Proof of Theorem 5.2:

Replace from "In the second case ..." until the end of the proof by

In the second case, we note that the local Galois module $G(\overline{\mathbf{Q}}_2)$ is unramified. More precisely, $G(\overline{\mathbf{Q}}_2)$ becomes constant over an unramified cubic extension H' of $\mathbf{Q}_2$. Let M be the Zariski closure of an order 2 subgroup of $G(\overline{\mathbf{Q}}_2)$. It is a finite flat group scheme over the ring of integers $O_{H'}$ of H'. Since H' is unramified over $\mathbf{Q}_2$, Oort-Tate implies that M is isomorphic to either $\mathbf{Z}/2\mathbf{Z}$ or μ_2. The same is true for the quotient G/M.

Since the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_2/\mathbf{Q}_2)$ is irreducible, the exactness of the connected-étale sequence of G over $\mathbf{Z}_2$ implies that G is either étale or local. In other words, M and G/M are either both isomorphic to $\mathbf{Z}/2\mathbf{Z}$ or to μ_2. If G is étale, we have $G \cong V$ by Galois theory. If G is local, we have an exact sequence over $O_{H'}$ of the form

$$0 \longrightarrow \mu_2 \longrightarrow G \longrightarrow \mu_2 \longrightarrow 0$$

It follows that the Cartier dual $G^\vee$ is étale over $O_{H'}$ and hence over $\mathbf{Z}_2$. It follows that $G^\vee$ is isomorphic to V and hence that G is isomorphic to $V^\vee$. This proves the theorem