Approximation of complex algebraic numbers by algebraic numbers of bounded degree.

Jan-Hendrik Evertse (Leiden).

Let \( n \) be a positive integer. For a given complex number \( \xi \), denote by \( k_n(\xi) \) the supremum of all \( k \) such that the inequality

\[
0 < |\xi - \alpha| < H(\alpha)^{-k}
\]

has infinitely many solutions in algebraic numbers \( \alpha \) of degree at most \( n \). Here \( H(\alpha) \) denotes the height of \( \alpha \), that is the maximum of the absolute values of the coefficients of the minimal polynomial of \( \alpha \). In 1965, Sprindzhuk proved that \( k_n(\xi) = n + 1 \) for almost all \( \xi \in \mathbb{R} \) (with respect to the Lebesgue measure on \( \mathbb{R} \)) and \( k_n(\xi) = \frac{n+1}{2} \) for almost all \( \xi \in \mathbb{C} \) (with respect to the Lebesgue measure in \( \mathbb{C} \)). In 1971 Schmidt proved that if \( \xi \) is a real algebraic number of degree \( d \), then \( k_n(\xi) = \min(d, n + 1) \). So as for approximation by algebraic numbers of degree at most \( n \), real algebraic numbers \( \xi \) of degree larger than \( n \) show the same behaviour as almost all real numbers. Up to now, nobody had computed \( k_n(\xi) \) for complex algebraic numbers \( \xi \). I will present some new results in this direction, obtained jointly with Yann Bugeaud. These results show that if \( \xi \) is complex algebraic of degree \( d \) then \( k_n(\xi) = \frac{d}{2} \) if \( d \leq n \) and \( k_n(\xi) \in \{ \frac{n+1}{2}, \frac{n+2}{2} \} \) if \( d > n \). The hard core of the proof of this result is a central result in Diophantine approximation, Schmidt’s Subspace Theorem.