Instabilities and Phase Transitions in the Ising Model. A Review (*).

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1. – Introduction.

The Ising model plays a very special role in statistical mechanics and provides the simplest nontrivial example of a system undergoing phase transitions [1].

The analysis of this model has provided deep insight into the general nature of the phase transitions which are certainly better understood nowadays

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after the publication of the hundreds of papers which followed the pioneering work of ISING, PEIERLS, ONSAGER, and LEE and YANG [1-4].

The main reason why so much attention has been given to this very special model lies in its simplicity and in the fact that, in spite of it, it first provided firm and quantitative indications that a microscopic short-ranged interaction can produce phase transitions which, furthermore, deeply differ in character from the classical Van der Waals (or Curie-Weiss or mean field) type of transitions [5].

It should also be mentioned that the two-dimensional Ising model in zero external field is exactly solvable [6], this fact has very often been used as a check of the validity of numerical approximations devised to be applied to more complicated models [7].

In some cases the Ising model is a good phenomenological model for antiferromagnetic materials [8].

Finally, last but not least, we mention that the Ising model has given rise to a number of interesting developments and reinterpretations of old results in the theory of Markov chains [9], information theory and ergodic theory [10], random walks [11], and therefore constitutes a remarkable example of a subject which has simultaneously been the object of advanced research in physics, mathematics and mathematical physics.

In this review article we hope to give a self-contained, though certainly not exhaustive, description of the model and of some selected rigorous results illustrating properties which throw some light on the general nature of the phenomenon of the phase transitions far from the critical point and which, hopefully, should not be a peculiarity of the simplicity of the model.

There exist some very good accounts on the theoretical arguments leading to the consideration of the Ising model in the context of physical problems [7, 12]. Therefore we shall completely skip this aspect of the matter and refer the interested reader to the literature.

2. – The model. Grand canonical and canonical ensembles. Their inequivalence.

We consider a δ-dimensional ($\delta = 1, 2, 3$) square lattice Z^δ and a finite square $\Lambda \subset Z^\delta$ centred around the origin, containing $|\Lambda| = L^\delta$ lattice sites.

On each site $x \in \Lambda$ is located a classical spin $\sigma_x = \pm 1$. The « configurations » of our system will, therefore, consist in a set $\underline{\sigma} = (\sigma_{x_1}, \ldots, \sigma_{x_{|\Lambda|}})$ of $|\Lambda|$ numbers $\sigma_x = \pm 1$; the number of these configurations is $2^{|\Lambda|}$. The ensemble of these configurations will be denoted as $\mathscr{U}(\Lambda)$.

To each spin configuration is assigned a certain « energy »

$$H_\Lambda(\underline{\sigma}) = -J \sum_{\langle i,j \rangle} \sigma_{x_i} \sigma_{x_j} - h \sum_i \sigma_{x_i} - \mathscr{B}_\Lambda(\underline{\sigma}) , \tag{2.1}$$

where $\sum_{\langle i,j \rangle}$ means that the sum is over pairs (x_i, x_j) of neighbouring points, h is an external magnetic field and $\mathscr{B}_\Lambda(\sigma)$ describes the interaction of the spins in the box Λ with the «rest of the world» [13].

For simplicity we shall treat, in this paper, only the case $J > 0$.

Of course $\mathscr{B}_\Lambda(\sigma)$ in (2.1) can be rather arbitrary and, actually, depends on the particular physical problem under investigation. It is subject, however, to one constraint of physical nature: in case we were interested in letting $\Lambda \to \infty$, we should impose the condition

$$\lim_{\Lambda\to\infty} \frac{\max\limits_{\underline{\sigma}} |\mathscr{B}_\Lambda(\underline{\sigma})|}{|\Lambda|} = 0 \,, \tag{2.2}$$

i.e. we want the energy due to $\mathscr{B}_\Lambda(\sigma)$ should not to be of the same order as the volume of the box. In other words it should be a «surface term».

The laws of statistical mechanics provide a relationship between the microscopic Hamiltonian (2.1) and the macroscopic quantities appearing in the thermodynamical theory of the system.

The free energy per unit volume is given by

$$f_\Lambda(\beta, h) = \frac{\beta^{-1}}{|\Lambda|} \log Z(\beta, h, \Lambda, \mathscr{B}) \,, \tag{2.3}$$

where $\beta = T^{-1}$ is the inverse temperature and

$$Z(\beta, h, \Lambda, \mathscr{B}) = \sum_{\underline{\sigma}\in\mathscr{U}(\Lambda)} \exp[-\beta H_\Lambda(\underline{\sigma})] \tag{2.4}$$

is the grand canonical partition function. Furthermore the probability of finding the system in a configuration $\underline{\sigma}$ of the grand canonical ensemble $\mathscr{U}(\Lambda)$ is given by the Boltzmann factor

$$\frac{\exp[-\beta H_\Lambda(\underline{\sigma})]}{Z(\beta, h, \Lambda, \mathscr{B})} \,, \qquad \underline{\sigma}\in\mathscr{U}(\Lambda) \,. \tag{2.5}$$

For a theoretical foundation of (2.3), (2.5) see [14]. The grand-canonical-ensemble formalism based on (2.3), (2.5) corresponds to the physical situation in which there are no constraints on the system. If one could, by some experimental arrangement, regard, for example, the total magnetization $M(\underline{\sigma}) = \sum_{x\in\Lambda} \sigma_x$ as fixed: $M(\underline{\sigma}) = M = m|\Lambda|$, then the expression (2.3) for the free energy would no longer be appropriate nor would the predictions based on (2.5) be appropriate. One should rather consider the canonical ensemble, *i.e.* the set of the allowed configurations would be the set $\mathscr{U}(\Lambda, m) \subset \mathscr{U}(\Lambda)$ consisting of all the $\underline{\sigma}\in\mathscr{U}(\Lambda)$ such that $\sum_{x\in\Lambda} \sigma_x = m|\Lambda|$ $(|m| \leqslant 1)$, and the thermodynamics would be described by the function

$$g_\Lambda(\beta, m, h) = \frac{\beta^{-1}}{|\Lambda|} \log Z(\beta, h, \Lambda, \mathscr{B}, m) , \tag{2.6}$$

where

$$Z(\beta, h, \Lambda, \mathscr{B}, m) = \sum_{\underline{\sigma} \in \mathscr{U}(\Lambda, m)} \exp[-\beta H_\Lambda(\underline{\sigma})] , \tag{2.7}$$

and the free energy would be $\tilde{f}(\beta, h)$:

$$\tilde{f}(\beta, h) = m(h) + g_\Lambda(\beta, 0, m(h)) , \tag{2.8}$$

where $m(h)$ is the solution of the equation [15]

$$h = \frac{\partial g_\Lambda(\beta, 0, m)}{\partial m} . \tag{2.9}$$

There is no reason for having $\tilde{f}_\Lambda = f_\Lambda$ since they correspond to different physical problems; it is only when, *in some sense*, the fluctuations become negligible (*i.e.* in the limit $\Lambda \to \infty$) that one can expect the identity between $\tilde{f}$ and f.

Of course in general the difference between $\tilde{f}_\Lambda$ and f_Λ should vanish as $O(|\Lambda|^{-(\delta-1)/\delta})$ (and logarithmically for $\delta = 1$); but, as we shall see on many occasions, the situation is not so simple for other quantities such as the correlation functions or the average magnetization.

The inequivalence, for finite volume, of the predictions of the canonical and grand canonical ensembles should not be interpreted as meaning that statistical mechanics is only approximate when applied to finite systems; it simply means that in dealing with finite systems care must be paid not only to the boundary conditions but also to the actual physical situation from which the problem under consideration arises.

We conclude by observing that in the canonical ensemble the probability of a spin configuration will be given by an expression similar to (2.5):

$$\frac{\exp[-\beta H_\Lambda(\underline{\sigma})]}{Z(\beta, h, \Lambda, \mathscr{B}, m)} , \qquad \underline{\sigma} \in \mathscr{U}(\Lambda, m) . \tag{2.10}$$

3. – Boundary conditions. Equilibrium states.

Formula (2.5) or (2.10) provides a complete statistical description of the properties of the system. An alternative and often more convenient, though equally complete, description is provided by the so-called correlation functions

$$\langle \sigma_{x_1}\sigma_{x_2} \dots \sigma_{x_n}\rangle_{\Lambda,\mathscr{B}_\Lambda} = \frac{\sum_{\underline{\sigma}} \sigma_{x_1} \dots \sigma_{x_n} \exp\left[-\beta H_\Lambda(\underline{\sigma})\right]}{\sum_{\underline{\sigma}} \exp\left[-\beta H_\Lambda(\underline{\sigma})\right]}, \tag{3.1}$$

where the $\sum_{\underline{\sigma}}$ is extended to the appropriate statistical ensemble.

For instance the average magnetization in the grand canonical ensemble $\mathscr{U}(\Lambda)$ is

$$m_\Lambda(\beta, h) = \frac{\partial f_\Lambda(\beta, h)}{\partial h} = \frac{\sum_{x\in\Lambda} \langle \sigma_x \rangle_{\Lambda,\mathscr{B}}}{|\Lambda|}. \tag{3.2}$$

We shall refer to the family of correlation functions (3.1) (regarded as a whole) as the « equilibrium state of the system in the box Λ ».

We shall call equilibrium state of the infinite system any family $\{\langle \sigma_{x_1} \dots \sigma_{x_n}\rangle\}$ of functions such that, for a suitable choice of the $\mathscr{B}_\Lambda(\underline{\sigma})$,

$$\langle \sigma_{x_1} \dots \sigma_{x_n}\rangle = \lim_{\Lambda\to\infty} \langle \sigma_{x_1} \dots \sigma_{x_n}\rangle_{\Lambda,\mathscr{B}_\Lambda} \tag{3.3}$$

for all $n \geqslant 1$ and all $x_1, x_2, \dots, x_n \in Z^\delta$, *simultaneously* [16].

An equilibrium state for the infinite system will simply be called an equilibrium state and is specified by a suitable choice of a sequence $\{\mathscr{B}_\Lambda(\underline{\sigma})\}$ of boundary conditions satisfying the requirement (2.2).

Let us list a number of remarkable boundary conditions:

1) *Open boundary condition* (also called perfect-wall boundary conditions): this name will be given to the case

$$\mathscr{B}_\Lambda(\underline{\sigma}) \equiv 0 \qquad \text{for all } \underline{\sigma} \in \mathscr{U}(\Lambda). \tag{3.4}$$

2) *Periodic boundary conditions*: this corresponds to allowing spins on opposite faces of the box Λ to interact through a coupling $-J$ (*i.e.* as the bulk spins). Clearly this can be obtained by a suitable choice of $\mathscr{B}_\Lambda(\underline{\sigma})$; to this choice we shall refer as « periodic boundary conditions ».

3) $(\underline{\varepsilon})$-*boundary conditions*: let $(\xi_1, \xi_2, \dots)$ be the $2\delta|\Lambda|^{(\delta-1)/\delta}$ lattice points adjacent to the boundary of Λ. Let $\underline{\varepsilon} = (\varepsilon_{\xi_1}, \varepsilon_{\xi_2}, \dots)$, $\varepsilon_{\xi_i} = \pm 1$, be fixed. We shall call $(\underline{\varepsilon})$-boundary condition the choice

$$\mathscr{B}_\Lambda(\sigma) = -J \sum_{x_i \in \partial\Lambda} \sigma_{x_i} \varepsilon_{\xi_i}, \tag{3.5}$$

where (x_i, ξ_i) are nearest neighbours.

The physical meaning of this boundary condition is clear: we imagine that the sites neighbouring the boundary $\partial\Lambda$ of Λ are occupied by a spin configuration $\underline{\varepsilon}$ and these spins interact with the spins $\underline{\sigma}$ through the same coupling constant of the bulk spins.

The cases $\underline{\varepsilon} = (+1, +1, \ldots)$ or $\underline{\varepsilon} = (-1, -1, \ldots)$ will be, respectively, referred to as the $(+)$-*boundary condition* or the $(-)$-*boundary condition.*

4) In the two-dimensional case we shall be interested in another boundary condition. Suppose the spins on the opposite *vertical* sides of Λ are allowed to interact through a coupling $-J$ (*i.e.* we impose periodic boundary conditions along the rows of Λ *only*); and suppose that a set $\underline{\varepsilon}_u$ of fixed spins is located on the lattice sites adjacent to the upper base of Λ and, similarly, a set $\underline{\varepsilon}_l$ of fixed spins is adjacent to the lower base of Λ. The spins $\underline{\varepsilon}_u$, $\underline{\varepsilon}_l$ are allowed to interact with the nearest spins in Λ with a coupling $-J$. We shall naturally refer to this choice of $\mathscr{B}_\Lambda(\underline{\sigma})$ as the $(\underline{\varepsilon}_u, \underline{\varepsilon}_l)$-*cylindrical boundary condition.*

The particular cases

$$\underline{\varepsilon}_u = (+1, +1, \ldots), \qquad \underline{\varepsilon}_l = (+1, +1, \ldots)$$

or

$$\underline{\varepsilon}_u = (+1, +1, \ldots), \qquad \underline{\varepsilon}_l = (-1, -1, \ldots)$$

will be referred to respectively as $(+, +)$-*cylindrical boundary condition* or $(+, -)$-*cylindrical boundary condition.*

4. – The Ising model in 1 and 2 dimensions and zero field.

To acquire some familiarity with the model let us examine some of the simplest cases.

Consider the one-dimensional Ising chain with periodic boundary conditions. If we label the points of Λ as $1, 2, \ldots, L$, the Hamiltonian in zero field is

$$H_\Lambda(\underline{\sigma}) = -J\sum_{i=1}^{L}\sigma_i\sigma_{i+1}\,, \qquad \sigma_{L+1} \equiv \sigma_1\,, \tag{4.1}$$

(clearly $\mathscr{B}_\Lambda(\underline{\sigma}) = -J\sigma_L\sigma_1$). The grand canonical partition function can be written

$$Z_\Lambda(\beta) = \sum_{\underline{\sigma}}\exp\left[\beta J\sum_{i=1}^{L}\sigma_i\sigma_{i+1}\right] = \sum_{\underline{\sigma}}\prod_{i=1}^{L}\exp\left[J\beta\sigma_i\sigma_{i+1}\right]. \tag{4.2}$$

Remarking that $(\sigma_i\sigma_{i+1})^2 \equiv 1$ and, therefore,

$$\exp\left[\beta J\sigma_i\sigma_{i+1}\right] \equiv \cosh\beta J + \sigma_i\sigma_{i+1}\sinh\beta J\,,$$

eq. (4.2) can be rewritten as

$$Z_\Lambda(\beta) = (\cosh \beta J)^L \sum_{\underline{\sigma}} \prod_{i=1}^{L} (1 + (\operatorname{tgh} \beta J)\sigma_i \sigma_{i+1}) \,. \tag{4.3}$$

If one develops the product in (4.3) one gets a sum of terms of the form

$$(\operatorname{tgh} \beta J)^k \sigma_{i_1}\sigma_{i_1+1}\sigma_{i_2}\sigma_{i_2+1} \ldots \sigma_{i_k}\sigma_{i_k+1} \,. \tag{4.4}$$

It is clear that, unless $k = 0$ or $k = L$, each of the terms (4.4) contains at least one index i_j which appears only once. Therefore, after performing the sum over the σ's, all the terms (4.4) give a vanishing contribution to $Z_\Lambda(\beta)$ except the two with $k = 0$, $k = 1$ which are, respectively, 1 and $(\operatorname{tgh} \beta J)^L \cdot \sigma_1\sigma_2\sigma_2\sigma_3 \ldots \sigma_{L-1}\sigma_L\sigma_L\sigma_1 \equiv (\operatorname{tgh} \beta J)^L$.

This implies

$$Z_\Lambda(\beta) = (\cosh \beta J)^L 2^L (1 + (\operatorname{tgh} \beta J)^L) \,. \tag{4.5}$$

Hence [17],

$$\beta f_\Lambda(\beta) = \log (2 \cosh \beta J) + \frac{1}{L} \log (1 + (\operatorname{tgh} \beta J)^L) \,. \tag{4.6}$$

It has to be remarked that $\beta f_\Lambda(\beta)$ as well as $\beta f(\beta) = \lim_{L\to\infty} \beta f_\Lambda(\beta) = \log 2 \cosh \beta J$ is analytic in β; this fact is usually referred to as the absence of phase transitions in the one-dimensional Ising model.

The reader can check, using the above scheme, that the partition function in the grand canonical ensemble and zero field but open boundary conditions (see p. 137) is slightly different from (4.5) and, precisely, is equal to $(\cosh \beta J)^L 2^L$.

Consider now the two-dimensional Ising model in a zero field and with open boundary conditions:

$$H_\Lambda(\underline{\sigma}) = -J \sum_{i=1}^{L} \sum_{j=1}^{L-1} \sigma_{ij}\sigma_{ij+1} - J \sum_{i=1}^{L-1} \sum_{j=1}^{L} \sigma_{ij}\sigma_{i+1j} \,, \tag{4.7}$$

A better form for $H_\Lambda(\underline{\sigma})$ is the following:

$$H_\Lambda(\underline{\sigma}) = -J \sum_b \tilde{\sigma}_b \,, \tag{4.8}$$

where $\sum_b$ denotes sum over the bonds, *i.e.* over the segments $b = [(i, j), (i, j+1)]$ or $b = [(i, j), (i+1, j)]$, and $\tilde{\sigma}_b$ is the product of the two spins at the extremes of b (*e.g.*, if $b = [(i, j), (i+1, j)]$ then $\tilde{\sigma}_b = \sigma_{ij}\sigma_{i+1j}$).

The partition function can be written, as in the one-dimensional case, as

$$Z_\Lambda(\beta) = (\cosh \beta J)^{2L(L-1)} \sum_{\underline{\sigma}} \prod_{b} (1 + (\operatorname{tgh} \beta J)\,\tilde{\sigma}_b) \,. \tag{4.9}$$

Developing the product we are led to a sum of terms of the type

$$(\operatorname{tgh} \beta J)^k \tilde{\sigma}_{b_1} \tilde{\sigma}_{b_2} \dots \tilde{\sigma}_{b_k} \,, \tag{4.10}$$

and we can conveniently describe this term through the geometric set of lines $b_1, b_2, \dots, b_k$. After the $\sum_{\sigma}$ is taken, many terms of the form (4.10) give a vanishing contribution. The ones that give a nonvanishing contribution are the ones in which the vertices of the geometric figure $b_1 \cup b_2 \cup \dots \cup b_k$ belong to an even number of b_j's (two or four). These terms are the ones such that $\tilde{\sigma}_{b_1} \cdot \tilde{\sigma}_{b_2} \dots \tilde{\sigma}_{b_k} \equiv 1$. In Fig. 1 we give a typical nonvanishing term and in Fig. 2 an example of a vanishing term ($k = 30$).

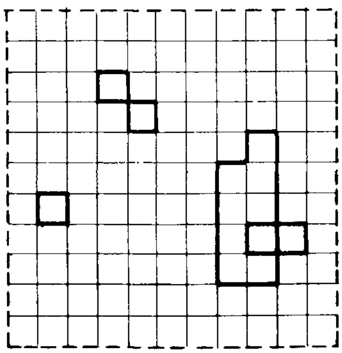

Fig. 1.

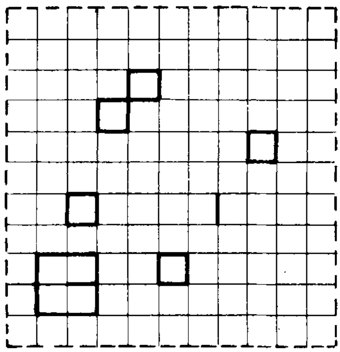

Fig. 2.

Fig. 1. – The dashed line is the boundary of Λ.

Fig. 2. – The dashed line is the boundary of Λ.

We shall, in the following, call a geometric figure built with k segments $b_1, \dots, b_k$ such that $\tilde{\sigma}_{b_1} \cdot \tilde{\sigma}_{b_2} \dots \tilde{\sigma}_{b_k} \equiv 1$ a *k-sided multipolygon* on the box Λ (needless to say that all the $b_1, \dots, b_k$ are different from each other). Let $P_k(\Lambda)$ be the number of these polygons.

The partition function is now easily written as [18]

$$Z_\Lambda(\beta) = (\cosh \beta J)^{2L(L-1)} 2^{L^2} \sum_{k \geqslant 0} P_k(\Lambda) (\operatorname{tgh} \beta J)^k \,. \tag{4.11}$$

This formula is not so simple as in the one-dimensional case. However it is surprisingly useful, as we shall see later.

5. – Phase transitions. Definitions.

We have already seen, in the preceding Section, that the one-dimensional Ising model has no phase transitions in zero field since both $f_\Lambda(\beta)$ and $f(\beta)$ are analytic in β.

We wish to discuss in more detail what is meant by a « phase transition ». It should be said at the outset that there is no universally agreed upon definition of such a concept. Intuitively, from everyday experience, one would say that a phase transition is a phenomenon of macroscopic instability: slight changes of external conditions should imply dramatic changes of some macroscopic variables; it is hard to imagine how in such a situation thermodynamic functions like the free energy, etc., could be analytic functions of the parameters in terms of which they are expressed (say, temperature, chemical potential or magnetic field, etc.).

For the above reason an analytic singularity in the thermodynamic functions is usually thought of as a « symptom » of a phase transition and on this idea it would be possible to base a definition and a theory of the phenomenon of phase transitions.

In this paper, however, we will not base the investigation of the nature of the phase transitions in the Ising model on the search for the singularities of the thermodynamic functions; we shall rather adopt and make more precise the other, perhaps more immediate and intuitive, approach based on the detection of « macroscopic instabilities ».

This way of proceeding is more convenient for the simple reason that a number of very clear and rather deep results have been obtained along these lines. But it should be understood that this second approach does not « brilliantly » avoid the problems of the first. It is simply an approach to the theory of phase transitions which, so far, has asked and provided a less refined description of the phenomena of interest as compared to the description which would be expected from the analysis of the singularities of appropriate analytic functions (an analysis which is still in a very primitive stage and whose problems are not well formulated even in the simplest cases) [19].

Let us now discuss in a more precise way the concept of macroscopic instability.

Consider the Ising model and define that a phase transition takes place at the values (β, h) of the thermodynamic parameters if the system is unstable with respect to boundary perturbations; *i.e.* if there are at least two sequences $\mathscr{B}_\Lambda(\underline{\sigma})$ and $\mathscr{B}'_\Lambda(\underline{\sigma})$ of boundary terms (see (2.1)) such that (say, in the grand canonical ensemble)

$$\lim_{\Lambda\to\infty} \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\Lambda,\mathscr{B}_\Lambda} \neq \lim_{\Lambda\to\infty} \langle \sigma_{x_1} \dots \sigma_{x_n} \rangle_{\Lambda,\mathscr{B}'_\Lambda} \tag{5.1}$$

for a suitable choice of $x_1, x_2, \dots, x_n$.

We first clarify why we say that, if (5.1) holds, we have a macroscopic instability.

We remark that a change in the boundary conditions does not change extensive properties of the system such as the free energy. In fact, from the definition (2.4),

$$\frac{Z(\beta, h, \Lambda, \mathscr{B}_\Lambda)}{Z(\beta, h, \Lambda, \mathscr{B}'_\Lambda)} \leqslant \exp\left[\max_{\underline{\sigma}\in\mathscr{U}(\Lambda)} |\mathscr{B}_\Lambda(\underline{\sigma})| + |\mathscr{B}'_\Lambda(\underline{\sigma})|\right] \tag{5.2}$$

and therefore (2.2) implies

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \log Z(\beta, h, \Lambda, \mathscr{B}_\Lambda) \equiv \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \log Z(\beta, h, \Lambda, \mathscr{B}'_\Lambda)\,. \tag{5.3}$$

On the other hand, if (5.1) is true, intensive quantities like the correlation functions are sensible to the boundary conditions: for instance if

$$\lim_{\Lambda\to\infty} \langle \sigma_{x_1} \rangle_{\Lambda,\mathscr{B}_\Lambda} \neq \lim_{\Lambda\to\infty} \langle \sigma_{x_1} \rangle_{\Lambda,\mathscr{B}'_\Lambda},$$

we realize that the local magnetization changes as a consequence of a change in the boundary condition *even* if the boundary is very remote.

Of course, once provided with a « definition » of what a phase transition is, one has not gone very far. The real question is whether the definition reflects what is physically expected; this implies, in particular, that one should at least be able to prove the existence of a phase transition in the above sense in cases in which one expects a transition. Hopefully the definition and its physical interpretation should allow one to do more; for instance to provide the tools for a closer description of typical phenomena (like the phase separation).

We end here the above, somewhat philosophical and necessary, discussion and, in the next Sections, we start describing in some concrete examples the results that have been obtained in the last decade when the above point of view was starting to be developed, rather independently, by several people.

6. – Geometric description of the spin configurations.

In this Section we introduce a new description of the spin configurations which we shall use to derive in a very elegant way the exact value of the critical temperature of the two-dimensional Ising model. In the next Sections the geometric representation, introduced below, will be widely used for other purposes [20].

Consider an Ising model with boundary conditions of the type (3.5) (($\underline{\varepsilon}$)-boundary conditions) or with periodic boundary conditions (see p. 137).

Given a configuration $\sigma \in \mathscr{U}(\Lambda)$ we draw a unit segment perpendicular to the centre of each bond b having opposite spins at its extremes (in three dimensions we draw a unit square surface element perpendicular to b). A two-dimensional example of this construction is provided by Fig. 3 (($\underline{\varepsilon}$)-boundary condition).

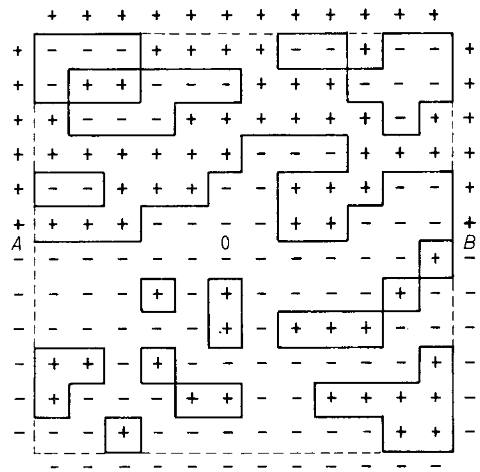

Fig. 3. – The dashed line is the boundary of Λ; the outer spins are the ones fixed by the boundary condition. The points A, B are points where an open line ends.

The set of segments group into lines (or surfaces) which separate regions where the spins are positive from regions where they are negative.

It is clear that some of the lines (or surfaces, if $\delta = 3$) are « closed polygons » (« closed polyhedra »), while others are not closed. It is perhaps worth stressing that our polygons are not really such in a geometrical sense since they are not necessarily self-avoiding (see Fig. 3); however they are such that they can intersect themselves only on vertices (and not on sides). From a geometrical point of view a family of disjoint polygons (in the above sense and in two dimensions) is the same thing as a multipolygon in the sense discussed in Sect. **4** (see Fig. 1).

In two dimensions instead of saying that a polygon is closed we could equivalently say that its vertices belong to either two or four sides.

We notice that the $(+)$-boundary conditions, the $(-)$-boundary conditions and the periodic boundary conditions are such that lines (surfaces) associated to the spin configurations are *all closed* polygons (polyhedra). In the periodic case some polygons might wind around the two holes of the torus.

In the two-dimensional case and if the boundary conditions are the $(+, +)$-cylindrical or the $(+, -)$-cylindrical ones (see p. 138) a geometric construction of the above type can still be performed and, also in this case, the lines are closed polygons (some of which may « wind around » the cylinder Λ).

For a fixed boundary condition let $(\gamma_1, \gamma_2, \ldots, \gamma_k, \lambda_1, \ldots, \lambda_h)$ be the disjoint components of the set of lines (surfaces) associated by the above construction to a spin configuration $\underline{\sigma} \in \mathscr{U}(\Lambda)$. $\gamma_1, \ldots, \gamma_k$ are closed lines and $\lambda_1, \ldots, \lambda_h$ are not closed.

Clearly the correspondence between $(\gamma_1, \ldots, \gamma_k, \lambda_1, \ldots, \lambda_h)$ and $\underline{\sigma}$ is, *for a fixed boundary condition*, one-to-one except for the case of the periodic boundary conditions, when it is one-to-two. Changing the boundary condition implies changing the set of lines (surfaces) which describe the same spin configuration $\underline{\sigma}$.

A very important property of the above geometric description is that, if $|\gamma|$ ($|\lambda|$) denote the length (area) of the lines (surfaces) γ and λ, then the energy of a spin configuration is, in a zero field, given by

$$H_\Lambda(\underline{\sigma}) = -J\cdot(\text{number of bonds in } \Lambda) + 2J\Big[\sum_i |\gamma_i| + \sum_j |\lambda_j|\Big]. \tag{6.1}$$

This remark easily follows from the fact that each bond b contributing $-J$ to the energy has equal spins at its extremes, while the bonds contributing $+J$ have opposite spins at their extremes and, therefore, are cut by a segment of unit length belonging to some γ_i or λ_i.

If $\mathscr{N}_\Lambda =$ number of bonds in Λ, the partition function becomes (in a zero field and with nonperiodic boundary conditions)

$$Z_\Lambda(\beta) = \sum_{\gamma_1\ldots\gamma_k} \sum_{\lambda_1\ldots\lambda_h} \Big(\exp\Big[-2\beta J \sum_i |\gamma_i|\Big] \exp\Big[-2\beta J \sum_j |\lambda_j|\Big]\Big) \exp[\beta J \mathscr{N}_\Lambda], \tag{6.2}$$

where the sums run over the set of lines associated with a spin configuration $\underline{\sigma} \in \mathscr{U}(\Lambda)$ and the boundary condition under consideration.

In the case of periodic boundary conditions there are no λ's and there is an extra factor of two (due to the two-to-one correspondence between $\underline{\sigma}$ and $(\gamma_1, \ldots, \gamma_n)$):

$$Z_\Lambda(\beta) = 2 \sum_{\gamma_1\ldots\gamma_n} \exp\Big[-2\beta J \sum_i |\gamma_i|\Big] \exp[\beta J \mathscr{N}_\Lambda], \tag{6.3}$$

where $\mathscr{N}_\Lambda = 2L^2$.

From the above considerations we draw two important consequences:

I) If the boundary condition is fixed, the probability of a spin configuration $\underline{\sigma}$ described by $\gamma_1, \ldots, \gamma_n, \lambda_1, \ldots, \lambda_h$ is proportional to

$$\exp\Big[-2\beta J\Big(\sum_i |\gamma_i| + \sum_j |\lambda_j|\Big)\Big]. \tag{6.4}$$

II) In the case of $(+)$ or $(-)$ boundary conditions and 2 dimensions we notice that $\sum_{\gamma_1,\ldots,\gamma_n}$ in (6.2) is a sum over «multipolygons» lying on a shifted lattice and in a box Λ' containing $(L-1)^2$ spins (see definition in Sect. **4**,

p. 140) and, therefore, if $\sum_i |\gamma_i| = k$ we have

$$Z_\Lambda(\beta) = \exp[2L(L-1)\beta J] \sum_{k\geqslant 0} P_k(\Lambda') \exp[-2\beta J k] , \tag{6.5}$$

where $P_k(\Lambda')$ is the number of different multipolygons with perimeter k (cf. (4.11)).

If we now define β^* through

$$\operatorname{tgh} \beta J = \exp[-2\beta^* J] , \tag{6.6}$$

then comparison of (6.5) with (4.11) yields

$$\frac{Z_\Lambda(\beta)}{(\cosh \beta J)^{2L(L+1)} 2_{L_2}} = \frac{Z_{\Lambda'}(\beta^*)}{\exp[2\beta^* J L(L-1)]} . \tag{6.7}$$

Here $Z_\Lambda(\beta)$ is computed with open boundary conditions while $Z_{\Lambda'}(\beta^*)$ is computed with $(+)$-boundary conditions.

If we assume that the bulk free energy $f(\beta) = \lim_{\Lambda\to\infty} (1/|\Lambda|) \log Z_\Lambda(\beta)$ has one and only one singularity as a function of β, for β real, then (6.7) can be used to locate this singularity. In fact it implies

$$f(\beta) - \log 2(\cosh \beta J)^2 = -2\beta^* J + f(\beta^*) , \tag{6.8}$$

hence a singularity in β can take place only when $\beta = \beta^*$, *i.e.* for $\beta = \beta_{c,0}$ such that

$$\operatorname{tgh} \beta_{c,0} J = \exp[-2\beta_{c,0} J] , \tag{6.9}$$

which, indeed, has been shown by ONSAGER [3] to be the exact value of the critical temperature defined as the value of β where $f(\beta)$ is singular [20].

In the next Section we outline the theory of the phase transitions in the Ising model as a macroscopic instability and a spontaneous breakdown of the up-down symmetry. We shall concentrate, for geometric reasons, on the two-dimensional Ising model, but, unless explicitly stated, the results hold in any dimension $\delta \geqslant 2$.

7. – Phase transitions. Existence.

In this Section we shall show that the $(+)$-boundary conditions and the $(-)$-boundary conditions (see Sect. **3**) produce, if the temperature is low enough, different equilibrium states (see Sect. **3**), *i.e.* for large β the correlation functions are different and the difference does not vanish in the limit $\Lambda \to 0$ (cf. (5.1)).

More precisely we shall prove that, if $h=0$ and β is large enough,

$$\lim_{\Lambda\to\infty} \langle\sigma_x\rangle_{\Lambda,\pm} = \pm m^*(\beta) \neq 0 , \tag{7.1}$$

where the index $\pm$ refers to the boundary conditions.

Clearly (7.1) shows that the magnetization is unstable (in zero field and at low temperature) with respect to boundary perturbations. We also remark that, using periodic boundary conditions, one would obtain still another result:

$$\lim_{\Lambda\to\infty} \langle\sigma_x\rangle_{\Lambda,\,\text{periodic}} \equiv 0 , \qquad \text{if } h=0 , \tag{7.2}$$

since $\langle\sigma_x\rangle_{\Lambda,\,\text{periodic}} \equiv 0$, if $h=0$, for symmetry reasons.

After a description of the very simple and instructive proof of (7.1) we shall go further and discuss more deeply the character of the phase transition.

As already remarked, the spin configurations $\underline{\sigma}\in\mathscr{U}(\Lambda)$ are described in terms of closed polygons $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ if the boundary condition is $(+)$ or $(-)$ and the probability of a configuration $\underline{\sigma}$ described by $\gamma_1, \ldots, \gamma_n$ is proportional to (see (6.4))

$$\exp\Big[-2\beta J \sum_i |\gamma_i|\Big] . \tag{7.3}$$

Below we identify $\underline{\sigma}$ with $(\gamma_1, \ldots, \gamma_n)$ (with fixed boundary conditions).

Let us estimate $\langle\sigma_x\rangle_{\Lambda,+}$. Clearly $\langle\sigma_x\rangle_{\Lambda,+} = 1-2P_{\Lambda,+}(-)$, where $P_{\Lambda,+}(-)$ is the probability that in the site x the spin is -1.

Notice that if the site x is occupied by a negative spin the x is *inside* some contour γ associated to the spin configuration $\underline{\sigma}$ under consideration. Hence if $\varrho(\gamma)$ is the probability that a given contour belongs to the set of contours describing some configuration $\underline{\sigma}$ we deduce

$$P_{\Lambda,+}(-) \leqslant \sum_{\gamma\circ x} \varrho(\gamma) , \tag{7.4}$$

where $\gamma\circ x$ means that γ surrounds x.

Let us now estimate $\varrho(\gamma)$: if $\Gamma = (\gamma_1, \ldots, \gamma_n)$ is a spin configuration and if the symbol $\Gamma\,\text{comp}\,\gamma$ means that the contour γ is disjoint from $\gamma_1, \ldots, \gamma_n$ (*i.e.* $\{\gamma\cup\Gamma\}$ is a new spin configuration), then

$$\varrho(\gamma) = \frac{\sum\limits_{\Gamma\ni\gamma} \exp\Big[-2\beta J \sum\limits_{\gamma'\in\Gamma} |\gamma'|\Big]}{\sum\limits_{\Gamma} \exp\Big[-2\beta J \sum\limits_{\gamma'\in\Gamma} |\gamma'|\Big]} \equiv \exp[-2\beta J|\gamma|]\, \frac{\sum\limits_{\Gamma\,\text{comp}\,\gamma} \exp\Big[-2\beta J \sum\limits_{\gamma'\in\Gamma} |\gamma'|\Big]}{\sum\limits_{\Gamma} \exp\Big[-2\beta J \sum\limits_{\gamma'\in\Gamma} |\gamma'|\Big]} . \tag{7.5}$$

Before continuing the proof let us remark that if $\underline{\sigma} = (\gamma, \gamma_1, \gamma_2, \ldots, \gamma_n)$ then $\underline{\sigma}' = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is obtained from $\underline{\sigma}$ by reversing the sign of the spins inside γ; this can be used for an intuitive picture of the second equation in (7.5).

Clearly the last ratio in (7.5) does not exceed 1; hence

$$\varrho(\gamma) \leqslant \exp\left[-2\beta J|\gamma|\right]. \tag{7.6}$$

Calling $p = |\gamma|$ and observing that there are at most 3^p different shapes of γ with perimeter p and at most p^2 congruent γ's containing (in their interior) x, we deduce from (7.4), (7.6)

$$P_{\Lambda,+}(-) \leqslant \sum_{p=4}^{\infty} p^2\, 3^p \exp\left[-2\beta J p\right]. \tag{7.7}$$

Hence if $\beta \to \infty$ (*i.e.* the temperature $T \to 0$) this probability can be made as small as we like and, therefore, $\langle\sigma_x\rangle_{\Lambda,+}$ is as close to 1 as we like provided β is large enough. It is of fundamental importance that the closeness of $\langle\sigma_x\rangle_{\Lambda,+}$ to one is both x and Λ independent.

A similar argument for the (—)-boundary condition, or the remark that $\langle\sigma_x\rangle_{\Lambda,-} = -\langle\sigma_x\rangle_{\Lambda,+}$, allows us to conclude that, at large β, $\langle\sigma_x\rangle_{\Lambda,-} \neq \langle\sigma_x\rangle_{\Lambda,+}$ and the difference between these two quantities is uniform in Λ.

Hence we have completed the proof of the fact that there is a strong instability with respect to the boundary conditions of some correlation functions [21].

We can look upon the above phenomenon as a spontaneous break-down of the up-down symmetry: the Hamiltonian of the model is symmetric, in a zero field, with respect to spin reversal if one neglects the boundary terms; the phase transition manifests itself in the fact that there are equilibrium states in which the symmetry is violated only on the boundary and which are not symmetric even in the limit when the boundary recedes to infinity.

8. – Microscopic description of the pure phases.

The description of the phase transition presented in Sect. **7** can be made much more precise from the physical point of view as well as from the mathematical point of view. A deep and physically clear description of the phenomenon is provided by the theorem below, which, also, makes precise some ideas familiar from the droplet model [22].

Assume that the boundary condition is the (+)-boundary condition and describe a spin configuration $\underline{\sigma} \in \mathscr{U}(\Lambda)$ by means of the associated closed disjoint polygons $(\gamma_1, \ldots, \gamma_n)$.

We regard the ensemble $\mathscr{U}(\Lambda)$ as equipped with the probability distribution attributing to $\underline{\sigma} = (\gamma_1, \ldots, \gamma_n)$ a probability proportional to (7.3).

Then the following theorem holds [23].

Theorem. If β is large enough there exist positive numbers

$$\varrho(\gamma) \leqslant \exp\left[-2\beta J|\gamma|\right]$$

such that a spin configuration $\underline{\sigma}$ randomly chosen out of the ensemble $\mathscr{U}(\Lambda)$ will contain, with a probability approaching 1 as $\Lambda \to \infty$, a number $K_{(\gamma)}(\underline{\sigma})$ of contours congruent to γ such that

$$|K_{(\gamma)}(\underline{\sigma}) - \varrho(\gamma)|\Lambda|| \leqslant C\sqrt{|\Lambda|}\exp\left[-\beta J|\gamma|\right], \qquad C>0, \tag{8.1}$$

and this relation is to be interpreted as holding simultaneously for all γ's. (In three dimensions one has $|\Lambda|^{\frac{2}{3}}$ instead of $\sqrt{|\Lambda|}$.)

It is clear that the above Theorem means that there are very few contours (and that the larger they are the smaller is, in absolute and relative value, their number). The inequality (8.1) also implies that for some $C(\beta)$ there are *no* contours with perimeter $|\gamma| > C(\beta)\log|\Lambda|$. Hence a typical spin configuration in the grand canonical ensemble with $(+)$-boundary conditions is such that the large majority of the spins is «positive» and, in this «sea» of positive spins, there are a few negative spins distributed in small and rare regions (in a number, however, still of the order of $|\Lambda|$).

Another nice result which follows from the results of Sect. **7** and from some improvement [24] of them concerns the behaviour of the equation of state near the phase transition region at low (enough) temperatures.

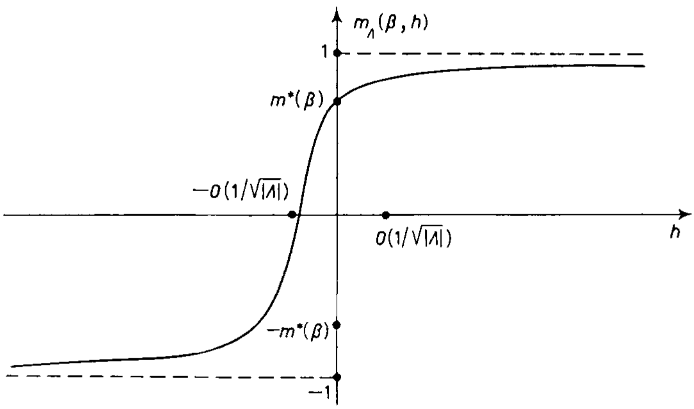

Fig. 4.

If Λ is finite the graph of $m_\Lambda(\beta, h)$ as a function of h will have a rather different behaviour depending on the possible boundary conditions; *e.g.*, if the boundary condition is $(+)$ or $(-)$, one gets respectively the results depicted in Fig. 4 and 5.

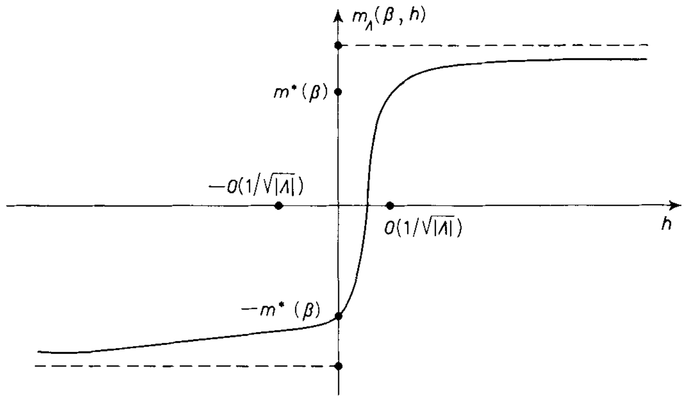

Fig. 5.

With the periodic boundary conditions the state diagram changes as in Fig. 6.

The thermodynamic limit $m(\beta, h) = \lim_{\Lambda\to\infty} m_\Lambda(\beta, h)$ exists for all $h \neq 0$ and the resulting graph is as shown in Fig. 7.

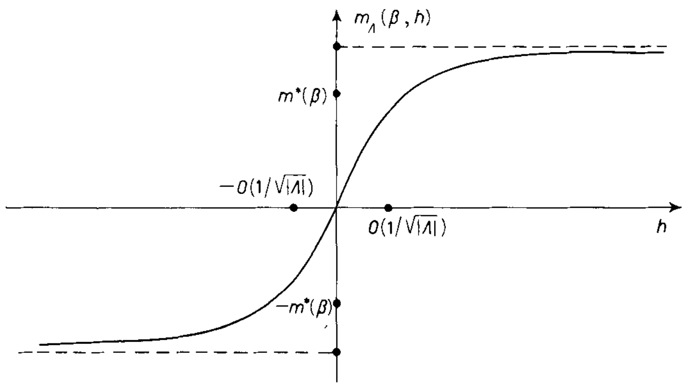

Fig. 6.

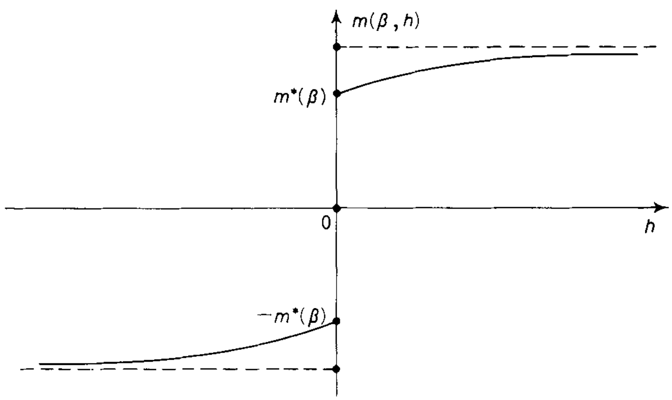

Fig. 7.

At $h=0$ the limit is not well defined and depends on the boundary condition (as it must). It can be proven that $\lim_{h\to 0^{\pm}} (\partial m(\beta, h)/\partial h) = \chi(\beta)$ is a finite number (*i.e.* the angle between the vertical part of the graph and the rest is sharp [24]).

The above considerations and results also furnish a clear idea of what a phase transition for a finite system means.

It is often stated that a finite system « does not » show « sharp » phase transitions; however this statement is always made when considering one fixed boundary condition, usually of periodic or perfect-wall type. By taking into account the importance of the boundary terms we see what are the phenomena that occur in a finite system if the corresponding infinite system has a sharp phase transition.

The next Section is devoted to the discussion of a number of problems concerning the generality of the definition of a phase transition as an instability with respect to the boundary perturbations and other related problems. Notice that an unpleasant limitation on the results discussed in this Section is the condition of low temperature (« β large enough »).

9. – Results on phase transitions in a wider range of temperature.

The results of the preceding Sections show that, at a low enough temperature, the Ising model is unstable with respect to changes in the boundary conditions. A natural question is whether one can go beyond the low-temperature region and fully describe the phenomena in the region where the instability takes place. In the particular case of 2 dimensions it would also be natural to ask whether the maximum value of β to which an instability is associated is the one given by eq. (6.9) which corresponds to the value of β where the infinite-volume free energy $f(\beta)$ has a singularity.

The above types of questions are very difficult and are essentially related to the, already mentioned, theory of the phase transitions based on the search and study of analytic singularities of the thermodynamic functions (which is a theory, however, that has still to be really developed).

Nevertheless a number of interesting partial results are known which considerably improve the picture of the phenomenon of the phase transitions as we can see from the preceding Sections. A list of these results follows:

1) It can be shown that the zeros of the polynomial in $z=\exp[\beta h]$ given by the product of $z^{|A|}$ times the partition function (2.4) with periodic or perfect-wall boundary conditions lie on the unit circle: $|z|=1$. It is easy to deduce, with the aid of Vitali's convergence theorem for equibounded analytic functions, that this implies that the only singularities of $f(\beta, h)$ in the region $0<\beta<\infty$, $-\infty<h<+\infty$ can be found at $h=0$.

A singularity appears if and only if the point $z=1$ s an accumulation point of the limiting (as $\Lambda\to\infty$) distribution of the zeros on the unit circle.

In fact if the zeros in question are $z_1, \ldots, z_{2|\Lambda|}$

$$\frac{1}{|\Lambda|}\log z^{|\Lambda|} Z(\beta, h, \Lambda, \text{periodic}) = 2\beta J + \frac{1}{|\Lambda|}\sum_{i=1}^{2|\Lambda|}\log(z - z_i) \tag{9.1}$$

and if $|\Lambda|^{-1}\cdot$(number of zeros of the form $z_j = \exp[i\theta_j]$ with $\theta \leqslant \theta_j \leqslant \theta + \mathrm{d}\theta$) $\xrightarrow[\Lambda\to\infty]{}$ $\xrightarrow[\Lambda\to\infty]{} \varrho_\beta(\theta)(\mathrm{d}\theta/2\pi)$ in a suitable sense, we get, from (9.1),

$$\beta f(\beta, h) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log(z - \exp[i\theta])\,\varrho_\beta(\theta)\,\mathrm{d}\theta + 2\beta J - \beta h\,, \tag{9.2}$$

where the last term comes from the $|z|^{|\Lambda|}$ inserted in (9.1).

The existence of the measure $\varrho_\beta(\theta)(\mathrm{d}\theta/2\pi)$ such that (9.2) is true follows, after some thought, from the existence of the thermodynamic limit $\lim\limits_{\Lambda\to\infty} f_\Lambda(\beta, h) = f(\beta, h)$ [25].

2) It can be shown that the zeros of the partition function do not move too much under small perturbations of the spin-spin potential even if one allows « many spin » interactions, *i.e.* even if one perturbs the Hamiltonian (2.1) with perfect-wall boundary conditions into

$$\begin{cases} H'_\Lambda(\underline{\sigma}) = H_\Lambda(\underline{\sigma}) + (\delta H_\Lambda)(\underline{\sigma})\,, \\ (\delta H_\Lambda)(\underline{\sigma}) = \sum\limits_{k\geqslant 1}\ \sum\limits_{x_1,\ldots,x_k\in\Lambda}\frac{1}{k!}J'(x_1, \ldots, x_k)\,\sigma_{x_1}\ldots\sigma_{x_k}\,, \end{cases} \tag{9.3}$$

where $J'(X)$ is a function of the set $X = (x_1, \ldots, x_k)$ such that

$$\|J'\| = \sup_{y\in Z^\delta}\sum_{X\ni y}|J'(X)| \tag{9.4}$$

is small enough.

More precisely, if one knows that, when $J'=0$, the zeros in $z=\exp[\beta h]$ of the partition function lie in a certain closed set N of the z-plane then, if $J'\neq 0$, they lie in a set N^1 contained in a neighbourhood of N which can be made as small as we please when $\|J'\|\to 0$.

This result allows us to make a connection between the analyticity properties and the boundary condition instability as described below in point 3) [26].

3) There can be a boundary condition instability only in zero field and, in this case, if and only if the spectrum $\varrho_\beta(\theta)$ has no gap around $\theta=0$.

The proof of this result relies upon 2) and the remark that the correlation functions are the functional derivatives with respect to $J'(x_1, \dots, x_k)$ of the free energy defined by the Hamiltonian (9.3) [26].

4) Another question is whether the boundary condition instability is always revealed by the one-spin correlation function (as in Sect. 7) or whether it might be shown only by some correlation functions of higher order. This question is answered by the following result.

There can be a boundary condition instability (at $h = 0$ and β fixed) if and only if

$$\lim_{h\to 0^+} m(\beta, h) \neq \lim_{h\to 0^-} m(\beta, h) . \tag{9.5}$$

Notice that, in view of what was said above (point 3)), $m(\beta, h) = \lim_{\Lambda\to\infty} m_\Lambda(\beta, h)$ is boundary condition independent as long as $h \neq 0$.

In other words there is a boundary condition instability if and only if there is spontaneous magnetization. This rules out the possibility that the phase transition could manifest itself through an instability of some high-order correlation function which, practically, might be unobservable from an experimental point of view [27].

5) Point 4) implies that a natural definition of the critical temperature T_c is the least upper bound of the T's such that (9.5) is true ($T = \beta^{-1}$). It is clear that, at this temperature, the gap around $\theta = 0$ closes and the function $f(\beta, h)$ has a singularity at $h = 0$ for $\beta > \beta_c = T_c^{-1}$; it can in fact be proven that if (9.5) is true for a given β_0 then it is true for all $\beta > \beta_0$ [28].

6) The location of the singularities of $f(\beta, 0)$ as a function of β remains on open question, see however FISHER [28]. In particular the question of whether there is a singularity of $f(\beta, 0)$ at β_c is open. This implies that, at least in principle, it is still unproven that the singularity of the Onsager solution of the two-dimensional Ising model takes place at the critical point as defined in 5). It is, however, clear from the above considerations and from the fact, proven by YANG (cited in ref. [6]), that for $\beta > \beta_{c,0}$ (9.5) certainly holds, that $\beta_{c,0} \geqslant \beta_c$ (see (6.9)).

7) Finally another interesting question can be raised. For $\beta < \beta_c$ we have instability with respect to the boundary conditions (see 6) above): How strong is this instability? In other words, how many « pure » phases can exist?

Our intuition, in the case of the Ising model suggests that there should be only two different phases: the positively and the negatively magnetized ones.

To answer the above question in a rigorous way it is necessary to agree on what a pure phase is [29]. We shall call an equilibrium state a « pure phase » if it is translationally invariant and if its correlation functions have a cluster property of the form

$$\langle \sigma_{x_1} \cdots \sigma_{x_n} \sigma_{y_1+a} \cdots \sigma_{y_m+a} \rangle \xrightarrow[a\to\infty]{} \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle \langle \sigma_{y_1} \cdots \sigma_{y_m} \rangle , \tag{9.6}$$

where the convergence is understood in a very weak sense, *i.e.* the weakest sense which still permits one to deduce that the fluctuations of extensive quantities are $o(|\Lambda|)$ [28], *i.e.*

$$(9.7)\qquad \frac{1}{|\Lambda|}\sum_{a\in\Lambda}\langle\sigma_{x_1}\dots\sigma_{x_n}\sigma_{y_1+a}\dots\sigma_{y_n+a}\rangle \xrightarrow[\Lambda\to\infty]{} \langle\sigma_{x_1}\dots\sigma_{x_n}\rangle\langle\sigma_{y_1}\dots\sigma_{y_m}\rangle\,,$$

i.e. the convergence in (9.6) takes place in the Cesaro-limit sense.

It can be proved that, in the case of the Ising model, the two states obtained as limits for $\Lambda\to\infty$ of finite-volume states (cf. Sect. **3**) corresponding to (+)- or (—)-boundary conditions are different for $\beta>\beta_c$ and are pure phases in the sense of (9.7) above [30].

Actually it can be proved that, in this case, the limits (9.6) exist in the ordinary sense [30] rather than in the Cesaro sense, and that, at low temperature, they are approached exponentially fast [31].

Furthermore, if β is large enough (in 2 dimensions 10% larger than β_c), these two pure phases exhaust the set of pure phases. For β close to β_c, however, the question is still open [32].

Having discussed the rigorous results about the structure of the phase transition and the nature of the pure phases, we shall turn, in the next Section, to the phenomenon of coexistence of two pure phases.

10. – Separation and coexistence of pure phases. Phenomenological considerations.

Our intuition about the phenomena connected with the classical phase transitions is usually based on the properties of the liquid-gas phase transition; this transition is experimentally investigated in situations in which the total number of particles is fixed (canonical ensemble) and in the presence of an external field (gravity).

The importance of these experimental conditions is obvious; the external field produces a nontranslationally invariant situation and the separation of the two phases. The fact that the total number of particles is fixed determines, on the other hand, the fraction of volume occupied by the two phases. For a discussion of the phenomenon of phase separation in the absence of an external field see the brief discussion in Sect. **14**.

In the frame of the Ising model it will be convenient to discuss the phenomenon of the phase coexistence in the analogue of the canonical ensemble $\mathcal{U}(\Lambda, m)$ introduced and discussed in Sect. **2** where the total magnetization $M=m|\Lambda|$ is held fixed.

To put ourselves in the phase transition region we shall take β large enough and

$$(10.1)\qquad m=\alpha m^*(\beta)+(1-\alpha)(-m^*(\beta))=(1-2\alpha)m^*(\beta)\,,$$

i.e. we put ourselves in the vertical « plateau » of the diagram $(m, h)_\beta$ (see Fig. 7).

Having fixed m as in (10.1) does not yet determine the phenomenon of the separation of the phases in two different regions; to obtain this effect it will be necessary to introduce some external cause favouring the occupation of a part of the volume by a single phase. Such an asymmetry can be obtained in at least two ways: through a weak uniform external field (in complete analogy with the gravitational field of the liquid-vapour transition) or through an asymmetric field acting only on the boundary spins. This second way should have the same qualitative effect as the first, since, in a phase transition region, a boundary perturbation produces volume effects (this last phenomenon, which has been investigated in the previous Sections, is often referred to as the « long-range order » of the correlations).

From the mathematical point of view it is simpler to use a boundary asymmetry to produce the phase separation since it corresponds to a break-down of the up-down symmetry due only to the boundary spins (whose number is relatively small).

To obtain a further, but not really essential, simplification of the problem consider the two-dimensional Ising model with $(+, -)$-cylindrical or $(+, +)$-cylindrical boundary conditions.

The spins adjacent to the bases of Λ act as symmetry-breaking external fields.

The $(+, +)$-cylindrical boundary condition should, clearly, favour the formation inside Λ of the positively magnetized phase; therefore it will be natural to consider, in the canonical ensemble, this boundary condition only in the case that the total magnetization is fixed to be $+m^*(\beta)$ (see Fig. 7).

On the other hand the boundary condition $(+, -)$ favours the separation of phases (positively magnetized phase near the top of Λ and negatively magnetized phase near the bottom).

Therefore it will be natural to consider this boundary condition in the case of a canonical ensemble with magnetization $m = (1-2\alpha)m^*(\beta)$ (cf. (10.1)).

In this last case one expects, as already mentioned, the positive phase to adhere to the top of Λ, to extend, in some sense to be discovered, up to a distance αL from it, and then to change into the negatively magnetized pure phase.

To make precise the above phenomenological description we shall describe the spin configurations $\underline{\sigma} \in \mathscr{U}(\Lambda, m)$ through the associated sets of disjoint polygons (cf. Sect. **6**).

Fix the boundary conditions to be the $(+, +)$- or $(+, -)$-cylindrical boundary condition and observe that the polygons associated to a spin configuration $\underline{\sigma} \in \mathscr{U}(\Lambda, m)$ are *all closed* and of two types: the ones of the first type, denoted by $\gamma_1, \ldots, \gamma_n$, are polygons which do not encircle Λ, the second type of polygons, denoted by the symbol λ_α, are the ones which wind up around Λ.

So a spin configuration $\underline{\sigma}$ will be described by a set of polygons $(\gamma_1, \ldots, \gamma_n, \lambda_1, \ldots, \lambda_h)$. It is, perhaps, useful to remark once more that the same configuration $\underline{\sigma}$ will be described by different sets of polygons according to which boundary condition is used. However, for a fixed boundary condition, the correspondence between spin configuration and sets of disjoint closed contours is one-to-one and the statistical weight of a configuration $\underline{\sigma} = (\gamma_1, \ldots, \gamma_n, \lambda_1, \ldots, \lambda_h)$ is (cf. (6.4))

$$\exp\left[-2\beta J\left(\sum_i |\gamma_i| + \sum_j |\lambda_j|\right)\right].$$

It should also be remarked that the above notation is not coherent with the notation of Sect. **6**, where the symbol λ is used for open polygons (absent here); we hope that this will not cause any confusion. The reason we call λ the contours that go around the cylinder Λ is that they look like open contours if one forgets that the opposite vertical sides of Λ have to be identified.

It is very important to remark that if we consider the $(+,-)$-boundary conditions then the number of polygons of λ-type must be *odd*, while, if we consider the $(+,+)$-boundary condition, then the number of λ-type polygons must be *even*.

11. – Separation and coexistence of phases. Results.

Bearing in mind the geometric description of the spin configuration in the canonical ensembles considered with the $(+,+)$-cylindrical or the $(+,-)$-cylindrical boundary conditions (which we shall denote briefly as $\mathscr{U}^{++}(\Lambda, m)$, $\mathscr{U}^{+-}(\Lambda, m)$) we can formulate the following Theorem [33]:

Theorem. For $0 < \alpha < 1$ fixed, then, if β is large enough, a spin configuration $\underline{\sigma} = (\gamma_1, \ldots, \gamma_n, \lambda_1 \ldots, \lambda_{2h+1})$ randomly chosen out of $\mathscr{U}^{+-}(\Lambda, m)$ (where $m = (1-2\alpha)m^*(\beta)$) enjoys the properties 1)-4) below with a probability (in $\mathscr{U}^{+-}(\Lambda, m)$) approaching 1 as $\Lambda \to \infty$:

1) $\underline{\sigma}$ contains *only one* contour of λ-type and

$$\big||\lambda| - (1 + \varepsilon(\beta))L\big| \leqslant o(L)\,, \tag{11.1}$$

where $\varepsilon(\beta) > 0$ is a suitable (α-independent) function of β tending to zero exponentially fast as $\beta \to \infty$.

2) If Λ_λ, Λ'_λ denote the regions above and below λ we have

$$\big||\Lambda'_\lambda| - \alpha|\Lambda|\big| < \varkappa(\beta)|\Lambda|^{\frac{3}{4}}\,, \tag{11.2}$$

$$\big||\Lambda_\lambda| - (1-\alpha)|\Lambda|\big| < \varkappa(\beta)|\Lambda|^{\frac{3}{4}}\,, \tag{11.3}$$

where $\varkappa(\beta) \to 0$ exponentially fast as $\beta \to \infty$.

3) If $M_\lambda = \sum_{x \in \Lambda_\lambda} \sigma_x$, we have

$$|M_\lambda - \alpha m^*(\beta)|\Lambda|| < \varkappa(\beta)|\Lambda|^{\frac{3}{4}} \tag{11.4}$$

and a similar inequality holds for $M'_\lambda = \sum_{x \in \Lambda'_\lambda} \sigma_x = m|\Lambda| - M_\lambda$.

4) If $K^\lambda_{(\gamma)}(\underline{\sigma})$ denotes the number of contours congruent to a given γ and lying in Λ_λ, then, simultaneously for all the shapes of γ,

$$|K^\lambda_{(\gamma)}(\underline{\sigma}) - \varrho(\gamma)\alpha|\Lambda|| \leqslant C \exp[-\beta J|\gamma|] \sqrt{|\Lambda|}\,, \qquad C > 0\,, \tag{11.5}$$

where $\varrho(\gamma) \leqslant \exp[-2\beta J|\gamma|]$ is the same as the one in the text of the theorem of Sect. **8**. A similar result holds for the contours below λ (cf. the comments on (8.1)).

It is clear that the above theorem not only provides a detailed and rather satisfactory description of the phenomenon of phase separation, but also furnishes a precise microscopic definition of the line of separation between the two phases which should be identified with λ.

A very similar result holds in the ensemble $\mathscr{U}^{++}(\Lambda, m^*(\beta))$; in this case 1) is replaced by

1′) *no* λ-type polygon is present,

2) and 3) become superflous and 4) is modified in the obvious way. In other words a typical configuration in the canonical ensemble $\mathscr{U}^{++}(\Lambda, m^*(\beta))$ has the same appearance as a typical configuration of the grand canonical ensemble $\mathscr{U}(\Lambda)$ with $(+)$-boundary condition (which is described by the Theorem of Sect. **8**).

We conclude this Section with a remark about the condition that $0 < \alpha < 1$ has to be fixed beforehand in the above Theorem. Actually the results of the Theorem hold at fixed β for all the α's such that $\varepsilon(\beta) < \min(\alpha, (1-\alpha))$, *i.e.* for all the α's such that the line λ cannot touch the bases of Λ (in which case there would be additional physical phenomena).

12. – Surface tension in two dimensions. Alternative descriptions of the separation phenomena.

A remarkable application of the above theorem is the possibility of giving a microscopic definition of surface tension between the two pure phases [34].

We have seen that the partition functions

$$Z^{++}(\Lambda, \beta) = \sum_{\sigma \in \mathscr{U}^{++}(\Lambda, m^*(\beta))} \exp\left[-2\beta J\left(\sum_i |\gamma_i| + \sum_j |\lambda_j|\right)\right] \tag{12.1}$$

and (if $m = (1-2\alpha)m^*(\beta)$, $0<\alpha<1$)

$$Z^{+-}(\Lambda, \beta) = \sum_{\underline{\sigma}\in\mathscr{U}^{+-}(\Lambda,m)} \exp\left[-2\beta J\left(\sum_i |\gamma_i| + \sum_j |\lambda_j|\right)\right] \tag{12.2}$$

will essentially differ, at low enough temperature, only because of the line λ (present in $\mathscr{U}^{+-}(\Lambda, m)$ and absent in $\mathscr{U}^{++}(\Lambda, m^*(\beta))$, see the preceding Section).

A natural definition (in two dimensions) of surface tension between the phases, based on obvious physical considerations, can therefore be given in terms of the different asymptotic behaviours of $Z^{++}(\Lambda, \beta)$ and $Z^{+-}(\Lambda, m)$:

$$\tau(\beta) = \lim_{\Lambda\to\infty} \frac{1}{L} \log \frac{Z^{+-}(m, \Lambda)}{Z^{++}\big(m^*(\beta), \Lambda\big)}\,. \tag{12.3}$$

The above limit (which should be α-independent for $\varepsilon(\beta) < \min(\alpha, (1-\alpha))$ (cf. the concluding remarks of the preceding Section)) can be exactly computed at low enough temperature and furnishes

$$\tau(\beta) = -2\beta J - \log \operatorname{tgh} \beta J\,, \tag{12.4}$$

which is the value computed by ONSAGER [3] using a different definition not based on the above detailed microscopic description of the separation of the phases and of the line of separation [35].

We conclude this Section with a brief discussion on one particular but very convenient alternative way of investigating the phenomenon of the coexistence of two phases. Another, still different, way of investigating the phenomenon will be discussed in Sect. **14**.

Consider the grand canonical ensemble, but impose as boundary conditions the following: the spins adjacent to the upper half of the boundary of Λ are fixed to be $+1$, while the ones adjacent to the lower half are -1. This is a $\underline{\varepsilon}$-type boundary condition (see Sect. **3**).

It is clear that a configuration $\underline{\sigma} \in \mathscr{U}(\Lambda)$ is described, under the above boundary condition, by one open polygon (surface in 3 dimensions) going from one side of Λ to the opposite side and by a set of disjoint closed polygons (polyhedra) $\gamma_1, \dots, \gamma_n$.

Clearly λ plays now the role of the polygons encircling Λ in the case of cylindrical boundary conditions (and 2 dimensions) and it is also clear that a theorem very similar to the ones already discussed holds in this case. The above point of view is more relevant in the three-dimensional case where a « cylindrical » boundary condition would have a less clear physical meaning.

In the three-dimensional case λ is a « surface » with a boundary formed by the square on $\partial\Lambda$, where the « break » between the spins fixed to be $+1$ and the ones fixed to be -1 is located.

In the next Section we investigate in more detail the structure of the line or surface of separation between the phases.

13. – The structure of the line of separation. What a straight line really is.

The Theorem of Sect. **11** tells us that, if β is large enough, then the line λ is almost straight (since $\varepsilon(\beta)$ is small). It is a natural question to ask whether the line λ is straight in the following more precise sense: suppose that λ, thought of as being a polygon belonging to a $\underline{\sigma}\in\mathscr{U}^{+-}(\Lambda, m)$ (cf. Sect. **11**), passes through a point $q\in\Lambda$; then we shall say that λ is straight or rigid if the probability $\mathscr{P}_\Lambda$ that λ passes also through the site q', opposite to q on the cylinder Λ, *does not* tend to zero as $\Lambda\to\infty$, otherwise we shall say that λ is *not rigid* or fluctuates. Of course the above probabilities are to be computed in the ensemble $\mathscr{U}^{+-}(\Lambda, m)$.

It is rather clear what the above notion of rigidity means: the « excess » length $\varepsilon(\beta)L$ can be obtained in two ways: either the line λ is essentially straight (in the geometric sense) with a few « bumps » distributed with a density of order $\varepsilon(\beta)$ or, otherwise, the line λ is bent and, therefore, it is only locally straight and part of the excess length is gained through the bending.

In three dimensions a similar phenomenon is possible. As remarked at the end of the last Section, in this case λ becomes a surface with square boundary fixed at a certain height (*i.e.* zero), and we ask whether the centre of the square belongs to λ with a nonvanishing probability in the limit $\Lambda\to\infty$.

The rigidity or not of λ can, in principle, be investigated by optical means; one can have interference of coherent light scattered by regions of λ separated by a macroscopic distance only if λ is rigid in the above sense.

It has been rigorously proven that, at least at low temperature, the line of separation λ is *not* rigid in 2 dimensions $\big($and the fluctuation of the height of the middle point is of the order $O(\sqrt{L})\big)$. On the contrary, in 3 dimensions it has been shown that the surface λ is rigid at low temperature.

An interesting question remains open in the three-dimensional case and is the following: it is conceivable that the surface, although rigid at low temperature, might become loose at a temperature $\tilde{T}_c$ smaller than the critical temperature T_c (defined as the largest temperature below which there are at least two pure phases).

It would be interesting to examine the available experimental data on the structure of the surface of separation to set limits on $T_c-\tilde{T}_c$ in the case of the liquid-gas phase transition where such a phenomenon can conceivably occur even though a theory of it is far from being in sight, at least if one requires a degree of rigour comparable to that displayed in the treatment of the results so far given for the Ising model.

We conclude by remarking that the rigidity of λ is connected with the existence of translationally noninvariant equilibrium states (see Sect. **3**).

It seems almost certain that, in 2 dimensions, because of the discussed nonrigidity of λ there are no translationally noninvariant states [36].

Notice that the existence of translationally noninvariant equilibrium states is not necessary for the description of the coexistence phenomena. The theory of the 2-dimensional Ising model developed in the preceding Sections is a clear proof of such a statement [36].

14. – Phase separation phenomena and boundary conditions. Further results.

The phenomenon of phase separation described in Sect. **10** and **11** is the ferromagnetic analogue to the phase separation between a liquid and its vapour in the presence of the gravitational field.

It is relevant to ask to what extent an external field (or some equivalent boundary condition) is really necessary; for instance one can imagine a situation in which two phases coexist in the absence of any external field.

Let us discuss first some phenomenological aspects of the liquid-gas phase separation in the absence of outer fields. One imagines that, if the density is fixed and corresponds to some value on the « plateau » of the phase diagram, then the space will be filled by vapour and drops of liquid in equilibrium. Observe that the drops will move and, from time to time, collide; since the surface tension is negative the drops will tend to cluster together and, eventually, in an equilibrium situation there will be just one big drop. The location of the drop in the box Λ will depend on how the walls are made and how they interact with the particles within Λ.

Let us consider some extreme cases:

1) the walls « repel » the drops,

2) the walls « attract » the drops,

3) the wall is perfect and does not distinguish between the vapour and the liquid.

In the first case the drops will stay away from the boundary $\partial\Lambda$ of Λ. In the second case the drop will spread on the walls, which will be wet as much as possible. In the third case it will not matter where the drop is; the drop will be located in a position that minimizes the « free » part of its boundary (*i.e.* the part of the boundary of the drop not on $\partial\Lambda$). This means that the the drop will prefer to stay near a corner rather than wetting all the wall.

Let us translate the above picture into Ising-model language. Assume β is large and $m=(1-2\alpha)m^*(\beta)$ (see Fig. 7) (*i.e.* assume that the magnetization is on the vertical plateau of the $(m, h)_\beta$ diagram (see Fig. 7)).

Then conditions 1), 2), 3) can be realized as:

1) The spins adjacent to the boundary are all fixed to be $+1$. This favours the adherence to the boundary of the positively magnetized phase.

2) The spins adjacent to the boundary are all fixed to be -1. This favours the adherence to the boundary of the negatively magnetized phase.

3) Then are no spins adjacent to the boundary, *i.e.* we consider *perfect-wall* boundary conditions.

The rigorous results available in the case of the Ising model confirm the phenomenological analysis based on the liquid-vapour coexistence [23]:

Theorem. Fix $0<\alpha<1$ and consider $(+)$-boundary conditions. Then a spin configuration $\underline{\sigma}$ randomly extracted from the canonical ensemble with magnetization $m=(1-2\alpha)m^*(\beta)$ has, if β is large enough, properties 1)-3) below with a probability tending to 1 as $\Lambda\to\infty$.

1) There is *only one* γ such that $|\gamma|>(1/333)\log|\Lambda|$ and it has the property

$$\big||\gamma|-4\sqrt{(1-\alpha)|\Lambda|}\big|\leqslant\delta(\beta)\sqrt{|\Lambda|} \tag{14.1}$$

with $\delta(\beta)\to 0$ as $\beta\to\infty$ (exponentially fast);

2) the area enclosed by γ is $\theta(\gamma)$:

$$\big|\theta(\gamma)-(1-\alpha)|\Lambda|\big|\leqslant\varkappa(\beta)|\Lambda|^{\frac{3}{4}}\,; \tag{14.2}$$

3) the magnetization $M(\theta(\gamma))$ inside γ is on the average equal to $-m^*(\beta)$ and, more precisely,

$$\big|M(\theta(\gamma))+m^*(\beta)(1-\alpha)|\Lambda|\big|\leqslant\varkappa(\beta)|\Lambda|^{\frac{3}{4}}, \tag{14.3}$$

and therefore the average magnetization outside γ is $+m^*(\beta)$.

This Theorem holds also in 3 dimensions but the exponents of $|\Lambda|$ on the r.h.s. of (14.1)-(14.3) change.

The above Theorem shows that a typical configuration consists of a positively magnetized pure phase *adherent* to the boundary and of a « drop » of negatively magnetized phase *not adhering* to the boundary (since γ is closed). The size of the drop is $\sim\sqrt{(1-\alpha)|\Lambda|}$ (as it should be).

Notice that the drop is almost square in shape (as follows from (14.1), (14.2)); this should not be astonishing since the space is discrete and the isoperimetric problem on a square lattice has the square as a solution (rather than a circle).

The opposite situation is realized if one fixes a $(-)$-boundary condition; a square drop forms in the middle of the box with side $\sim\sqrt{\alpha|\Lambda|}$ and average magnetization $+m^*(\beta)$.

Finally, if the boundary condition is of the perfect-wall type $(\mathscr{B}_\Lambda(\underline{\sigma})\equiv 0)$, then the above Theorem does not hold and one can except to prove (say, in

2 dimensions) that a typical spin configuration has just one open contour λ (with ends on $\partial\Lambda$) which separates the space in two parts which are occupied by the opposite phases; the line λ should be the shortest possible compatible with the condition that the volume Λ is divided by it into two regions of volume $\alpha|\Lambda|$ and $(1-\alpha)|\Lambda|$ (respectively occupied by the positively magnetized phase and by the negatively magnetized phase). The results just described for the case $\mathscr{B}_\Lambda(\underline{\sigma})\equiv 0$ have never been proven though there is evidence of their truth [32].

If one interprets the spins equal to $+1$ as particles and the spins equal to -1 as empty sites, then one has a lattice gas model which undergoes a liquid-vapour phase transition which presents the phenomenological aspects outlined at the beginning of this Section for these transitions.

To conclude we remark that, in the phase separation phenomenon, the finiteness of the box only plays the role of fixing the density and keeping the vapour tension. The detailed structure of the phenomenon depends on the boundary conditions which, in experimental situations, turn out to be something intermediate between the three extreme cases discussed above.

Notice that (14.1) does not provide a satisfactor ymeasure of $|\lambda|$ since the allowed error is still of the order of $\sqrt{|\Lambda|}$; it is an open problem to obtain a better estimate of $|\lambda|$ of the type (11.1). It is also an open problem to find an expression for the surface tension of the square drop (which is expected to be the same as (12.3) in two dimensions); see the Introduction to [34]. A third problem is the investigation of the dependence of the correlation functions on the distance from the surface of the drop.

The analogues of the first two questions just raised have been satisfactorily answered rigorously in the 2-dimensional Ising model with cylindrical boundary conditions (see Sect. **11** and **12**), *i.e.* in the case of an «infinite» drop with a flat surface.

The third problem has been only approximately studied even in the case of a flat drop [37].

15. – Conclusions and open problems.

In the preceding Sections we have dealt with the case of a nearest-neighbour Ising model. It has become customary, in the literature, to call with the name Ising model more general models in which the «bulk» Hamiltonian has the form

$$h\sum_{x_i}\sigma_{x_i}+\sum_{i<j}J_2(x_i,x_j)\sigma_{x_i}\sigma_{x_j}+\sum_{i<j<k}J_3(x_i,x_j,x_k)\sigma_{x_i}\sigma_{x_j}\sigma_{x_k}+\dots,\tag{15.1}$$

where the potentials $J_n(x_1,\dots,x_n)$ are translationally invariant functions of

$(x_1, \ldots, x_n)$ and satisfy certain decrease conditions of the type

$$(15.2)\qquad \begin{cases} \sum_{x} |J_2(0, x)| < +\infty\,, \\ \sum_{x,y} |J_3(0, x, y)| < +\infty\,, \end{cases}$$

etc.

If only pair potentials are present, *i.e.* if the bulk Hamiltonian is of the form

$$(15.3)\qquad h \sum \sigma_{x_i} + \sum_{i<j} J(x_i - x_j)\sigma_{x_i}\sigma_{x_j}$$

and if $J(r) \leqslant 0$, then most of the results described in this paper and appropriately reformulated have either already been proved, or are being proved or are very reasonable conjectures [38].

Many results shall stay true for more general pair potentials and for other models (like continuous gases) at least from the qualitative point of view; in fact it is reasonable that the selected rigorous results should have, at least qualitatively, an analogue in the « general » case of a classical (as opposed to quantum) phase transition.

Results such as analyticity and absence of phase transitions at high temperature are a peculiarity of the lattice models and have been, therefore, left out [39]. We have made some exceptions to the above rule of selection of results by quoting some of the exact results from Onsager's solution of the 2-dimensional Ising model.

Below we list a number of rather randomly chosen interesting and open problems suggested by the topics of this article:

1) It would be interesting to fill the gap between T_c and the maximum value of T ($\sim 10\,\%$ of T_c in 2 dimensions) for which one can prove that there are only two pure phases. This is related to other problems such as the conjectured identity, in 2 dimensions, of β_c and $\beta_{c,0}$ (see (6.9) and point 4) in Sect. **9**).

2) The solution of the 2-dimensional Ising model is based on the so-called « transfer matrix ». The investigation of the transfer matrix has been pursued in some detail in the case of periodic or open boundary conditions in two or three dimensions [40], see also [3, 41].

It would be of interest to study the transfer matrix with nonsymmetric boundary conditions. In particular it would be of interest to study the transfer matrix between two rows (or planes) where the line (or surface) of separation should pass (if straight). A qualitative difference should arise between two and three dimensions (see, for more details, the Appendix).

3) In Fig. 7 we see that the isotherm $m(\beta, h)$ as a function of $h \geqslant 0$ abruptly ends at $h = 0$. It is still an open question whether $h = 0$ is an analytic singularity of $m(\beta, h)$ or whether $m(\beta, h)$ can be analytically continued to $h < 0$. There is strong evidence for the existence of a singularity [42].

4) In the case the answer to 3) is in agreement with the conjecture, how one can explain the metastability phenomena [43]?

5) It would be of interest to find generalizations of the phase coexistence theory to other lattice models for which phase transitions are proven to take place [44].

6) The existence of phase transitions has recently been proved, for the first time, for a continuous system. It would be of interest to analyse the phenomenon of the phase coexistence in this case [45].

7) If, for a system, a phase transition is known to take place, when can one answer the question of how many pure phases exist?

8) A detailed description of the correlation functions near the line or surface of separation between two phases has still to be presented (see [36, 37]).

9) It would be of interest to investigate the microscopic definition of surface tension in the particular case of the 3-dimensional Ising model (which, so far, has not been studied).

10) It would be of interest to prove that, in 3 dimensions, the surface tension $\tau(\beta)$ is such that $\tau(\beta) + 2\beta J$ is analytic in $\exp[-\beta J]$.

11) It would be of interest to investigate the phase transitions in models not showing the up-down symmetry like the ones obtained by choosing in (15.1) $J_3 \neq 0$ [46].

12) Three more open problems are listed at the end of Sect. **14**.

APPENDIX

Transfer matrix in the Ising model.

Consider the one-dimensional Ising model with periodic boundary conditions. The partition function can be written as (if $\sigma_{L+1} \equiv \sigma_1$)

$$\text{(A.1)} \qquad Z(\Lambda, \beta, h) = \sum_{\sigma_1 \dots \sigma_L} \prod_{i=1}^{L} \left(\exp[\beta J \sigma_i \sigma_{i+1} + \beta h \sigma_i]\right) =$$

$$= \sum_{\sigma_1 \dots \sigma_L} \prod_{i=1}^{L} \left(\exp\left[\frac{\beta}{2} h\sigma_i\right] \exp[\beta J \sigma_i \sigma_{i+1}] \exp\left[\frac{\beta}{2} h \sigma_{i+1}\right]\right) =$$

$$= \sum_{\sigma_1 \dots \sigma_L} V_{\sigma_1 \sigma_2} V_{\sigma_2 \sigma_3} \dots V_{\sigma_L \sigma_1} = \operatorname{Tr} V^L ,$$

where V is a two-by-two matrix such that

$$\text{(A.2)}\qquad \left\{\begin{aligned} V_{\sigma\sigma'} &= \exp\left[\frac{\beta}{2}h\sigma + \beta J\sigma\sigma' + \frac{\beta}{2}h\sigma'\right], \qquad \sigma, \sigma' = \pm 1\,,\\ V &= \begin{pmatrix} \exp[\beta h + \beta J] & \exp[-\beta J] \\ \exp[-\beta J] & \exp[-\beta h - \beta J] \end{pmatrix}. \end{aligned}\right.$$

If λ_+ and λ_- $(\lambda_+ > \lambda_-)$ are the two eigenvalues of V, we find

$$\text{(A.3)}\qquad Z(\Lambda, \beta, h) = \lambda_+^L + \lambda_-^L\,,$$

$$\text{(A.4)}\qquad \beta f(\beta, h) = \lim_{L\to\infty} \frac{1}{L} \log Z = \log \lambda_+\,.$$

It is easy to check that $\lambda_+(\beta, h)$ is analytic in β and h for $0 < \beta < +\infty$ and $-\infty < h < +\infty$, *i.e.* there are no phase transitions (as singularities of $f(\beta, h)$).

A similar method can be applied to the two-dimensional Ising model (Λ is now an $M \times N$ box). Suppose, for simplicity, $h = 0$:

$$\text{(A.5)}\qquad \begin{aligned} Z(\beta, \Lambda) &= \sum_{\underline{\sigma}} \prod_{i=1}^{M} \prod_{j=1}^{N} \left(\exp[\beta J\sigma_{i,j}\sigma_{i+1,j} + \beta J\sigma_{i,j}\sigma_{i,j+1}]\right) = \\ &= \sum_{\underline{\sigma}_1} \cdots \sum_{\underline{\sigma}_M} \prod_{i=1}^{M} \left\{\prod_{j=1}^{N} \exp\left[\frac{\beta J}{2}\sigma_{i,j}\sigma_{i,j+1} + \beta J\sigma_{i,j}\sigma_{i+1,j} + \frac{\beta J}{2}\sigma_{i+1,j}\sigma_{i+1,j+1}\right]\right\}, \end{aligned}$$

where in the second line we denote by $\underline{\sigma}_i = (\sigma_{i1}, \ldots, \sigma_{iN})$ all the spins on the i-th row of Λ; the boundary conditions impose $\underline{\sigma}_1 \equiv \underline{\sigma}_{M+1}$ and $\sigma_{i1} \equiv \sigma_{iN+1}$.

Clearly, if we define the $2^N \times 2^N$ matrix

$$\text{(A.6)}\qquad \begin{aligned} V_{\underline{\sigma}\underline{\sigma}'} &= \prod_{i=1}^{N} \left(\exp\left[\frac{\beta J}{2}\sigma_j\sigma_{j+1}\right] \exp[\beta J\sigma_j\sigma'_j] \exp\left[\frac{\beta J}{2}\sigma'_j\sigma'_{j+1}\right]\right) = \\ &= \exp\left[\sum_{i=1}^{N} \left\{\frac{\beta J}{2}(\sigma_j\sigma_{j+1} + \sigma'_j\sigma'_{j+1}) + \beta J\sigma_j\sigma'_j\right\}\right], \end{aligned}$$

where $\sigma_1 \equiv \sigma_{N+1}$, $\sigma'_1 \equiv \sigma'_{N+1}$, we realize that

$$\text{(A.7)}\qquad Z(\Lambda, \beta) = \operatorname{Tr} V^M\,.$$

We have dealt, so far, only with periodic boundary conditions. We could introduce transfer matrices also in the case of other boundary conditions.

For instance, assume, for simplicity, that there are periodic boundary conditions along the columns; we shall consider the three cases below:

1) « perfect wall » boundary conditions along the rows;

2) boundary conditions on the rows corresponding to the existence on the lattice sites adjacent to the end points of the rows, of fixed spins $\varepsilon_i = +1$ (or $\varepsilon_i = -1$) for all i's;

3) boundary conditions which are of the same type as in 2) but *half* of the rows end in a positive spin (say the upper half) and *half* in a negative spin.

We shall now write down a transfer matrix expression for $Z(\Lambda, \beta)$ in the above cases.

In case 1)

$$Z(\Lambda, \beta) = \operatorname{Tr} V^{(1)M}, \tag{A.8}$$

where

$$V^{(1)}_{\underline{\sigma}\underline{\sigma}'} = \exp\left[\sum_{j=1}^{N-1}\left\{\frac{\beta J}{2}(\sigma_j\sigma_{j+1} + \sigma'_j\sigma'_{j+1})\right\} + \sum_{j=1}^{N}\beta J\sigma_j\sigma'_j\right]. \tag{A.9}$$

In case 2)

$$Z(\Lambda, \beta) = \operatorname{Tr} V^{(\pm)M},$$

where

$$V^{(\pm)}_{\underline{\sigma}\underline{\sigma}'} = \exp\left[\pm\beta J(\sigma_1 + \sigma'_1 + \sigma_N + \sigma'_N)\right] V^{(1)}_{\underline{\sigma}\underline{\sigma}'}. \tag{A.10}$$

In case 3), assuming here the height of Λ to be $M+1$ and M even, we have

$$Z(\Lambda, \beta) = \operatorname{Tr}(V^+)^{M/2} V^{(3)}(V^-)^{M/2}, \tag{A.11}$$

where

$$V^{(3)}_{\underline{\sigma}\underline{\sigma}'} = \exp\left[\beta J(\sigma'_1 + \sigma'_N - \sigma_1 - \sigma_N)\right] V^{(1)}_{\underline{\sigma}\underline{\sigma}'}. \tag{A.12}$$

The transfer matrix V in (A.7) is the one that has been diagonalized exactly in the famous paper by ONSAGER [3]. The matrix $V^{(1)}$ has been diagonalized exactly in ref. [47].

The matrices $V^{(\pm)}$ have, so far, never been studied, neither has been studied the matrix $V^{(3)}$.

A similar formulation of the problem of the computation of Z can be formulated in three dimensions.

Some very interesting results on the spectral properties of the generalization to three dimensions of the matrix V (periodic boundary conditions) have been obtained in ref. [48].

In three dimensions one expects that the analogue of $V^{(3)}$ (in contrast to $V^{(1)}$, V^+, V) has spectral properties which radically differ from those of V. In 2 dimensions this phenomenon should not occur and all the above matrices should have the same spectrum (asymptotically as $\Lambda \to \infty$). As mentioned in Sect. **15**, problem 2), this should be related to the fact that $V^{(3)}$ should contain some information about the rigidity of the line or surface of phase separation which is « sitting » right near the two lines between which $V^{(3)}$ « transfers ».

A very interesting nonrigorous analysis of the spin correlation functions in terms of the transfer matrix has been done in ref. [49]. The paper of ref. [48] (written independently of [49]) has been devoted to trying to make this analysis rigorous.

NOTES AND REFERENCES

[1] For a history of the Ising model see S. G. Brush: *Rev. Mod. Phys.*, **39**, 883 (1969).

[2] R. Peierls: *Proc. Cambridge Phil. Soc.*, **32**, 477 (1936).

[3] L. Onsager: *Phys. Rev.*, **65**, 117 (1944).

[4] T. D. Lee and C. N. Yang: *Phys. Rev.*, **87**, 410 (1952).

[5] The mean field theory, as it appears from the literature quoted below, is a theory of phase transitions due to very-long-range and weak forces. The Ising model is, in contrast, a theory based on strong and short-ranged forces. The original approach to the Van der Waals theory (also called the mean field theory) can be found in the book by S. Chapman and T. Cowling: *The Mathematical Theory of Nonuniform Gases* (Cambridge, 1953), p. 284. A more refined and interesting formulation is in N. G. Van Kampen: *Phys. Rev.*, **135**, A 362 (1964). A rigorous and very clear theory is in J. L. Lebowitz and O. Penrose: *Journ. Math. Phys.*, **7**, 98 (1966). The first to understand how to rigorously formulate (and prove in particular cases) the mean field theory have been P. Hemmer, M. Kac and G. E. Uhlenbeck in a series of papers appeared in *Journ. Math. Phys.* and reproduced, with introductory remarks, in E. Lieb and D. C. Mattis: *Mathematical Physics in One Dimension* (New York, 1966). A more phenomenological but very interesting and original theory is in the book of R. H. Brout: *Phase Transitions* (New York, 1965), where the most common phase transitions are treated from the unifying point of view of the mean field theory.

[6] The original solution of the free energy of the Ising model in 2 dimensions can be found in [3]. It was preceded by the exact location of the critical temperature by H. A. Kramers and G. H. Wannier: *Phys. Rev.*, **60**, 252 (1941). The spontaneous magnetization was found by L. Onsager: *Suppl. Nuovo Cimento*, **6**, 261 (1949), but never published; it was subsequently rediscovered by C. N. Yang: *Phys. Rev.*, **85**, 809 (1952). A modern derivation of the solution is to be found in the review article of T. D. Schultz, D. C. Mattis and E. Lieb: *Rev. Mod. Phys.*, **36**, 856 (1964). Another interesting older review article is the paper by G. F. Newell and E. W. Montroll: *Rev. Mod. Phys.*, **25**, 353 (1953). A combinatorial solution has been found by M. Kac and M. Ward and can be found in the book by L. Landau and E. L. Lifschitz: *Physique Statistique* (Moscow, 1967), p. 538. This derivation is not above criticism; an up-to-date paper on the subject is due to F. A. Berezin: *Russian Math. Surveys*, **24**, 1 (1969). Another approach to the solution (the « Pfaffian method ») can be found in P. W. Kasteleyn: *Physica*, **27**, 1209 (1961).

[7] See the review paper by M. E. Fisher: *Rep. Progr. Theor. Phys.*, **30**, 615 (1967), pag. 677-702 and appended references.

[8] It is the case of $MnCl_2 \cdot 4H_2O$ see M. E. Fisher and M. F. Sykes: *Physica*, **28**, 939 (1962); see also [7].

[9] R. L. Dobrushin: *Theory of probability and its applications*, **13**, 197 (1968); *Functional analysis and its applications*, **2**, 292, 302 (1968); **3**, 22 (1968); F. Spitzer: *Am. Math. Monthly*, **78**, 142 (1971).

[10] D. Ruelle: *Statistical Mechanics* (New York, 1969), p. 168, 161.

[11] R. B. Griffiths: *Journ. Math. Phys.*, **8**, 478 (1967); M. E. Fisher: *Phys. Rev.*, **162**, 475 (1967); H. S. Green and C. H. Hurst: *Order Disorder Phenomena* (New York, 1964).

[12] D. C. MATTIS: *Theory of Magnetism* (London, 1965).

[13] This term is usually omitted and in some sense its importance has only recently been recognized after the works of R. L. DOBRUSHIN: see ref. [9], and of O. LANFORD and D. RUELLE: *Comm. Math. Phys.*, **13**, 194 (1969). It is one of the purposes of this article to emphasize the role of this term in the theory of phase transitions.

[14] K. HUANG: *Statistical Mechanics* (New York, 1963).

[15] Here we are cavalier about the problem of what $\partial/\partial m$ means, since $g_\Lambda(\beta, m)$ is only defined for certain rational values of m (finite in number). One could, for instance, extend $g_\Lambda(\beta, m)$ to all m's by considering instead of $g_\Lambda(\beta, m)$ its convex envelope (other people prefer to consider the g_Λ obtained by linear interpolation from (2.6)). This is not very satisfactory but should not be really important for large systems (it would be interesting to make precise this point). With the above definition of g_Λ the function $m(h)$ might have jumps.

[16] This definition is inspired by O. LANFORD and D. RUELLE: *Comm. Math. Phys.*, **13**, 194 (1969), where the equivalence of the above definition with a number of other possible definitions is shown. For instance the definition in question is equivalent to the one based on the requirement that the correlation functions should be a solution of the Kirkwood-Solsburg equations. It is also equivalent to the definition of equilibrium state in terms of tangent planes (*i.e.* functional derivatives of a suitable functional: see D. RUELLE: *Statistical Mechanics* (New York, 1969), p. 184). It should be said that these proofs of equivalence are not always explicitly derived in the quoted paper by LANFORD and RUELLE; they are, however, an easy corollary of their results and appear, derived in detail, in the, so far, unpublished lecture notes of the lectures delivered by the author at the Courant Institute, September 1971, preprint.

[17] This solution of the Ising model in a zero field can be found in the paper by G. F. NEWELL and E. W. MONTROLL: *Rev. Mod. Phys.*, **25**, 353 (1953).

[18] This expansion can be used as a starting point for the combinatorial solution mentioned in [6]. See L. LANDAU and E. L. LIFSHITZ: *Physique Statistique* (Moscow, 1967).

[19] Of course we do not really attach a deep physical meaning to the difference between these two approaches. Clearly they should become equivalent if one pretended to extract all the possible information from them. What is really important is that the first questions raised by both approaches are very interesting and relevant from a physical point of view. One of the goals of the analytic theory of the phase transitions is to understand the nature of the singularity at the critical point and along the break of the isotherms. A lot of interest has been devoted to this point and a number of enlightening phenomenological results are available. However the number of rigorous results on the matter is rather limited. An idea of the type of problems that are of interest can be gotten by reading the papers of P. W. KASTELEYN: in *Fundamental Problems in Statistical Mechanics, II*, edited by E. G. D. COHEN (Amsterdam, 1968), or the more detailed paper by M. E. FISHER: *Rep. Progr. Theor. Phys.*, **30**, 615 (1967), and the paper by M. E. FISHER: *Physics, Physica, Fizika*, **3**, 255 (1967).

[20] This geometric picture of the spin configurations can be traced back at least as far as Peierls' paper, ref. [2], and has been used, together with formula (4.11), to derive (6.8) (« Kramer's and Wannier's duality relation ») and (6.9) by H. A. KRAMERS and G. H. WANNIER: ref. [6]. A recent interesting generalization of the duality concept has been given by F. J. WEGNER: *Journn. Math. Phys.*, **12**, 2259 (1971), where some very interesting applications can be found as well as references to earlier works. The duality relation between (+)-

or (—)-boundary conditions and open boundary conditions (which is used here) has been developed in a conversation with BENETTIN, JONA-LASINIO and STELLA. The reader can find other interesting applications of the duality relation in their paper to appear in *Lett. Nuovo Cimento* (June 1972).

[21] The above proof is due to R. B. GRIFFITHS and, independently, to R. L. DOBUSHIN and is inspired by R. PEIERLS: *Proc. Cambridge Phil. Soc.*, **32**, 477 (1936).

[22] M. E. FISHER: *Physics, Physica, Fizika*, **3**, 255 (1967).

[23] This theorem is due to R. A. MINLOS and J. G. SINAI: *Trans. Moscow Math. Soc.*, **19**, 121 (1968); *Math. USSR Sbornik*, **2**, 335 (1967).

[24] F. A. BEREZIN and J. G. SINAI: *Trans. Moscow Math. Soc.*, **19**, 219 (1967).

[25] Here the symbol $\varrho_\beta(\theta)(\mathrm{d}\theta/2\pi)$ has not to be taken too seriously; it really denotes a measure on the circle and this measure is not necessarily $\mathrm{d}\theta$-continuous. Also the « convergence » statement really means the existence of a measure such that (9.2) holds for all real z. The original proof of this theorem is due to T. D. LEE and C. N. YANG: *Phys. Rev.*, **87**, 410 (1952). A much stronger and general theorem leading, in particular, to Lee-Yang's theorem is in D. RUELLE: *Phys. Rev. Lett.*, **26**, 303 (1971). Ruelle's theorem has been the last of a series of improvements and generalizations of Lee-Yang's theorem; see references in Ruelle's paper.

[26] D. RUELLE: *The use of small external fields etc.*, preprint (1971), to appear in *Journ. Math. Phys.*

[27] A. MARTIN-LÖF and J. L. LEBOWITZ: *Comm. Math. Phys.*, **25**, 276 (1972).

[28] K. B. GRIFFITHS: *Journ. Math. Phys.*, **8**, 478 (1967); M. E. FISHER: *Lectures in Physics*, Vol. **7** C (Boulder Colo., 1965).

[29] The definition below is due to D. RUELLE: *Statistical Mechanics* (New York, 1969), p. 161.

[30] This is an unpublished result of R. B. GRIFFITHS. His proof is reported in G. GALLAVOTTI, A. MARTIN-LÖF and S. MIRACLE-SOLE: to appear in the *Lecture Notes of the 1971 Battelle-Seattle Summer Rencontres in Mathematics and Physics*, edited by A. LENARD (Berlin).

[31] R. A. MINLOS and J. G. SINAI: *Trans. Moscow Math. Soc.*, **19**, 121 (1968); *Math. USSR Sbornik*, **2**, 335 (1967).

[32] G. GALLAVOTTI and S. MIRACLE-SOLE: *Phys. Rev. B*, **5**, 2555 (1972); see also A. MARTIN-LÖF: *Comm. Math. Phys.*, **24**, 253 (1972).

[33] This theorem is due to R. A. MINLOS and J. G. SINAI: *Trans. Moscow Math. Soc.*, **19**, 121 (1968); *Math. USSR Sbornik*, **2**, 335 (1967). Actually MINLOS and SINAI prove a more difficult theorem under slightly different conditions. The adaptation to the deduction of the results given here can be found in G. GALLAVOTTI and A. MARTIN-LÖF: *Comm. Math. Phys.*, **25**, 87 (1972); or, better, in G. GALLAVOTTI, A. MARTIN-LÖF and S. MIRACLE-SOLE: to appear in the *Lecture Notes of the 1971 Battelle-Seattle Summer Rencontres in Mathematics and Physics*, edited by A. LENARD (Berlin).

[34] G. GALLAVOTTI and A. MARTIN-LÖF: *Comm. Math. Phys.*, **25**, 87 (1972).

[35] For a comparison of various old definitions of surface tension, new ones and a proof of their equivalence see D. ABRAHAM, G. GALLAVOTTI and A. MARTIN-LÖF: *Surface tension in the two-dimensional Ising model*, preprint (1972).

[36] The results of this Section are due to G. GALLAVOTTI and H. VAN BEYEREN for the two-dimensional case and to R. L. DOBRUSHIN for the 3-dimensional case. See G. GALLAVOTTI and H. VAN BEYREN: preprint (1972); G. GALLAVOTTI: *Comm. Math. Phys.*, to appear; R. L. DOBRUSHIN: to appear in *Teor. Mat. Fiz.*

[37] R. J. BURFORD and M. E. FISHER: *Phys. Rev.*, **156**, 583 (1967).

[38] The reader may consult the book of D. RUELLE: *Statistical Mechanics* (New

York, 1969), p. 125; R. A. MINLOS and J. G. SINAI: *Trans. Moscow Math. Soc.*, **19**, 237 (1967); F. A. BEREZIN and J. G. SINAI: *Trans. Moscow Math. Soc.*, **19**, 219 (1967); R. L. DOBRUSHIN: *Functional analysis and its applications*, **3**, 22 (1968); and the review article by J. GINIBRE: in *Colloques du CNRS* (Gif-sur-Yvette, 1970).

[39] D. RUELLE: *Statistical Mechanics* (New York, 1969), p. 112.

[40] R. L. MINLOS and J. G. SINAI: *Teor. Mat. Fiz*, **2**, No. 2 (1970); W. J. CAMP and M. E. FISHER: *Phys. Rev. Lett.*, **26**, 73, 565 (1971).

[41] D. ABRAHAM: *Studies Appl. Math.*, **50**, 71 (1971).

[42] O. LANFORD and D. RUELLE: *Comm. Math. Phys.*, **13**, 194 (1969).

[43] Metastability should be a dynamical phenomenon as suggested, for instance, by O. LANFORD and D. RUELLE: *Comm. Math. Phys.*, **13**, 194 (1969). An interesting rigorous treatment of the metastability phenomenon in the case of very weak and very-long-ranged forces can be found in J. L. LEBOWITZ and D. PENROSE: to appear.

[44] See the review paper by J. GINIBRE: in *Colloques du CNRS* (Gif-sur-Yvette, 1970). See also J. L. LEBOWITZ and G. GALLAVOTTI: *Journ. Math. Phys.*, **12**, 1129 (1971).

[45] D. RUELLE: *Phys. Rev. Lett.*, **27**, 1041 (1971).

[46] Some interesting results are to be expected in these cases. See, for instance, D. MERMIN: *Phys. Rev. Lett.*, **26**, 168 (1971).

[47] D. ABRAHAM: *Studies Appl. Math.*, **50**, 71 (1971).

[48] R. L. MINLOS and J. G. SINAI: *Teor. Mat. Fiz.* **2**, No. 2 (1970); W. J. CAMP and M. E. FISHER: *Phys. Rev. Lett.*, **26**, 73, 565 (1971).

[49] W. J. CAMP and M. E. FISHER: *Phys. Rev. Lett.*, **26**, 73, 565 (1971).

GENERAL REFERENCES

D. RUELLE: *Statistical Mechanics* (New York, 1969).

R. A. MINLOS: *Lectures in Statistical Physics*, in *Russian Math. Surveys*, **23**, 137 (1968).

The interested reader will be helped by the very recent and complete review of rigorous results (many of which refer to the Ising model and its generalizations) in the article by:

R. B. GRIFFITHS: *Rigorous results and theorems*, to appear in *Phase Transitions and Critical Points*, edited by C. DOMB and M. S. GREEN (New York,).