Cyclic polynomials and the multiplication matrices of their roots

Let D be an integrally closed, characteristic zero domain, K its field of fractions, $m \geq 2$ an integer and $P(x) = \sum_{k=0}^{m} c_k x^k = \prod_{i=0}^{m-1}(x - \theta_i) \in D[x]$ a cyclic polynomial. Let τ be a generator of $\mathrm{Gal}(K(\theta_0)/K)$ and suppose the θ_i are labeled so that $\tau(\theta_i) = \theta_{i+1}$ (indices mod m). Suppose that the discriminant $\mathrm{discr}_{K(\theta_0)/K}(\theta_0, \theta_1, \ldots, \theta_{m-1})$ is nonzero. For $0 \leq i, j \leq m-1$, define the elements $a_{i,j} \in K$ by $\theta_0 \theta_i = \sum_{j=0}^{m-1} a_{i,j} \theta_j$. Let $A = [a_{i,j}]_{0 \leq i,j < m}$. We call A the multiplication matrix of the θ_i. We have that $P(x)$ is the characteristic polynomial of A. We study the relations between $P(x)$ and A. We show how to factor $P(x)$ in the field $K[A]$ and how to construct A in terms of the coefficients c_i. We give methods to construct matrices A, with entries in K, such that the characteristic polynomial of A belongs to $D[x]$, is cyclic, and has A as the multiplication matrix of its roots. One of these methods derives from a natural composition of multiplication matrices. The other method gives matrices A, that are generalizations of matrices of cyclotomic numbers of order m, whose characteristic polynomials have roots that are generalizations of Gaussian periods of degree m.