Approximation of complex algebraic numbers by algebraic numbers of bounded degree.

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Let n be a positive integer. For a given complex number ξ, denote by $k_n(\xi)$ the supremum of all k such that the inequality

$$0 < |\xi - \alpha| < H(\alpha)^{-k}$$

has infinitely many solutions in algebraic numbers α of degree at most n. Here $H(\alpha)$ denotes the height of α, that is the maximum of the absolute values of the coefficients of the minimal polynomial of α. In 1965, Sprindzhuk proved that $k_n(\xi) = n+1$ for almost all $\xi \in \mathbf{R}$ (with respect to the Lebesgue measure on $\mathbf{R}$) and $k_n(\xi) = \frac{n+1}{2}$ for almost all $\xi \in \mathbf{C}$ (with respect to the Lebesgue measure in $\mathbf{C}$). In 1971 Schmidt proved that if ξ is a real algebraic number of degree d, then $k_n(\xi) = \min(d, n+1)$. So as for approximation by algebraic numbers of degree at most n, real algebraic numbers ξ of degree larger than n show the same behavious as almost all real numbers. Up to now, nobody had computed $k_n(\xi)$ for complex algebraic numbers ξ. I will present some new results in this direction, obtained jointly with Yann Bugeaud. These results show that if ξ is complex algebraic of degree d then $k_n(\xi) = \frac{d}{2}$ if $d \leq n$ and $k_n(\xi) \in \{\frac{n+1}{2}, \frac{n+2}{2}\}$ if $d > n$. The hard core of the proof of this result is a central result in Diophantine approximation, Schmidt's Subspace Theorem.