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GRH AND FINITE FLAT GROUP SCHEMES OVER Z

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Abstract. Since simple commutative finite flat group schemes G over $\mathbf{Z}$ are killed by a prime number p, their order is a power of p. Tate asked whether a simple group scheme G is necessarily equal to $\mathbf{Z}/p\mathbf{Z}$ or μ_p. This has been proved for primes $p \leq 19$. Under assumption of the Generalized Riemann Hypothesis we extend this result to primes $p \leq 37$.

1. Introduction

A finite flat commutative group scheme G over $\mathbf{Z}$ is simple if and only if the set of its points $G(\overline{\mathbf{Q}})$ is an irreducible Galois module. Therefore G is annihilated by some prime number p and G is a p-group scheme in the sense that its order is a power of p. Group schemes of prime order p are simple. By a theorem of Oort and Tate [14], the only ones over $\mathbf{Z}$ are the constant group scheme $\mathbf{Z}/p\mathbf{Z}$ and its Cartier dual μ_p. Tate asked the following question [13].

Question 1.1. *Let p be a prime number. Are $\mathbf{Z}/p\mathbf{Z}$ and μ_p the only simple p-group schemes defined over $\mathbf{Z}$?*

Simple p-group schemes over $\mathbf{Z}$ cannot have order p^2. Indeed, suppose that such a group scheme G existed. Let G^D be the Cartier dual of G, and $G(\overline{\mathbf{Q}})$ and $G^D(\overline{\mathbf{Q}})$ their respective $\overline{\mathbf{Q}}$-points. Then, the determinant of the Galois action on $G(\overline{\mathbf{Q}})$ would be a power of the mod p cyclotomic character ω. Over the ring $\mathbf{Z}_p$ (and up to an unramified twist), the group scheme G would be either a (p,p)-group scheme in the sense of Raynaud [7], in which case the Galois action on the determinant is via ω, or an extension of group schemes isomorphic to $\mathbf{Z}/p\mathbf{Z}$ or μ_p. However, the Galois action on $G(\overline{\mathbf{Q}})$ and on $G^D(\overline{\mathbf{Q}})$ cannot be unramified at every prime. Therefore, over $\mathbf{Z}_p$ the group scheme G is an extension of $\mathbf{Z}/p\mathbf{Z}$ by μ_p, so that also in this case the action on the determinant would be via ω. Since ω is an odd character, Khare's proof of the level 1 Serre conjecture implies then that $G(\overline{\mathbf{Q}})$ must be reducible [4]. Therefore G cannot simple.

It is not known whether simple p-group schemes over $\mathbf{Z}$ can have order p^k for some $k \geq 3$. However, for small primes p this cannot happen. Abrashkin [1] and Fontaine [3] both showed that Question 1.1 has an affirmative answer for

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primes $p \leq 17$. In [2] this result is extended to $p = 19$. In this paper we prove the following theorem under the Generalized Riemann Hypothesis (GRH).

Theorem 1.2 (GRH)**.** *For every prime $p \leq 37$ the only simple finite flat commutative group schemes over $\mathbf{Z}$ of p-power order are $\mathbf{Z}/p\mathbf{Z}$ and μ_p.*

The main idea in the proof of Theorem 1.2 is also at the heart of the proof of the non-existence of abelian varieties with everywhere good reduction over $\mathbf{Q}$ by Abrashkin and Fontaine. It is the fact that for p a prime, and G a p-group scheme over $\mathbf{Z}$ killed by p, the field F generated by the points of G is a finite Galois extension of $\mathbf{Q}$ which has very little ramification. More precisely, F has the following two properties, the second of which follows from works of Abrashkin [1] and Fontaine [3].

(A) F is unramified outside p and ∞;
(B) $u_{F_\mathfrak{p}/\mathbf{Q}_p} \leq 1 + \frac{1}{p-1}$ for all primes $\mathfrak{p}$ of F lying over p.

Here, the invariant $u_{F_\mathfrak{p}/\mathbf{Q}_p}$, which only depends on the structure of the group scheme G over $\mathbf{Z}_p$, is defined as follows. Let $\mathfrak{p}$ be a prime of F lying over p, and let $F_\mathfrak{p}$ denote the completion of F at $\mathfrak{p}$. Let $\mathcal{O}_{F_\mathfrak{p}}$ be the ring of integers of $F_\mathfrak{p}$, and $\mathfrak{d}_\mathfrak{p}$ the different of $F_\mathfrak{p}$ over $\mathbf{Q}_p$. Then, by Kummer, we can write $\mathcal{O}_{F_\mathfrak{p}} = \mathbf{Z}_p[\alpha]$, for some $\alpha \in \mathcal{O}_{F_\mathfrak{p}}$. We define

$$u_{F_\mathfrak{p}/\mathbf{Q}_p} = i_\mathfrak{p} + v(\mathfrak{d}_\mathfrak{p}), \text{ where } i_\mathfrak{p} = \max\big\{v(\sigma(\alpha) - \alpha) : \sigma \in \mathrm{Gal}(F_\mathfrak{p}/\mathbf{Q}_p),\ \sigma \neq 1\big\}.$$

The valuation v is normalized by setting $v(p) = 1$. It follows that both $i_\mathfrak{p}$ and $v(\mathfrak{d}_\mathfrak{p})$ are in $\frac{1}{e_\mathfrak{p}}\mathbf{Z}$, where $e_\mathfrak{p}$ is the ramification index of $F_\mathfrak{p}$ over $\mathbf{Q}_p$. Note that the local root discriminant $\delta_{F_\mathfrak{p}}$ of F is equal to $p^{v(\mathfrak{d}_\mathfrak{p})}$ for each $\mathfrak{p}$. Therefore, by [3, sect. 3.3], the global root discriminant δ_F satisfies

$$\delta_F < p^{1+\frac{1}{p-1}}.$$

See [2] for a characterization of the invariant $u_{F_\mathfrak{p}/\mathbf{Q}_p}$ in terms of the upper numbering of the higher ramification groups.

The *Abrashkin-Fontaine field* is the maximal extension $\mathbf{Q} \subset F$ having the two properties (A) and (B) above. It contains the cyclotomic field $\mathbf{Q}(\zeta_p)$ as well as its Hilbert class field H. More generally, F does not admit any proper extensions that are unramified outside p and ∞ and are at most tamely ramified at p. In Section 3, we prove under the assumption of GRH that for $p \leq 37$ the Abrashkin-Fontaine field is actually equal to H. It follows that for $p \leq 37$, the points of a simple finite flat commutative p-power order group scheme G over $\mathbf{Z}$ are defined over the Hilbert class field H. A little bit of group theory then implies that the points of G are actually defined over the subfield $\mathbf{Q}(\zeta_p)$. The theorems of Oort and Tate [14] then imply Theorem 1.2. In other words, the group scheme G is isomorphic to $\mathbf{Z}/p\mathbf{Z}$ or μ_p.

For $p \geq 41$, bounds on the root discriminant of F are too large for the GRH Odlyzko bounds to apply. Therefore we do not have any bound on the degree of the Abrashkin-Fontaine field F. However, for larger p, we expect that F is usually strictly larger than the Hilbert class field H. This already happens for small values of p. Indeed, for $p = 53$, the splitting field H' of $x^8 - x^7 + 3x^6 - 3x^5 + 2x^4 - 2x^3 + 5x^2 + 5x + 1$ is unramified outside 53 and ∞. Since it is only tamely ramified at 53, it is contained in the Abrashkin-Fontaine field for $p = 53$. However, the Galois group of H' over $\mathbf{Q}$ is the non-solvable group $\mathrm{PGL}_2(\mathbf{F}_7)$. This implies that H' cannot be a subfield of H.

Similarly, for $p = 59$ the splitting field H'' of $x^4 - x^3 - 7x^2 + 11x + 3$ is unramified outside 59 and ∞. Therefore it is contained in the Abrashkin-Fontaine field F for $p = 59$. The Galois group of H'' over $\mathbf{Q}$ is isomorphic to the symmetric group S_4.

Since the Galois group of the Hilbert class field H of $\mathbf{Q}(\zeta_{59})$ is metabelian, and S_4 is not, H'' is not contained in H.

The Abrashkin-Fontaine field F may not even be a finite extension of $\mathbf{Q}$. Indeed, the entire Hilbert class field tower of $\mathbf{Q}(\zeta_p)$ is contained in F. The smallest prime p for which we know this tower to be infinite is $p = 401$. Since the degree 8 subfield of $\mathbf{Q}(\zeta_{401})$ has class number 45, this follows from the following result. The proof is similar to the proof of [8, Theorem 5.1].

Proposition 1.3. *Let $p > 2$ be a prime. If there exists a divisor d of $p-1$ for which $(p-1)/d$ is odd and for which the class number of the degree $d/2$ subfield of $\mathbf{Q}(\zeta_p)$ exceeds $2d+5$, then the class field tower of $\mathbf{Q}(\zeta_p)$ is infinite.*

2. The Abrashkin-Fontaine field F

Let p be a prime and let H be the Hilbert class field of $\mathbf{Q}(\zeta_p)$. It is contained in the Abrashkin-Fontaine field F defined in the introduction. The root discriminants of F and its subfields are smaller than $p^{1+\frac{1}{p-1}}$. In particular, for $p = 2$ the root discriminant of F is < 4. Odlyzko's discriminant bounds easily imply that $F = \mathbf{Q}(i)$ in this case. For $p > 2$ we have the following results.

Proposition 2.1. *Let $p > 2$ be prime and let F be the Abrashkin-Fontaine field. Then*

(a) *the maximal abelian extension of $\mathbf{Q}$ inside F is $\mathbf{Q}(\zeta_p)$;*
(b) *the maximal abelian extension of $\mathbf{Q}(\zeta_p)$ inside F is the Hilbert class field H of $\mathbf{Q}(\zeta_p)$.*

Proof. (a) By the Kronecker-Weber theorem the maximal abelian extension of $\mathbf{Q}$ that is unramified outside p is contained in a cyclotomic field $\mathbf{Q}(\zeta_{p^v})$ for some $v \geq 0$. Since $u_{\mathbf{Q}_p(\zeta_{p^v})/\mathbf{Q}_p}$ is equal to v, we must have $v = 1$.

(b) By class field theory the maximal abelian extension of $\mathbf{Q}(\zeta_p)$ inside F is contained in a ray class field K of conductor π^a for some $a \geq 0$. Here π denotes the prime $\zeta_p - 1$ over p. It splits completely in the Hilbert class field H. Let $\mathfrak{p}$ be a prime over π in K and let $K_{\mathfrak{p}}$ be the completion of K at $\mathfrak{p}$. We have $u_{\mathbf{Q}_p(\zeta_p)/\mathbf{Q}_p} = 1$ and $i_{\mathbf{Q}_p(\zeta_p)/\mathbf{Q}_p} = \frac{1}{p-1}$. It follows from [11, Lemma 2.1] that $u_{K_{\mathfrak{p}}/\mathbf{Q}_p} = 1 + \max(0, \frac{a}{p-1} - \frac{1}{p-1})$. Since $K \subset F$, this is at most $1 + \frac{1}{p-1}$. Therefore we have $a \leq 2$. The proposition now follows from the following lemma. □

Lemma 2.2. *Let p be an odd prime and let π denote the prime above p in $\mathbf{Q}(\zeta_p)$. Then the ray class fields of conductor π and π^2 are equal to the Hilbert class field H of $\mathbf{Q}(\zeta_p)$.*

Proof. Let $a \in \mathbf{Z}$ be coprime to p. Then the cyclotomic unit $\eta_a = (\zeta_p^a - 1)/(\zeta_p - 1)$ is congruent to a modulo π. Class field theory therefore implies that the ray class field of conductor π is equal to the Hilbert class field H. To prove that the ray class field of conductor π^2 is also equal to H, we compute η_a^{p-1} modulo π^2. We find

$$\eta_a^{p-1} = \left(\frac{\zeta_p^a - 1}{\zeta_p - 1}\right)^{p-1} = \left(\frac{(1+\pi)^a - 1}{\pi}\right)^{p-1} \equiv 1 - \frac{a-1}{2}\pi \mod \pi^2.$$

Since we can take $a = 2$, the lemma follows. □

In the rest of this section we collect some numerical data regarding the primes $p = 23$, 29, 31 and 37. The root discriminants of the corresponding Abrashkin-Fontaine fields are too large for the unconditional Odlyzko bounds to apply. However, under the Generalized Riemann Hypothesis (GRH), Odlyzko obtained bounds that still apply. See [5, Tables 1 and 3]. Table 1 contains data that are used in the proofs in the final section.

p	h	$\delta_{\mathbf{Q}(\zeta_p)}$	$p^{1+\frac{1}{p-1}}$	$[H^{\mathrm{unr}}:\mathbf{Q}]$	$[H^{\mathrm{unr}}:H]$	$[H^{\mathrm{tame}}:\mathbf{Q}]$	$[H^{\mathrm{tame}}:H]$	$[F:\mathbf{Q}]$	$[F:H]$
23	3	19.94	26.52	120	1	220	3	600	9
29	8	25.71	32.71	400	1	1000	4	4800	21
31	9	27.64	34.76	720	2	1900	7	10000	37
37	37	33.46	40.93	5300	3	42700	32	5684000	4267

TABLE 1. Odlyzko bounds

We use the following notations:

- p is a prime between 23 and 37.
- h is the class number of the cyclotomic field $\mathbf{Q}(\zeta_p)$.
- $\delta_{\mathbf{Q}(\zeta_p)}$ is the root discriminant of $\mathbf{Q}(\zeta_p)$.
- the root discriminant bound $p^{1+\frac{1}{p-1}}$ of the Abrashkin-Fontaine field F.
- H is the Hilbert class field of $\mathbf{Q}(\zeta_p)$.
- H^{unr} is the maximal unramified extension of H; and the column $[H^{\mathrm{unr}}:\mathbf{Q}]$ (resp. $[H^{\mathrm{unr}}:H]$) gives an upper bound on the degree of H^{unr} (resp. relative degree of H^{unr} over H).
- H^{tame} is the maximal tamely ramified extension of H; and the column $[H^{\mathrm{tame}}:\mathbf{Q}]$ (resp. $[H^{\mathrm{tame}}:H]$) gives an upper bound on the degree of H^{tame} (resp. relative degree of H^{tame} over H).
- The column $[F:\mathbf{Q}]$ (resp. $[F:H]$) is an upper bound on the degree of F (resp. $[F:H]$).

We note that H^{unr} has the same root discriminant as H, and that H^{tame} has root discriminant at most p. All the bounds assume GRH, and were computed using Odlyzko's bounds from [5, Table 1]. Only for $p = 37$, the degrees are outside the range of this table and we computed the Odlyzko bounds using [5, Table 3]. For instance, the bound 5684000 was obtained by choosing $b = 15.000$ and $E = 70185$.

3. **A representation-theoretic result**

In this section we fix a prime p and let k denote an algebraic closure of $\mathbf{F}_p$. Our proof of the main result of this paper depends on a representation theoretic result. For a finite abelian group G and a character $\chi : G \to k^*$, we let $k(\chi)$ denote the 1-dimensional k-vector space on which G acts through χ. The χ-*twist* of a $k[G]$-module M is the module $M \otimes_k k(\chi)$.

Proposition 3.1. *Let A be a finite abelian group and let Δ be a finite abelian group of order prime to p acting linearly on A. Let Q denote the semidirect product $A \rtimes \Delta$. Let V be a finite irreducible $k[Q]$-module. Then, as a Δ-module, V is isomorphic to a twist of $k[\Delta/H]$ for some subgroup $H \subset \Delta$.*

Proof. Let V be a finite irreducible $k[Q]$-module and let the action of Q on V be given by $\varrho : Q \longrightarrow \mathrm{Aut}(V)$. Then the p-Sylow subgroup P of A acts trivially on V. Since P is normal in Q, we may replace Q by Q/P and assume that p does not divide $\#Q$. Since A is abelian, $\varrho|_A$ is a sum of characters. Let $\psi : A \to k^*$ be one of those characters and let Δ_ψ be the stabilizer of ψ inside Δ and $H_\psi = A \rtimes \Delta_\psi$. Then, by Clifford theory there exists a representation $\theta : H_\psi \to k^*$, which extends ψ, such that

$$\varrho = \mathrm{Ind}_{H_\psi}^{Q} \theta.$$

Indeed, the proof in [12, §8.2, Proposition 25] not only works when the characteristic of the coefficient field is zero, but also when it is prime to the order of Q. Since Δ_ψ is the stabilizer of ψ, its action on V commutes with that of A. In fact, since Δ is

abelian, the action of Δ_ψ on V commutes with that of Q. So, by Schur's Lemma, Δ_ψ acts on V via a character $\chi_0 : \Delta_\psi \to k^*$. We extend χ_0 to a character of Δ. Then Δ_ψ acts trivially on the twisted module $V \otimes k(\chi_0^{-1})$. Since each character χ of Δ/Δ_ψ occurs at most once in V and since $\dim V = [\Delta : \Delta_\psi]$, it follows that each χ occurs *exactly* once, so that, as a Δ-module, $V \otimes k(\chi_0^{-1})$ is isomorphic to $k[\Delta/\Delta_\psi]$. This proves the proposition. □

Theorem 3.2. *Let p be a prime and let G be a simple finite flat commutative group scheme over $\mathbf{Z}$ of p-power order. If the field generated by the points of G is contained in the Hilbert class field of $\mathbf{Q}(\zeta_p)$, then $G \cong \mathbf{Z}/p\mathbf{Z}$ or $G \cong \mu_p$.*

Proof. Let G be a simple finite flat commutative group scheme over $\mathbf{Z}$ of p-power order, with the property that the field K generated by the points of G is contained in the Hilbert class field H of $\mathbf{Q}(\zeta_p)$. It suffices to show that K is contained in $\mathbf{Q}(\zeta_p)$. Indeed, in that case the group of points $G(\overline{\mathbf{Q}})$ is an irreducible module over $\mathbf{F}_p[\Delta]$, where $\Delta = \mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$. Since Δ is cyclic of order $p-1$, the group $G(\overline{\mathbf{Q}})$ has order p. It is then a consequence of the Oort-Tate classification of group schemes of order p that $G \cong \mathbf{Z}/p\mathbf{Z}$ or $G \cong \mu_p$.

This argument takes care of the primes p for which H is equal to $\mathbf{Q}(\zeta_p)$, in other words the primes $p \leq 19$. The following argument works for the primes $p > 19$. Let Q be the Galois group of the Hilbert class field H over $\mathbf{Q}$. Put $A = \mathrm{Gal}(H/\mathbf{Q}(\zeta_p))$ and $\Delta = \mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q}) \cong (\mathbf{Z}/p\mathbf{Z})^*$. Since the unique prime over p in $\mathbf{Q}(\zeta_p)$ is principal, it splits completely in H. Therefore, the decomposition group at each prime of H lying over p is isomorphic to Δ. It follows that Q is a semidirect product $A \rtimes \Delta$.

Let V be an irreducible constituent of the $k[Q]$-module $G(\overline{\mathbf{Q}}) \otimes k$. Then Proposition 3.1 implies that, as a Δ-module, V is a sum of characters on Δ, each of which appears with multiplicity at most *one*. But, by the Oort-Tate classification, only the trivial character 1 and the cyclotomic character ω can occur in V. It follows that the dimension of V is at most 2.

If $\dim V$ were 2, Proposition 3.1 would imply that ω is trivial on the unique index 2 subgroup of Δ, so that ω is quadratic. This is a contradiction since $p > 3$. Therefore we have $\dim V = 1$. Since k^* is abelian, Proposition 2.1 implies that the subgroup A acts trivially on V. It follows that A also acts trivially on $G(\overline{\mathbf{Q}}) \otimes k$ and hence on $G(\overline{\mathbf{Q}})$, as required.

□

4. End of proof

In this section we prove the following result. Together with Theorem 3.2 it implies Theorem 1.2.

Theorem 4.1 (GRH)**.** *Let $p = 23$, 29, 31 or 37. Then the Abrashkin-Fontaine field F is equal to the Hilbert class field H of $\mathbf{Q}(\zeta_p)$.*

Proof. We have inclusions

$$\mathbf{Q} \subset \mathbf{Q}(\zeta_p) \subset H \subset F.$$

Let $\Gamma = \mathrm{Gal}(F/\mathbf{Q})$. By Proposition 2.1 its commutator subgroup Γ' is $\mathrm{Gal}(F/\mathbf{Q}(\zeta_p))$ and the commutator subgroup Γ'' of Γ' is $\mathrm{Gal}(F/H)$. Below we show that for $p = 23$, 29, 31 and 37 the group Γ'' is trivial, or equivalently that Γ' is abelian. This implies that $F = H$. Since the details are different for each prime, we proceed case by case.

Case $p = 23$. From Table 1, we see that $[F : H] = \#\Gamma'' \leq 9$. This implies that F is tamely ramified at 23, so that Table 1 implies that $\#\Gamma'' \leq 3$. It follows that $\mathrm{Aut}(\Gamma'')$ is abelian. This implies that Γ' is in the kernel of the map $\Gamma \to \mathrm{Aut}(\Gamma'')$

given by conjugation. It follows that Γ'' is in the center of Γ'. Since Γ'/Γ'' is cyclic, we find that $\Gamma' = \mathrm{Gal}(F/\mathbf{Q}(\zeta_p))$ is abelian, as required.

Case $p = 29$**.** We have the following inclusions

$$\mathbf{Q} \underset{4}{\subset} \underbrace{K \underset{\Delta}{\subset} \mathbf{Q}(\zeta_{29}) \underset{V}{\subset} H}_{\Gamma_2} \underset{\Gamma''}{\subset} F.$$

Here $V = \mathrm{Gal}(H/\mathbf{Q}(\zeta_{29}))$ is a 3-dimensional $\mathbf{F}_2$-vector space on which the order 7 group $\Delta = \mathrm{Gal}(\mathbf{Q}(\zeta_{29})/K)$ acts non-trivially. See [9].

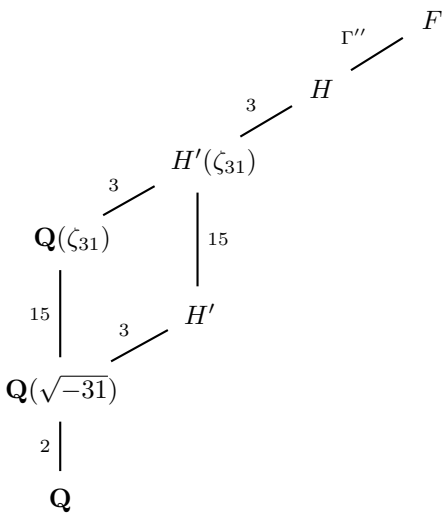
From Table 1, we see that $[F : H] = \#\Gamma'' \leq 21$. This implies that F is tamely ramified at 29, so that Table 1 implies that $\#\Gamma'' \leq 4$. So, Γ'' is either cyclic or isomorphic to Klein's four group. Since $[H : \mathbf{Q}] = 8 \cdot 28$ is not divisible by 3, it follows that the image of Γ in $\mathrm{Aut}(\Gamma'')$ has order at most 2. Therefore Γ' and hence its subgroup $\Gamma_2 = \mathrm{Gal}(H/K)$ act trivially on Γ''. We claim that the group extension

$$0 \longrightarrow \Gamma'' \longrightarrow \Gamma_1 \longrightarrow \Gamma_2 \longrightarrow 0$$

is split. It follows that K admits an extension E inside F with $\mathrm{Gal}(E/K) = \Gamma''$ of order at most 4. A short `pari`-computation [6] shows that the only such extension is K itself. This means that Γ'' is trivial and we are done.

It remains to show that the extension is split. Since $\#\Gamma_2 = 56$ is prime to 3, this is the case when $\#\Gamma'' = 3$. So, we may assume that Γ'' has order 1, 2 or 4. It suffices therefore to show that the cohomology group $H^2(\Gamma_2, \mathbf{Z}/2\mathbf{Z}) = 0$ vanishes. Since $H^2(\Gamma_2, \mathbf{Z}) = \mathrm{Hom}(\Gamma_2, \mathbf{Q}/\mathbf{Z})$ is trivial, our cohomology group is isomorphic to the 2-torsion of the Schur multiplier $H^3(\Gamma_2, \mathbf{Z})$. Since Δ and V have coprime order, the Hochschild-Serre spectral sequence degenerates and we find that $H^3(\Gamma_2, \mathbf{Z})$ is isomorphic to $H^3(V, \mathbf{Z})^\Delta$. The Schur multiplier of the abelian group V is $V \wedge V$ equipped with its natural Δ-action. In particular, the Δ-invariants are zero and we are done.

Case $p = 31$. From Table 1, we see that $[F : H] = \#\Gamma''$ is at most 37. If F is wildly ramified, the group Γ'' is cyclic of order 31 and its automorphism group is abelian. It follows that Γ' acts trivially on it. Since the class group of $\mathbf{Q}(\zeta_{31})$ is cyclic [9, Thm. III], this means that Γ' modulo its center is cyclic. It follows that Γ' is abelian and we are done.



If F is tamely ramified at 31, Table 1 implies that $[F : H] = \#\Gamma''$ is at most 7. It follows that $\mathrm{Aut}(\Gamma'')$ has order at most 6. If the image of $\Gamma \to \mathrm{Aut}(\Gamma'')$ is abelian, we find that Γ'' is contained in the center of Γ'. Since Γ'/Γ'' is cyclic, it follows that Γ' is abelian as required.

If the image of $\Gamma \to \mathrm{Aut}(\Gamma'')$ is not abelian, then Γ'' is isomorphic to V_4 or to S_3. Since S_3 is not the commutator subgroup of any group [10, Lemma 4.1], the group Γ'' must be isomorphic to Klein's group V_4. The group $\Gamma/\Gamma'' = \mathrm{Gal}(H/\mathbf{Q})$ has a unique quotient isomorphic to S_3. It is $\mathrm{Gal}(H'/\mathbf{Q})$, where H' is the Hilbert class field of $\mathbf{Q}(\sqrt{-31})$. It follows that the order 45 group $\mathrm{Gal}(H/H')$ acts trivially on Γ''.

This implies that H' admits an extension field E inside F for which $\mathrm{Gal}(E/H')$ is isomorphic to V_4. However, a short `pari` computation [6] shows that the ray class group of conductor $\sqrt{-31}$ of H' has odd order. Therefore this cannot occur.

Case $p = 37$. In Table 1, we find that $[F : H] \leq 4270$. Suppose that $F \neq H$. Then there a surjective homomorphism $\mathrm{Gal}(F/H) \to \Gamma_0$, where Γ_0 is a simple group. Let $H \subset L \subset F$ be the fixed field of its kernel, so that $\Gamma_0 = \mathrm{Gal}(L/H)$.

If the order of Γ_0 is divisible by 37, then Sylow theory implies that Γ_0 is either cyclic of order 37 or it is a non-commutative simple group of order $37(1 + 37k)$ for some $k \geq 1$. Since $\#\Gamma_0 \leq 4270$ this implies that $\#\Gamma_0 = 1406$, 2775 or 4144. We leave the exercise to show that groups of these orders cannot be simple to the reader. It follows that either Γ_0 is cyclic of order 37, or $\#\Gamma_0$ is prime to 37.

If Γ_0 is cyclic of order 37, we consider the maximal Galois extension $\mathbf{Q}(\zeta_{37}) \subset E$ inside F for which $P = \mathrm{Gal}(E/\mathbf{Q}(\zeta_{37}))$ is a 37-group. Then E contains the Hilbert class field H. The maximal Galois extension $H \subset E'$ inside F for which $\mathrm{Gal}(E'/H)$ is a 37-group, is Galois over $\mathbf{Q}(\zeta_{37})$. Therefore we have $E = E'$. It follows that E contains L. We have inclusions

$$\mathbf{Q} \subset \underbrace{\mathbf{Q}(\zeta_{37}) \subset H \subset L \subset E}_{P} \subset F.$$

By Proposition 2.1, the maximal *abelian* extension of $\mathbf{Q}(\zeta_{37})$ inside F is the Hilbert class field H. This means that $P/[P, P]$ is cyclic of order 37. By Burnside, P is also cyclic, and hence has order 37. It follows that $E = L = H$. Contradiction.

So $\#\Gamma_0 = [L : H]$ is prime to 37 and hence the extension $H \subset L$ is at most tamely ramified at the primes lying over 37. From Table 1, we see that $[L : H] \leq 32$. It follows that Γ_0 is cyclic of prime order $q \leq 31$.

Consider the maximal abelian exponent q-extension E'' of H inside F. Then we have

$$\mathbf{Q} \subset \mathbf{Q}(\zeta_{37}) \subset H \subset L \subset E'' \subset F.$$

and E'' is Galois over $\mathbf{Q}$. The degree $[E'' : H]$ is at most 32. The orders of the groups $\mathrm{GL}_d(\mathbf{F}_q)$ are not divisible by 37 when $q^d \leq 32$. It follows that conjugation by the cyclic order 37 group $\mathrm{Gal}(H/\mathbf{Q}(\zeta_{37}))$ is trivial on $\mathrm{Gal}(E''/H)$. This means that $\mathrm{Gal}(E''/H)$ is in the center of $\mathrm{Gal}(E''/\mathbf{Q}(\zeta_{37}))$. Since $\mathrm{Gal}(H/\mathbf{Q}(\zeta_{37}))$ is cyclic, the group $\mathrm{Gal}(E''/\mathbf{Q}(\zeta_{37}))$ is abelian. However, by Proposition 2.1 the field H is the maximal abelian extension of $\mathbf{Q}(\zeta_{37})$ inside F. Contradiction. So $F = H$ after all.

□

References

[1] Abrashkin, V.A.: Galois moduli of period p group schemes over a ring of Witt vectors, *Izv. Ak. Nauk CCCP*, Ser. Matem., **51** (1987). English translation in *Math. USSR Izvestiya*, **31** (1988), 1–46.

[2] Dembélé, L. and Schoof, R.: Finite flat group schemes over **Z** killed by 19, Journal of Number Theory, **260** (2024), 191–195.

[3] Fontaine, J.-M.: Il n'y a pas de variété abélienne sur **Z**, *Invent. Math.* **81**, (1985) 515–538.

[4] Khare, C.: Serre's modularity conjecture: The level one case, *Duke Mathematical Journal*, **134**, (2006) 557–589.
[5] Odlyzko, A.: Discriminant bounds, November 29, 1976. `http://www.dtc.umn.edu/∼odlyzko/unpublished/index.html`
[6] The PARI/GP computer algebra system, `https://pari.math.u-bordeaux.fr`
[7] Raynaud, M.: Schémas en groupes de type $(p, \ldots, p)$, Bull. Soc. Math. France, **102**, (1974) 241–280.
[8] Schoof, R.: Infinite class field towers of quadratic fields, *J. für die reine und angewandte Mathematik* **372**, (1986) 209–220.
[9] Schoof, R.: Minus class groups of cyclotomic fields of prime conductor, *Math. Comp.* **67**, (1998) 1225–1245.
[10] Schoof, R.: Semistable abelian varieties with good reduction outside 15, *Manuscripta Mathematica*, **139** (2012), 49–70.
[11] Schoof, R.: Abelian varieties over real quadratic fields with good reduction everywhere, in preparation.
[12] Serre, J.-P.: *Linear representations of finite groups*. Translated from the second French edition by Leonard L. Scott GTM, Vol. **42** Springer-Verlag, New York-Heidelberg, 1977. x+170 pp.
[13] Tate, J.T.: p-Divisible groups. Proc. of a Conference on Local Fields, Driebergen, Springer 1967.
[14] Tate, J.T. and Oort, F.: Group schemes of prime order, *Ann. Scient. École Norm. Sup.* **3** (1970), 1–21.
[15] Washington, L.C.: *Introduction to cyclotomic fields*, 2nd Ed, GTM **83**, Springer-Verlag 1997.

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