)(z_{m_0q+n_1q}-x_1)+\frac{\lambda^{2n_1}}{1-\lambda^2}\frac{\eta^2}{2}\mathcal{Z}_+\mathbb{W}(X_{p/q})\mathcal{Z}_+^T+\mathcal{O}(\lambda^{5n/4})\\ &=\partial_1 h(x_1,x_2)(z_{m_0q+n_1q}-x_1)+\frac{\lambda^{2n_1}}{1-\lambda^2}C_{q+}\eta^2+\mathcal{O}(\lambda^{5n/4}).\end{aligned}\tag{30}$$

Similarly, we can estimate the other tail and obtain

$$\begin{aligned}I_-&=\sum_{i=-\infty}^{-(n_0+n_2)q+1}h(z_{i-1},z_i)-h(x_{i-1},x_i)\\ &=\partial_2 h(x_0,x_1)(z_{-(m_0+n_2)q+1}-x_1)+C_{q-}\xi^2\frac{\lambda^{2(n_2+1)}}{1-\lambda^2}+\mathcal{O}(\lambda^{5n/4}).\end{aligned}\tag{31}$$

Summing up all contributions (28), (30) and (31) together

$$\begin{aligned}\mathbb{I}&:=I+I_-+I_+\\ &=\partial_1 h(y_{-(n_0+n_2)q+1},y_{-(n_0+n_2)q+2})(z_{-(n_0+n_2)q+1}-y_{-n_0q-n_2q+1})\\ &\quad+\partial_2 h(y_{(m_0+n_1)q-1},y_{(m_0+n_1)q})(z_{(m_0+n_1)q}-y_{(m_0+n_1)q})\\ &\quad+\partial_2 h(x_0,x_1)(z_{-(m_0+n_2)q+1}-x_1)+\partial_1 h(x_1,x_2)(z_{m_0q+n_1q}-x_1)\\ &\quad+2C_{q+}\eta^2\frac{\lambda^{2n_1}}{1-\lambda^2}+2C_{q-}\xi^2\frac{\lambda^{2(n_2+1)}}{1-\lambda^2}+\mathcal{O}(\lambda^{9n/8}).\end{aligned}\tag{32}$$

Since $y_{(m_0+n_1)q}=y_{-(n_0+n_2)q+1}$, by (15), we have

$$\begin{aligned}&\partial_2 h(y_{(m_0+n_1)q-1},y_{(m_0+n_1)q})(z_{(m_0+n_1)q}-y_{(m_0+n_1)q})\\ &+\partial_1 h(y_{-(n_0q+n_2)q+1},y_{-(n_0+n_2)q+2})(z_{-(n_0+n_2)q+1}-y_{-(n_0+n_2)q+1})\\ &=\partial_2 h(y_{(m_0+n_1)q-1},y_{(m_0+n_1)q})(z_{(m_0+n_1)q}-z_{-(n_0+n_2)q+1}).\end{aligned}$$

Notice that

$$\begin{aligned}y_{(m_0+n_1)q}-x_1&=y_{(m_0+n_1)q}-z_{(m_0+n_1)q}+z_{(m_0+n_1)q}-x_1\\ &=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\xi\lambda^{n-n_1}+\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\eta\lambda^{n_1}+\mathcal{O}(\lambda^{3n/4}),\end{aligned}$$

$$\begin{aligned}y_{(m_0+n_1)q-1}-x_0\quad&=\quad\begin{pmatrix}a'(\xi\lambda^{n_2}\cos\theta-\eta\lambda^{n_1}\sin\theta)+b'(\xi\lambda^{n_2}\sin\theta+\eta\lambda^{n_1}\cos\theta)\\ \gamma_0'\end{pmatrix}\\ &\quad+\mathcal{O}(\lambda^{3n/4}),\end{aligned}$$

and

$$z_{(m_0+n_1)q}-z_{-(n_0+n_2)q+1}=\begin{pmatrix}-\eta\lambda^{n_1}\sin\theta-\xi\lambda^{n_2}\cos\theta\\ \gamma_1'\end{pmatrix}+\mathcal{O}(\lambda^{3n/4}),$$

where a' and b' are from the expression

$$Df^{-1}(x_1)=\begin{pmatrix}a' & b'\\ * & *\end{pmatrix},$$

with

$$a'=\frac{-\partial_{22}h(x_0,x_1)}{\partial_{12}h(x_0,x_1)},\quad b'=\frac{1}{\partial_{12}h(x_0,x_1)},$$

(here we have used (10)). Thus we have

$$\begin{aligned}
&\partial_2 h(y_{(m_0+n_1)q-1}, y_{(m_0+n_1)q})\\
&= \partial_2 h(x_0,x_1) + \partial_{12}h(x_0,x_1)(y_{(m_0+n_1)q-1} - x_0) + \partial_{22}h(x_0,x_1)(y_{(m_0+n_1)q} - x_1) + \mathcal{O}(\lambda^n)\\
&= \partial_2 h(x_0,x_1) + \partial_{22}h(x_0,x_1)\big[\xi\lambda^{n_2}\cos\theta - \eta\lambda^{n_1}\sin\theta\big] + \mathcal{O}(\lambda^{3n/4})\\
&\quad + \partial_{12}h(x_0,x_1)\Big[-\frac{\partial_{22}h(x_0,x_1)}{\partial_{12}h(x_0,x_1)}(\xi\lambda^{n_2}\cos\theta - \eta\lambda^{n_1}\sin\theta) + \frac{1}{\partial_{12}h(x_0,x_1)}(\xi\lambda^{n_2}\sin\theta + \eta\lambda^{n_1}\cos\theta)\Big]\\
&= \partial_2 h(x_0,x_1) + (\xi\lambda^{n_2}\sin\theta + \eta\lambda^{n_1}\cos\theta) + \mathcal{O}(\lambda^{3n/4}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\partial_2 h(y_{(m_0+n_1)q-1}, y_{(m_0+n_1)q})(z_{(m_0+n_1)q} - z_{-(n_0+n_2)q+1})\\
&= \partial_2 h(x_0,x_1)(z_{(m_0+n_1)q} - z_{-(n_0+n_2)q+1})\\
&\quad + (\xi\lambda^{n_2}\sin\theta + \eta\lambda^{n_1}\cos\theta)(-\xi\lambda^{n_2}\cos\theta - \eta\lambda^{n_1}\sin\theta) + \mathcal{O}(\lambda^{5n/4})\\
&= \partial_2 h(x_0,x_1)(z_{(m_0+n_1)q} - z_{-(n_0+n_2)q+1})\\
&\quad - \sin\theta\cos\theta(\xi^2\lambda^{2n_2} + \eta^2\lambda^{2n_1}) - \xi\eta\lambda^n + \mathcal{O}(\lambda^{5n/4}).
\end{aligned} \tag{33}$$

Using (29),

$$\begin{aligned}
&\partial_2 h(x_0,x_1)(z_{(m_0+n_1)q} - z_{(n_0+n_2)q+1})\\
&+ \partial_2 h(x_0,x_1)(z_{-(m_0+n_2)q+1} - x_1) + \partial_1 h(x_1,x_2)(z_{m_0 q + n_1 q} - x_1) = 0,
\end{aligned}$$

hence we have,

$$\begin{aligned}
\mathbb{I} &= 2C_{q+}\eta^2\frac{\lambda^{2n_1}}{1-\lambda^2} + 2C_{q-}\xi^2\frac{\lambda^{2(n_2+1)}}{1-\lambda^2}\\
&\quad - \sin\theta\cos\theta(\xi^2\lambda^{2n_2} + \eta^2\lambda^{2n_1}) - \xi\eta\lambda^n + \mathcal{O}(\lambda^{9n/8}).
\end{aligned} \tag{34}$$

If n is even, then we have

$$\begin{aligned}
\mathbb{I} &= \Big(\frac{2C_{q+}\eta^2}{1-\lambda^2} + \frac{2C_{q-}\lambda^2\xi^2}{1-\lambda^2} - \xi^2\sin\theta\cos\theta - \eta^2\sin\theta\cos\theta - \xi\eta\Big)\frac{\lambda^{n_0+m_0+n}}{\lambda^{-m_0-n_0}}\\
&\quad + \mathcal{O}(\lambda^{9n/8})\\
&:= C_{p,q}\lambda^{n_0+m_0+n} + \mathcal{O}(\lambda^{9n/8}),
\end{aligned} \tag{35}$$

and shile n is odd, that is $n = 2n_1 + 1$, then

$$\begin{aligned}
\mathbb{I} &= \Big(\frac{2C_{q+}\eta^2}{\lambda(1-\lambda^2)} + \frac{2C_{q-}\xi^2\lambda^3}{1-\lambda^2} - \xi^2\lambda\sin\theta\cos\theta - \eta^2\lambda^{-1}\sin\theta\cos\theta - \xi\eta\Big)\frac{\lambda^{m_0+n_0+n}}{\lambda^{-m_0-n_0}}\\
&\quad + \mathcal{O}(\lambda^{9n/8})\\
&=: C'_{p,q}\lambda^{n_0+m_0+n} + \mathcal{O}(\lambda^{9n/8}).
\end{aligned} \tag{36}$$

Summarizing, the proof of the assertion follows by denoting $N = n_0 + m_0 + n$. This completes the proof of Theorem 16.

□

Remark 24. The constants in (34) are independent of the choice of the base point where we apply the normal form Lemma 23. Namely, if we choose x_2 as the base point, then in (23) and (30), the terms of the order λ^{2n_1} become

$$C_{q+}\frac{\eta^2\lambda^{2n_1}}{1-\lambda^2} - \frac{1}{2}\Big[h_{11}(x_1,x_2)(Z_0^+)^2 + 2h_{12}(x_1,x_2)Z_0^+Z_1^+ + h_{22}(x_1,x_2)(Z_1^+)^2\Big]\eta^2\lambda^{2n_1},$$

and the terms of order $\lambda^{2(n_2+1)}$ in (26) and (31) turn into

$$C_{q-}\frac{\xi^2\lambda^{2(n_2+1)}}{1-\lambda^2}+\frac{1}{2}\Big[h_{11}(x_1,x_2)(Z_0^-)^2+2h_{12}(x_1,x_2)Z_0^-Z_1^-+h_{22}(x_1,x_2)(Z_1^-)^2\Big]\xi^2\lambda^{2n_2}.$$

Those in (33) become

$$\Big[\partial_{12}h(x_1,x_2)(x_1,x_2)(Z_0^-Z_1^+ - Z_0^+Z_1^-)+\partial_{22}h(x_1,x_2)(Z_1^-Z_1^+ - Z_1^-Z_1^+)\Big]\xi\eta\lambda^n$$
$$+\Big(\partial_{12}h(x_1,x_2)Z_0^+Z_1^+ + \partial_{22}h(x_1,x_2)(Z_1^+)^2\Big)\eta^2\lambda^{2n_1}$$
$$+\Big(-\partial_{12}h(x_1,x_2)Z_0^-Z_1^- - \partial_{22}h(x_1,x_2)(Z_1^-)^2\Big)\xi^2\lambda^{2n_2}.$$

Then adding them up, using (9), (10), (17) and (24), we have exactly (34).

Appendix B. Proof of Lemma 21

In this appendix we want to prove Lemma 21, namely that constants $C_{p,q}(f)$, $C'_{p,q}(f)$ appearing in Theorem 16 are generically non-zero.

From now on, we use the notations $C_{p,q}(f)$, $C'_{p,q}(f)$, $\lambda(f)$, $\xi(f)$, etc..., to indicate explicitly the dependence on f.

We start first with the following Lemma.

Lemma 25. *There exist $\bar\epsilon_1>0$, $\bar\epsilon_2>0$ and a family of billiard maps $f_{\epsilon_1,\epsilon_2}$ parametrized by $\epsilon_1\in[-\bar\epsilon_1,\bar\epsilon_1]$ and $\epsilon_2\in[-\bar\epsilon_2,\bar\epsilon_2]$ such that*

$$f_{0,0}=f,\quad \lambda(f_{\epsilon_1,\epsilon_2})=\lambda(f),\quad \theta(f_{\epsilon_1,\epsilon_2})=\theta(f),$$

and

$$\frac{d}{d\epsilon_1}\xi(f_{\epsilon_1,\epsilon_2})\neq 0,\quad \frac{d}{d\epsilon_2}\eta(f_{\epsilon_1,\epsilon_2})\neq 0.$$

Moreover,

$$\|f_{\epsilon_1,\epsilon_2}-f\|_{C^\tau}\to 0,\quad \text{as}\quad \epsilon_1\to 0,\ \epsilon_2\to 0.$$

Proof. Let us denote

$$s_i'=\pi_1\big(f^i(z_0)\big),\quad i=-2,-1,0,1,2.$$

Because of the graph property of the orbit $X_{p/q+}$, for $i=-2,-1,0,1,2$, there exist $\gamma_i^-<0$, $\gamma_i^+>0$ and functions φ_i such that the following holds:

(1) $\big\{(s,r): s\in[s_i'+\gamma_i^-, s_i'+\gamma_i^+],\ r\in[0,1]\big\}\cap X_{p/q+}=\{z_i\}$.

(2) Denote

$$\Gamma_i:=\{(s,\varphi_i(s)): s\in[s_i'+\gamma_i^-, s_i'+\gamma_i^+]\},\quad i=\pm 2,$$

and

$$\Gamma_0^\pm:=\{(s,\varphi_0^\pm(s): s\in[s_0'+\gamma_0^-, s_0'+\gamma_0^+].$$

The graphs Γ_0^- and Γ_{-2} are the local graphs of the unstable manifold of x_0 near the points z_i, $i=0,-2$, and the graphs Γ_0^+ and Γ_2 are the local graphs of the stable manifold of x_1 near the points z_i, $i=0,2$.

(3) There exist strictly increasing C^τ functions

$$\eta_i(t) : [s'_i + \gamma_i^-, s'_i + \gamma_i^+] \to [s'_{i+2} + \gamma_{i+2}^-, s'_{i+2} + \gamma_{i+2}^+],\ i = -2, 0$$

such that $\eta(s'_i) = s'_{i+2}$, $i = -2, 0$,

$$f^2\big(s, \varphi_{-2}(s)\big) = \big(\eta_{-2}(s), \varphi_0^-(\eta_{-2}(s))\big), \quad s \in: [s'_{-2} + \gamma_{-2}^-, s'_{-2} + \gamma_{-2}^+],$$

and

$$f^2\big(s, \varphi_0^+(s)\big) = \big(\eta_0(s), \varphi_2(\eta_0(s))\big), \quad s \in: [s'_0 + \gamma_0^-, s'_0 + \gamma_0^+].$$

Let $0 < \bar\epsilon_1 < \frac{1}{3}\min\{|\gamma_{-2}^+|, |\gamma_{-2}^-|\}$, $0 < \bar\epsilon_2 < \frac{1}{3}\min\{|\gamma_0^+|, |\gamma_0^-|\}$ and $\bar\epsilon_1$, $\bar\epsilon_2$ be small enough. For $\epsilon_1 \in [-\bar\epsilon_1, \bar\epsilon_1]$ and $\epsilon_2 \in [-\bar\epsilon_2, \bar\epsilon_2]$, we define a deformation $\Omega_{\epsilon_1,\epsilon_2}$ of the domain Ω, with the corresponding billiard map $f_{\epsilon_1,\epsilon_2}$ such that

i. If $s \in [s'_{-2} - \epsilon_1, s'_{-2} + \epsilon_1]$ and $r = \varphi_{-2}(s)$, then

$$f^2_{\epsilon_1,\epsilon_2}(s, r) = \big(\eta_{-2}(s + \epsilon_1), \varphi_0^-(\eta_{-2}(s + \epsilon_1))\big).$$

ii. If $s \in [s'_0 - \epsilon_2, s'_0 + \epsilon_2]$, and $r = \varphi_0^+(s)$, then

$$f^2_{\epsilon_1,\epsilon_2}(s, r) = \big(\eta_0(s + \epsilon_2), \varphi_2(\eta_1(s + \epsilon_2)\big).$$

iii. Let

$$\epsilon'_1 = \max\{\big|\pi_1(f(s_{-2} \pm \epsilon_1, \varphi_{-2}(s_{-2} \pm \epsilon_1)) - s'_{-1}\big|\}$$

and

$$\epsilon'_2 = \max\{\big|\pi_1(f(s_0 \pm \epsilon_2, \varphi_0^+(s_0 \pm \epsilon_2)) - s'_1\big|\}.$$

If $s \notin [s'_{-1} - 3\epsilon'_1, s'_{-1} + 3\epsilon'_1] \cup [s'_1 - 3\epsilon'_2, s'_1 + 3\epsilon'_2]$, then

$$\partial\Omega_{\epsilon_1,\epsilon_2}(s) = \partial\Omega(s).$$

The existence of such domain is due to the implicit function theorem for small enough $\bar\epsilon_1$ and $\bar\epsilon_2$.

By the construction, we could see that for $f_{\epsilon_1,\epsilon_2}$:

a. $X_{p/q}$ is still the minimal periodic orbit in $\mathcal{M}_{\frac{p}{q}}$;
b. the orbit $\{f^i_{\epsilon_1,\epsilon_2}(z_0), i \in \mathbb{Z}\}$ is the minimal orbit in $\mathcal{M}_{\frac{p}{q}+}$;
c. near $X_{p/q}$, the billiard maps $f_{\epsilon_1,\epsilon_2}$ and f are the same;
d. the point $f^{-n_0q+1}_{\epsilon_1,\epsilon_2}(z_0)$ moves non-degenerately as ϵ_1 change. So does the point $f^{m_0q}_{\epsilon_1,\epsilon_2}$ with respect to ϵ_2.

These imply that the parametrized family of billiard maps $f_{\epsilon_1,\epsilon_2}$ satisfy the requirements of the Lemma. □

We can now prove Lemma 21.

Proof. [**Lemma 21**] For each $p/q \in \mathbb{Q} \cap (0, 1/2]$, let us denote $\mathcal{G}_{p/q}$ the set of billiard maps f such that $C_{p/q}(f) \neq 0$ and $C'_{p/q}(f) \neq 0$. Clearly $\mathcal{G}_{p/q}$ is an open set, since $C_{p/q}(f)$ and $C'_{p/q}(f)$ are continuous with respect to f in the C^τ-topology. If $C_{p/q}(f) = 0$, by Lemma 25, we could find a billiard map f', which is arbitrary close to f in the C^τ-topology, such that $C_{p/q}(f') \neq 0$. Therefore $\mathcal{G}_{p/q}$ is a dense open subset. Then we can choose the generic set to the residual set

$$\mathcal{G}' = \bigcap_{p/q \in \mathbb{Q} \cap (0,1/2]} \mathcal{G}_{p/q}. \tag{37}$$

In particular, each billiard map $f \in \mathcal{G}'$ verifies the assertion of the Lemma. □

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