Effectivity of Arakelov Divisors and the Theta Divisor of a Number Field

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1. Introduction

In the well known analogy between the theory of function fields of curves over finite fields and the arithmetic of algebraic number fields, the number theoretical analogue of a divisor on a curve is an Arakelov divisor. In this paper we introduce the notion of an *effective* Arakelov divisor; more precisely, we attach to every Arakelov divisor D its *effectivity*, a real number between 0 and 1. This notion naturally leads to another quantity associated to D. This is a positive real number $h^0(D)$ which is the arithmetic analogue of the dimension of the vector space $H^0(D)$ of sections of the line bundle associated to a divisor D on an algebraic curve. It can be interpreted as the logarithm of a value of a theta function. Both notions can be extended to higher rank Arakelov bundles.

In this paper we show that the *effectivity* and the numbers $h^0(D)$ behave in several respects like there traditional geometric analogues. We prove an analogue of the fact that $h^0(D) = 0$ for any divisor D on an algebraic curve with $\deg(D) < 0$. We provide evidence for a conjecture that would imply an analogue of the fact that $h^0(D) \leq \deg(D) + 1$ for divisors D on an algebraic curve with $\deg(D) \geq 0$. The Poisson summation formula implies a Riemann-Roch Theorem involving the numbers $h^0(D)$ and $h^0(\kappa - D)$ with κ the canonical class; it is a special case of Tate's Riemann-Roch formula. Unfortunately, we do not have a definition of $h^1(D) := h^0(\kappa - D)$ for an Arakelov divisor without recourse to duality. Following K. Iwasawa [Iw] or J. Tate [T] one derives in a natural way the finiteness of the class group and the unit theorem of Dirichlet from this analogue of the Riemann-Roch Theorem, avoiding the usual arguments from geometry of numbers.

The notion of effectivity naturally leads to a definition of the zeta function of a number field which is closely analoguous to the zeta function of a curve over a finite field. In this way the Dedekind zeta function, multiplied by the usual gamma factors, is recovered as an integral over the Arakelov class group.

There is a close connection between Arakelov divisors and certain lattices. The numbers $h^0(D)$ are closely related to the Hermite constants of these lattices. The value of h^0 on the canonical class is an invariant of the number field that can be viewed as an analogue of the genus of a curve. The quantity $h^0(D)$ defines a real analytic function on the Arakelov divisor class group. Its restriction to the group of Arakelov divisors of degree $\frac{1}{2}\deg(\kappa)$ can be viewed as the analogue of the set of rational points of the theta divisor of an algebraic curve over a finite field.

It is natural to try to obtain arithmetic analogues of various basic geometric facts like Clifford's Theorem. As is explained at the end of section 5, this comes down to studying the behaviour of the function h^0 on a space parametrizing bundles of rank 2.

We suggest in the same spirit a definition for an invariant $h^0(L)$ for a metrized line bundle L on an arithmetic surface. The definition for h^0 provided here hints at a further theory and we hope this paper will stimulate the readers to develop it.

In section 2 we recall some well known facts concerning Arakelov divisors. In section 3 we introduce the notion of effectivity and the definition of h^0. We apply these to the zeta function of a number field in section 4. In section 5 we give some estimates on $h^0(D)$. We introduce the analogue of the genus for number fields in section 6. In section 7 we briefly discuss a two variable zeta function. Finally, in section 8 and 9, we make some remarks about the higher dimensional theory.

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2. Arakelov Divisors

The similarity between the class group $Cl(O_F)$ of the ring of integers of a number field F and the Jacobian of an irreducible smooth curve is a particular aspect of the deep analogy between number fields and function fields of curves. However, the class group classifies isomorphism classes of line bundles on the *affine* scheme $\mathrm{Spec}(O_F)$ and algebraic geometry tells us that the Jacobian of a *projective* or complete curve is much better behaved than that of an affine one. Following S. Arakelov [A1,A2] one arrives at an improved analogy via a sort of compactification of the scheme $\mathrm{Spec}(O_F)$ by taking the archimedean primes of F into account. One thus obtains a generalization of the class group which is a compact group; it is an extension of the usual (finite) ideal class group by a real torus. In this section we recall the definitions and briefly discuss a variant of this theory.

Let F be a number field. An *Arakelov divisor* is a formal finite sum $\sum_P x_P P + \sum_\sigma x_\sigma \sigma$, where P runs over the prime ideals of the ring of integers O_F and σ runs over the infinite, or archimedean, primes of the number field F. The coefficients x_P are in $\mathbf{Z}$ but the x_σ are in $\mathbf{R}$. The Arakelov divisors form an additive group $\mathrm{Div}(F)$ isomorphic to $\sum_P \mathbf{Z} \times \sum_\sigma \mathbf{R}$. The first sum is infinite but the second is a real vector space of dimension $r_1 + r_2$. Here r_1 and r_2 denote the number of real and complex infinite primes, respectively. We have that $r_1 + 2r_2 = n$ where $n = [F : \mathbf{Q}]$. The degree $\deg(D)$ of an Arakelov divisor D is given by

$$\deg(D) = \sum_P \log(N(P))x_P + \sum_\sigma x_\sigma.$$

The norm of D is given by $N(D) = e^{\deg(D)}$.

An Arakelov divisor $D = \sum_P x_P P + \sum_\sigma x_\sigma \sigma$ is determined by the associated fractional ideal $I = \prod P^{-x_P}$ and by the $r_1 + r_2$ coefficients $x_\sigma \in \mathbf{R}$ at the infinite primes. For every $f \in F^*$ the principal Arakelov divisor (f) is defined by $(f) = \sum_P x_P P + \sum_\sigma x_\sigma \sigma$, where $x_P = \mathrm{ord}_P(f)$ and $x_\sigma = -\log|\sigma(f)|$ or $-2\log|\sigma(f)|$ depending on whether σ is real or complex. The ideal associated to D is the principal fractional ideal $f^{-1}O_F$. By the product formula we have $\deg(f) = 0$. The principal Arakelov divisors form a subgroup of the group $\mathrm{Div}(F)$. The quotient group is called the Arakelov divisor class group or Arakelov-Picard group. It is denoted by $\mathrm{Pic}(F)$. There is an exact sequence

$$0 \longrightarrow \mu_F \longrightarrow F^* \longrightarrow \mathrm{Div}(F) \longrightarrow \mathrm{Pic}(F) \longrightarrow 0.$$

Here μ_F denotes the group of roots of unity in F^*. Since the degree of a principal divisor is zero, the degree map $\mathrm{Div}(F) \longrightarrow \mathbf{R}$ factors through $\mathrm{Pic}(F)$. For $d \in \mathbf{R}$, we denote by $\mathrm{Pic}^{(d)}(F)$ the set of divisor classes of degree d. Forgetting the infinite components, we obtain a surjective homomorphism from $\mathrm{Pic}^{(0)}(F)$ to the ideal class group $Cl(O_F)$ of the ring of integers O_F which fits into an exact sequence

$$0 \longrightarrow V/\phi(O_F^*) \longrightarrow \mathrm{Pic}^{(0)}(F) \longrightarrow Cl(O_F) \longrightarrow 0.$$

Here $V = \{(x_\sigma) \in \prod_\sigma \mathbf{R} : \sum_\sigma x_\sigma = 0\}$ and $\phi : O_F^* \longrightarrow \prod_\sigma \mathbf{R}$ is the natural map $O_F^* \longrightarrow \mathrm{Div}(F)$ followed by the projection on the infinite components. The vector space V has dimension $r_1 + r_2 - 1$ and $\phi(O_F^*)$ is a discrete subgroup of V. In section 4 we show that the group $\mathrm{Pic}^{(0)}(F)$ is compact. This statement is equivalent to Dirichlet's Unit Theorem and the fact that the ideal class group $Cl(O_F)$ is finite. The volume of $\mathrm{Pic}^{(0)}(F)$ is equal to hR, where $h = \#Cl(O_F)$ and R denotes the regulator of F.

It is natural to associate a lattice to an Arakelov divisor $\sum_P x_P P + \sum_\sigma x_\sigma \sigma$. When σ is real, the coefficient x_σ determines a scalar product on $\mathbf{R}$ by setting $\|1\|_\sigma^2 = e^{-2x_\sigma}$. When σ is complex, x_σ determines a hermitian product on $\mathbf{C}$ by setting $\|1\|_\sigma^2 = 2e^{-x_\sigma}$. Taken together, these metrics induce a metric on the product $\mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ by

$$\|(z_\sigma)\|_D^2 = \sum_\sigma |z_\sigma|^2 \|1\|_\sigma^2 .$$

We view, as usual, the number field F as a subset of $\mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ via the embeddings σ. With these metrics the covolume of the lattice I is equal to $\sqrt{|\Delta|}/N(D)$, where Δ denotes the discriminant of F.

It is not difficult to see that the classes of two Arakelov divisors D and D' in $\mathrm{Pic}(F)$ are the same if and only if there is an O_F-linear isomorphism $I \longrightarrow I'$ that is compatible with the metrics on the associated lattices. Therefore, the group $\mathrm{Pic}(F)$ parametrizes isometry classes of lattices with compatible O_F-structures. The cosets $\mathrm{Pic}^{(d)}(F)$ parametrize such lattices of covolume $\sqrt{|\Delta|}e^{-d}$.

For an Arakelov divisor D one defines the Euler-Poincaré characteristic (cf. [Sz]):

$$\chi(D) = -\log(\mathrm{covol}(I)) = \deg(D) - \frac{1}{2}\log|\Delta|,$$

where the covolume is that of the ideal associated to D, viewed as a lattice $I \subset \mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ equipped with the metrics induced by D. Since $\chi(O_F) = -\frac{1}{2}\log|\Delta|$, we have for an Arakelov divisor D that

$$\chi(D) = \deg(D) + \chi(O_F).$$

Variant. We may also consider the following variant of Arakelov theory. Let

$$W = \prod_{\sigma} \mathbf{C}/\mathbf{Z}(1),$$

with $\mathbf{Z}(1) = 2\pi i\mathbf{Z}$ and where the product runs over *all* embeddings. Define a complex conjugation c op W via:

$$c : z_\sigma \mapsto \bar{z}_{\bar{\sigma}}.$$

We have

$$\begin{aligned} W^c &= \{(z_\sigma)_\sigma : \bar{z}_\sigma = z_{\bar{\sigma}}\} \\ &= (\mathbf{R} \times \pi i\mathbf{Z}/2\pi i\mathbf{Z})^{r_1} \times (\mathbf{C}/2\pi i\mathbf{Z})^{r_2}. \end{aligned}$$

Consider Arakelov divisors similar to the ones above but now of the form

$$D = \sum_P x_P\, P + \sum_\sigma \lambda_\sigma\, \sigma, \qquad \text{with } \bar{\lambda}_\sigma = \lambda_{\bar{\sigma}},$$

where the sum is again over all embeddings σ. The principal divisor (f) of an element $f \in F^*$ is defined by

$$(f) = \sum_P \mathrm{ord}_P(f)\, P + \sum_\sigma \log(\sigma(f)).$$

The resulting class group is denoted by $\widetilde{\mathrm{Pic}}(F)$. We have an exact sequence

$$1 \longrightarrow O_F^* \longrightarrow W^c \longrightarrow \widetilde{\mathrm{Pic}}(F) \longrightarrow Cl(O_F) \longrightarrow 1.$$

The real dimension of $\widetilde{\mathrm{Pic}}(F)$ is $n = [F : \mathbf{Q}]$. Comparison with the usual Arakelov-Picard group can be done by mapping the infinite coefficients λ_σ to their real parts. This induces a surjective map $\widetilde{\mathrm{Pic}}(F) \to \mathrm{Pic}(F)$ whose kernel is isomorphic to the group $\prod_\sigma (i\mathbf{R}/2\pi i\mathbf{Z})^c \cong (\mathbf{Z}/2\mathbf{Z})^{r_1} \times (\mathbf{R}/2\pi\mathbf{Z})^{r_2}$ modulo the group μ_F of the roots of unity of F.

Remark. Instead of the full ring of integers one can also consider orders $A \subset O_K$ and consider these as the analogues of singular curves. Define

$$\partial_{A/\mathbf{Z}}^{-1} := \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Z}) = \{x \in K : \mathrm{Tr}(xA) \subset \mathbf{Z}\}.$$

If A is Gorenstein this is a locally free A-module of rank 1 and we can define the canonical divisor to be the ideal $(\partial_{A/\mathbf{Z}}^{-1})^{-1}$ together with the standard metrics. For instance, if $A = \mathbf{Z}[\alpha] = \mathbf{Z}[x]/(f(x))$ then the canonical divisor is $(f'(\alpha))$ by Euler's identity. We can now develop the theory for these orders instead for O_F. But we can also change the metrics on $\partial_{F/\mathbf{Q}}$ which means a change of model at the infinite places. The interpretation of this is less clear.

3. Effectivity and an Analogue of the Theta Divisor

A divisor $D = \sum_P x_P P$ of a smooth complete absolutely irreducible algebraic curve X over a field k is called *effective* when $n_P \geq 0$ for all points P. The vector space $H^0(D)$ of sections of the associated line bundle is defined as

$$H^0(D) = \{f \in k(X)^* : (f) + D \text{ is effective}\} \cup \{0\}.$$

Here (f) denotes the principal divisor associated to a non-zero function $f \in k(X)$. The usual generalization [Sz] of the space $H^0(D)$ to Arakelov divisors $D = \sum_P x_P P + \sum_\sigma x_\sigma \sigma$ of a number field F is given by

$$\begin{aligned} H^0(D) &= \{f \in F^* : \text{all coefficients of the divisor } (f) + D \text{ are non-negative}\} \cup \{0\}, \\ &= \{f \in I \colon \|f\|_\sigma \leq 1 \ \text{ for all infinite primes } \sigma\}. \end{aligned}$$

Here $I = \prod_P P^{-x_P}$ denotes the ideal associated to D. The set $H^0(D)$ is the intersection of a lattice and a compact set. Therefore it is finite and one then puts

$$h^0(D) := \log(\#H^0(D)).$$

This is not very satisfactory. We introduce here a new notion of $h^0(D)$ for Arakelov divisors D. First we introduce the notion of *effectivity* of an Arakelov divisor. Let D be an Arakelov divisor of a number field F. We view F as a subset of the Euclidean space $\mathbf{R}^{r_1} \times \mathbf{C}^{r_2}$ via its real and complex embeddings σ. For an element $f \in F$ we write $\|f\|_D$ for $\|(\sigma(f))\|_D$.

We define the *effectivity* $e(D)$ of an Arakelov divisor D by

$$e(D) = \begin{cases} 0, & \text{if } O_F \not\subset I; \\ \exp(-\pi\|1\|_D^2), & \text{if } O_F \subset I. \end{cases}$$

We have that $0 \leq e(D) < 1$. Explicitly, for $D = \sum_P x_P P + \sum_\sigma x_\sigma \sigma$ we have that $e(D) = 0$ whenever $x_P < 0$ for some prime P. If $x_P \geq 0$ for all P, we have that

$$e(D) = \exp(-\pi\|1\|_D^2) = \exp(-\pi \sum_{\sigma \text{ real}} e^{-2x_\sigma} - \pi \sum_{\sigma \text{ complex}} 2e^{-x_\sigma}).$$

The effectivity of D is close to 1 when each x_σ is large. If one of the x_σ becomes negative however, the effectivity of D tends doubly exponentially fast to 0. For instance, for $F = \mathbf{Q}$ the effectivity of the Arakelov divisor D_x with finite part $\mathbf{Z}$ and infinite coordinate $x_\sigma = x$ is given by the function $e(D_x) = e^{-\pi e^{-2x}}$, see Fig. 1.

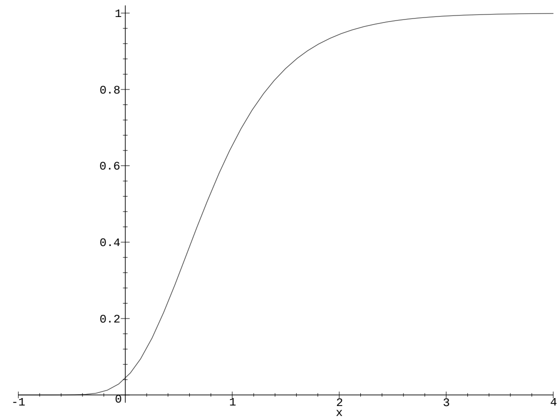

Fig.1. The function $e(D)$ for $\mathbf{Q}$.

We can use the notion of effectivity to define the analogue of the dimension $h^0(D)$ of the vector space $H^0(D)$ of sections of the line bundle associated to a divisor D of a curve. It seems natural to put for an Arakelov divisor D

$$H^0(D) = \{f \in F^* : e((f) + D) > 0\} \cup \{0\}.$$

The effectivity of $(f) + D$ is positive if and only if $f \in I$. Here I denotes the fractional ideal associated to D. Therefore $H^0(D)$ is equal to the infinite group I. The function e attaches a weight ('the effectivity') to non-zero functions $f \in H^0(D)$ viewed as sections of the bundle $O(D)$ via $f \mapsto e((f) + D)$. We consider $H^0(D)$ together with the effectivity as the analogue of the geometric $H^0(D)$. To measure its size, we weight the divisors $(f) + D$ to which the elements f give rise with their effectivity

$$e((f) + D) = e^{-\pi \|1\|^2_{(f)+D}} = e^{-\pi \|f\|^2_D}.$$

When we count elements f rather than the ideals they generate and add the element $0 \in I$ to the sum, we obtain the analogue of the order of the group $H^0(D)$:

$$k^0(D) = \sum_{f \in I} e^{-\pi \|f\|^2_D}.$$

See also [Mo, 1.4]. The analogue of the dimension of $H^0(D)$ is given by

$$h^0(D) = \log \left(\sum_{f \in I} e^{-\pi \|f\|^2_D} \right),$$

which we call the *size* of $H^0(D)$. Since two Arakelov divisors in the same class in $\mathrm{Pic}(F)$ have isometric associated lattices, the function $h^0(D)$ only depends on the class $[D]$ of D in $\mathrm{Pic}(F)$ and we may write $h^0[D]$. This is a function on the Picard group.

There is an analogue of the Riemann-Roch Theorem for the numbers $h^0(D)$. We define the *canonical divisor* κ as the Arakelov divisor whose ideal part is the inverse of the *different* ∂ of F and whose infinite components are all zero. We have that $N(\partial) = |\Delta|$, so that $\deg(\kappa) = \log|\Delta|$. Therefore $\frac{1}{2}\log|\Delta|$ may be seen as the analogue of the quantity $g-1$ that occurs in the Riemann-Roch formula for curves of genus g.

Proposition 1. *(Riemann-Roch) Let F be a number field with discriminant Δ and let D be an Arakelov divisor. Then*

$$h^0(D) - h^0(\kappa - D) = \deg(D) - \frac{1}{2}\log|\Delta|.$$

Proof. This is Hecke's functional equation for the theta function. The lattices associated to D and $\kappa - D$ are $\mathbf{Z}$-dual to one another and the formula follows from an application of the Poisson summation formula:

$$\sum_{f \in I} e^{-\pi\|f\|_D^2} = \frac{N(D)}{\sqrt{|\Delta|}} \sum_{f \in \partial I^{-1}} e^{-\pi\|f\|_{\kappa-D}^2}.$$

This Riemann-Roch theorem is a special case of Tate's [T]. Just like Tate had many choices for his zeta functions, we had a choice for our effectivity function. We chose the function $e^{-\pi\|x\|^2}$, because it gives rise to a symmetric form of the Riemann-Roch theorem and because it leads to the functional equation of the Dedekind zeta function. This is analogous to the geometric case, where there is a unique function h^0. We do not have a definition of $h^1(D)$ without recourse to duality. However, recently A. Borisov [Bo] proposed a direct definition of $h^1(D)$.

Recall that in the geometric case the Jacobian of a curve X of genus g together with its theta divisor $\Theta \subset \mathrm{Pic}^{(g-1)}(X)$ determines the curve and recall that the geometry of the curve (e.g. existence of linear systems) can be read off from the theta divisor. In particular, Riemann showed that the singularities of Θ determine the linear systems on X of degree $g-1$:

$$h^0(D) = \mathrm{ord}_{[D]}(\Theta),$$

where $\mathrm{ord}_{[D]}(\Theta)$ is the multiplicity of Θ at the point $[D] \in \mathrm{Pic}^{g-1}(X)$.

Let $d = \frac{1}{2}\log|\Delta|$. We view the restriction of the function h^0 to $\mathrm{Pic}^{(d)}(F)$ as the analogue of the theta divisor Θ. The function h^0 is a real analytic function on the space $\mathrm{Pic}^{(d)}(F)$. It should be possible to reconstruct the arithmetic of the number field F from $\mathrm{Pic}^{(d)}$ together with this function.

We give some numerical examples.

Example 1. For $F = \mathbf{Q}$ the function h^0 looks as follows. Since $\mathbf{Z}$ has unique factorization, the degree map $\mathrm{Pic}(\mathbf{Q}) \longrightarrow \mathbf{R}$ is an isomorphism. To $x \in \mathbf{R}$ corresponds the divisor D_x that has associated ideal equal to $\mathbf{Z}$ and infinite coordinate $x_\sigma = x$. We have that $h^0(D_x) = \log(\sum_{n\in\mathbf{Z}} e^{-\pi n^2 e^{-2x}})$. This function tends very rapidly to zero when x becomes negative. For instance, for $x = -3$ its value is smaller than 10^{-500}. The Riemann-Roch Theorem says in this case that $h^0(D_x) - h^0(D_{-x}) = x$ for all $x \in \mathbf{R}$.

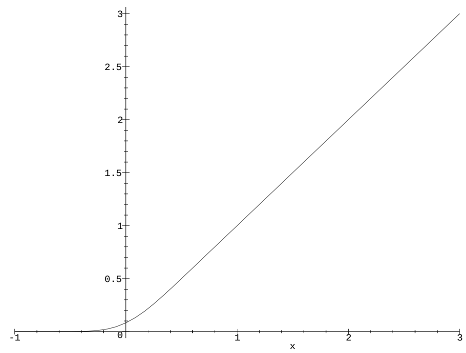

Fig.2. The function $h^0(D)$ for $\mathbf{Q}$.

Since D_x is the Arakelov divisor $x\,\sigma$ where σ is the unique infinite prime of $\mathbf{Q}$, the function $h^0(D_x)$ is the analogue of the function $h^0(nP) = \dim H^0(nP)$, where P is a point on the projective line $\mathbf{P}^1$. In that case one has that $h^0(nP) = \max(0, n+1)$.

Example 2. For a real quadratic field of class number 1, the Picard group is an extension of $\mathbf{R}$ by $\mathrm{Pic}^{(0)}(F)$. The group $\mathrm{Pic}^{(0)}(F)$ is isomorphic to $\mathbf{R}/R\mathbf{Z}$ where R is the regulator of F. We take $F = \mathbf{Q}(\sqrt{41})$ and we plot the function $h^0(D_x)$ on $\mathrm{Pic}^{(0)}(F)$. Here D_x denotes the divisor whose ideal part is the ring of integers O_F of F and whose infinite part has coordinates x and $-x$. It is periodic modulo $R = 4.15912713462618001310854497\ldots$ Note the relatively big maximum for $x = 0$. This phenomenon is the analogue of the geometric fact that for a divisor D of degree 0 on a curve we have $h^0(D) = 0$ unless D is the trivial divisor, for which $h^0(D) = 1$.

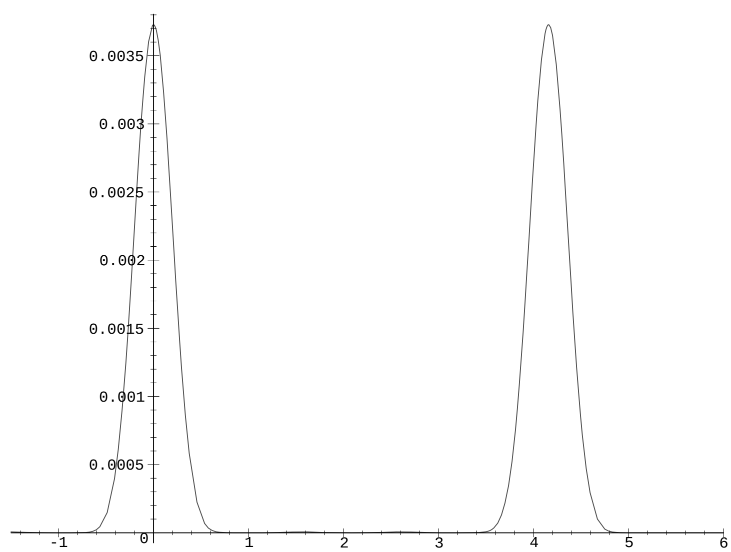

Fig.3. The function $h^0(D)$ on $\mathrm{Pic}^{(0)}(\mathbf{Q}(\sqrt{41}))$.

Example 3. We now take $F = \mathbf{Q}(\sqrt{73})$. The class number of F is 1 and the group $\mathrm{Pic}(F)$ is isomorphic to a cylinder. For every $d \in \mathbf{R}$, the coset $\mathrm{Pic}^{(d)}$ of classes of degree d is a circle whose circumference is equal to the regulator $R = 7.666690419258287747402075701\ldots$ of F. The classes of $\mathrm{Pic}^{(d)}$ are represented by the divisors D_x whose integral part is O_F and whose infinite components are $d/2-x$ and $d/2+x$ respectively. We depict the functions $h^0(D_x)$ restricted to $\mathrm{Pic}^{(d)}(F)$ for $d = \frac{i}{10}\log|\Delta|$ with $i = 0, 1, \ldots, 9$. The Riemann-Roch theorem says that the functions for i and $10-i$ are translates of one another by $\frac{|5-i|}{10}\log(73)$.

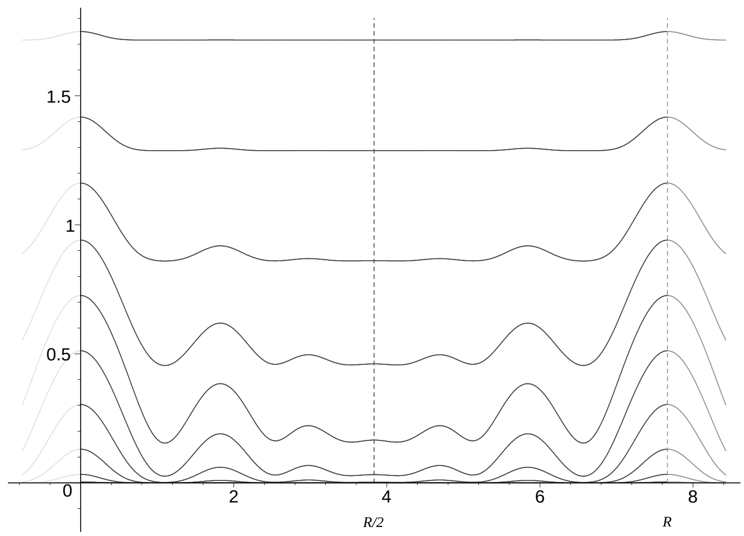

Fig.4. The function $h^0(x)$ for $\mathbf{Q}(\sqrt{73})$.

4. Zeta functions

In this section we recover the zeta function $\zeta_F(s)$ of a number field F as a certain integral of the "effectivity" function on the group of Arakelov divisors. It is a natural adaptation of the definition of the zeta function of a curve over a finite field. Following Iwasawa [Iw] and Tate [Ta], we use the Riemann-Roch Theorem to prove that the topological group $\mathrm{Pic}^{(0)}(F)$ is compact. This gives a proof of the finiteness of the class group and of Dirichlet's Unit Theorem that only makes use of the functional equation of the Theta function and is not based on the usual techniques from geometry of numbers. The geometric analogue of this result is the theorem that for a curve over a finite field the group $\mathrm{Pic}^{(0)}$ is finite. The usual proof of this fact exploits the Riemann-Roch theorem and the fact that the number of *effective* divisors of fixed degree is finite. From our point of view, the proof by Iwasawa and Tate is a natural generalization of this argument.

We briefly discuss the zeta function of an algebraic curve over a finite field. Let X be an absolutely irreducible complete smooth algebraic curve of genus g over $\mathbf{F}_q$. We denote the group of $\mathbf{F}_q$-rational divisors by $\mathrm{Div}(X)$. The degree of a point is the degree of its residue field over $\mathbf{F}_q$. The degree of a divisor $D = \sum_P n_P \cdot P \in \mathrm{Div}(X)$ is given by $\deg(D) = \sum_P n_P \deg(P)$. We let $N(D) = q^{\deg(D)}$ denote the norm of D. The zeta function $Z_X(s)$ of X is defined by

$$Z_X(s) = \sum_{D\geq 0} N(D)^{-s}, \qquad (s \in \mathbf{C},\ \mathrm{Re}(s) > 1).$$

Here the sum runs over the *effective* divisors D of X. It converges absolutely when $\mathrm{Re}\,(s) > 1$. In order to analyze the zeta function, one considers the Picard group $\mathrm{Pic}(X)$ of X. This is the group $\mathrm{Div}(X)$ modulo the group of $\mathbf{F}_q$-rational principal divisors $\{(f) : f \in \mathbf{F}_q(X)^*\}$. Since the degree of a principal divisor is zero, the degree $\deg(D)$ and the norm $N(D)$ only depend on the class $[D]$ of D in the Picard group. By $\mathrm{Pic}^{(d)}(X)$ we denote the divisor classes in the Picard group that have degree d.

We rewrite the sum, by first summing over divisor classes $[D] \in \mathrm{Pic}(X)$ and then counting the effective divisors in each $[D]$, i.e. counting $|D|$, the projectivized vector space of sections

$$H^0(D) = \{f \in \mathbf{F}_q(X)^* : \mathrm{div}(f) + D \geq 0\} \cup \{0\}$$

of dimension $h^0(D)$. We let $k^0(D) = \#H^0(D)$. Since the norm $N(D)$ only depends on the divisor class $[D]$ of D, we write $N[D]$. We have

$$Z_X(s) = \sum_{[D]\in\mathrm{Pic}(X)} \#\{D' \in [D] : D' \text{ effective}\} N[D]^{-s} = \sum_{[D]\in\mathrm{Pic}(X)} \frac{k^0(D)-1}{q-1} N[D]^{-s}.$$

The Riemann-Roch Theorem implies that $k^0(D) = q^{\deg(D)-g+1}$ when $\deg(D) > 2g-2$ and it relates $k^0(D)$ to $k^0(\kappa - D)$. Here κ denotes the canonical divisor of X. These facts imply that one can sum the series explicitly and that $Z_X(s)$ admits a meromorphic continuation to all of $\mathbf{C}$. Moreover, $Z_X(s)$ has a simple pole at $s = 1$ with residue given by

$$\operatorname*{Res}_{s=1} Z_X(s) = \frac{\#\mathrm{Pic}^{(0)}(X)}{(q-1)q^{g-1}\log(q)}.$$

In this way one can actually *prove* that $\mathrm{Pic}^{(0)}(X)$ is finite. In some sense, this proof is an analytic version of the usual argument that there are only finitely many effective divisors of fixed degree. The fact that the zeta function converges for $s > 1$ is a stronger statement that generalizes better to number fields.

Next we turn to the arithmetic situation. First we give a definition of the zeta function associated to a number field F that is natural from the point of view of section 3. Rather than summing $N(I)^{-s}$ over all non-zero ideals I of O_F, we take —just like we did for curves— the sum over all effective Arakelov divisors D. More precisely,

$$Z_F(s) = \int_{\mathrm{Div}(F)} N(D)^{-s} d\mu_{\mathrm{eff}},$$

where μ_{eff} denotes the measure on $\mathrm{Div}(F)$ that weights the Arakelov divisors with their effectivity. To see that this integral converges absolutely for $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1$, we split the integral into a product of an infinite sum and a multiple integral. Writing J for the *inverse* of the ideal associated to D and t_σ for e^{-x_σ}, where the x_σ denote the infinite components of D, we find

$$
\begin{aligned}
Z_F(s) &= \int_{\mathrm{Div}(F)} e(D)N(D)^{-s}\,dD,\\
&= \sum_{0\neq J\subset O_F} N(J)^{-s}\int_{t_\sigma}\left(\prod_\sigma t_\sigma^s\right)\exp(-\pi\sum_{\sigma\text{ real}} t_\sigma^2-\pi\sum_{\sigma\text{ complex}}2t_\sigma)\prod_\sigma\frac{dt_\sigma}{t_\sigma},\\
&= \left(2\pi^{-s/2}\Gamma(s/2)\right)^{r_1}\left((2\pi)^{-s}\Gamma(s)\right)^{r_2}\sum_{0\neq J\subset O_F}N(J)^{-s}.
\end{aligned}
$$

We see that our zeta function is precisely the Dedekind zeta function multiplied by the usual gamma factors. Therefore the integral converges absolutely for $s\in\mathbf{C}$ with $\mathrm{Re}(s)>1$. This way of writing $Z_F(s)$ may serve to motivate the definition of the effectivity e. As we remarked above, a priori other definitions of the effectivity are possible, and this leads to modified zeta functions similar to the integrals in Tate's thesis [T]. Our choice is the one made by Iwasawa in his 1952 letter to Dieudonné [Iw]. In this letter Iwasawa also showed that one can establish the compactness of $\mathrm{Pic}^{(0)}(F)$ by adapting the computations with the zeta function of a curve over a finite field that we indicated above. We briefly sketch his arguments.

Just as we did for curves, we write the zeta function as a repeated integral

$$Z_F(s)=\int_{\mathrm{Pic}(F)}N[D]^{-s}\left(\int_{[D]}e^{-\pi\|1\|_D^2}\,dD\right)d[D].$$

The divisors in the coset $[D]$ have the form $(f)+D$. Only the ones for which f is contained in I, where I is the ideal associated to D, have non-zero effectivity. Therefore

$$
\begin{aligned}
\int_{[D]}e^{-\pi\|1\|_D^2}\,dD &= \sum_{(0)\neq(f)\subset I}e^{-\pi\|1\|^2_{(f)+D}} = \sum_{(0)\neq(f)\subset I}e^{-\pi\|f\|_D^2},\\
&= \frac{1}{w}\left(-1+\sum_{f\in I}e^{-\pi\|f\|_D^2}\right),
\end{aligned}
$$

where w is the number of roots of 1 in O_F. This gives the following expression for $Z_F(s)$:

$$Z_F(s)=\frac{1}{w}\int_{\mathrm{Pic}(F)}(k^0[D]-1)N[D]^{-s}d[D].$$

An application of Proposition 1 (i.e. the arithmetic Riemann-Roch formula) easily implies the folowing

$$
\begin{aligned}
wZ_F(s) = \int_{\substack{[D]\in\mathrm{Pic}(X)\\ N[D]<\sqrt{|\Delta|}}}(k^0(D)-1)N[D]^{-s}d[D] &+ \int_{\substack{[D]\in\mathrm{Pic}(X)\\ N[D]\leq\sqrt{|\Delta|}}}(k^0(D)-1)\frac{N[D]^{s-1}}{\sqrt{|\Delta|}^{(2s-1)}}d[D]\\
&+\frac{\mathrm{vol}(\mathrm{Pic}^{(0)}(F))}{s(s-1)\sqrt{|\Delta|}^s}.
\end{aligned}
$$

For $s \in \mathbf{R}$, $s > 1$, all three summands are positive. Since $Z_F(s)$ converges for $s > 1$, substituting any such s gives therefore an upper bound for the volume of $\mathrm{Pic}^{(0)}(F)$. As explained in section 2 the fact that $\mathrm{Pic}^{(0)}(F)$ has finite volume, implies Dirichlet's Unit Theorem and the finiteness of the ideal class group Cl_F.

In contrast to the situation for curves over finite fields, this time the effective Arakelov divisors D of negative degree or, equivalently, with $N(D) < 1$, contribute to the integrals. However their contribution is very small because they have been weighted with their effectivity. We estimate the integrals in the next section. This estimate is also used there to deduce the meromorphic continuation of $Z_F(s)$.

5. Estimates for $h^0(D)$

In this section we give an estimate on $h^0(D)$ and discuss an analogue of the inequality $h^0(D) \leq \deg(D) + 1$ for divisors D with $\deg(D) \geq 0$. We also describe the relation between $h^0(D)$ and the Hermite constant of the lattice associated to D.

Proposition 2. *Let F be a number field of degree n. Let D be an Arakelov divisor D of F with $\deg(D) \leq \frac{1}{2}\log|\Delta|$ and let $f_0 \in I$ be the shortest non-zero vector in the lattice I associated to D. Then*

$$k^0(D) - 1 \leq \beta e^{-\pi\|f_0\|_D^2}.$$

for some constant β depending only on the field F.

Proof. Let D be a divisor with $N(D) \leq \sqrt{|\Delta|}$, or equivalently, $\deg(D) \leq \frac{1}{2}\log|\Delta|$. We define a positive real number u by $u = (\frac{1}{2}\log(|\Delta|) - \deg(D))/n$ with $n = [F : \mathbf{Q}]$. Define a new divisor D' of degree $\frac{1}{2}\log|\Delta|$ by

$$D' = D + \sum_{\sigma \text{ real}} u\,\sigma + \sum_{\sigma \text{ complex}} 2u\,\sigma.$$

For $0 \neq f \in I$, the ideal associated to D, we have $\|f\|_D^2 - \|f\|_{D'}^2 = \|f\|_D^2(1 - e^{-2u})$, hence

$$\begin{aligned} e^{-\pi\|f\|_D^2} &= e^{-\pi\|f\|_{D'}^2} \cdot e^{-\pi(\|f\|_D^2 - \|f\|_{D'}^2)} \\ &\leq e^{-\pi\|f\|_{D'}^2} \cdot e^{-\pi(\|f_0\|_D^2 - \|f_0\|_{D'}^2)}, \end{aligned}$$

where f_0 is a shortest non-zero vector for D, hence also for D'. We get

$$k^0(D) - 1 \leq (k^0(D') - 1) \cdot e^{-\pi\|f_0\|_D^2} \cdot e^{\pi\|f_0\|_{D'}^2}.$$

In section 4 we showed that the cosets $\mathrm{Pic}^{(d)}(F)$ are compact. The functions $k^0(D') - 1$ and $e^{\pi\|f_0\|_{D'}^2}$ are continuous on the coset of divisor classes of degree $d = \frac{1}{2}\log|\Delta|$, hence are bounded. This implies the Proposition.

Corollary 1. *Let F be a number field of degree n. Let D be an Arakelov divisor D of F with* $\deg(D) \leq \frac{1}{2}\log|\Delta|$. *Then*

$$h^0(D) < k^0(D) - 1 \leq \beta e^{-\pi n e^{-\frac{2}{n}\deg(D)}}$$

for some constant β depending only on the field F.

Proof. Let x_σ denote the infinite components of D and put $t_\sigma = e^{x_\sigma}$. Let $0 \neq f \in I$ be a non-zero vector in I. By the geometric-arithmetic mean inequality we have

$$\begin{aligned}\|f\|^2 &= \sum_{\sigma \text{ real}} t_\sigma^{-2}|\sigma(f)|^2 + \sum_{\sigma \text{ complex}} 2t_\sigma^{-1}|\sigma(f)|^2 \\ &\geq n(|N(f)|^2 \prod_\sigma t_\sigma^{-2})^{1/n} \geq n(|N(I)|^2 \prod_\sigma t_\sigma^{-2})^{1/n} = nN(D)^{-2/n}.\end{aligned}$$

It follows that $e^{-\pi\|f\|_D^2}$ is bounded from above by $e^{-\pi\, n e^{-\frac{2}{n}\deg(D)}}$. Proposition 2 now implies the result.

Corollary 1 is the analogue of the geometric fact that $H^0(D) = 0$ whenever D is a divisor on a curve with $\deg(D) < 0$. Indeed, if $\deg(D)$ becomes negative, then the proposition implies that $h^0(D)$ tends doubly exponentially fast to zero.

The estimate of Corollary 1 leads to the meromorphic continuation of the zeta function $Z_F(s)$. Indeed, since there exists a constant β only depending on the number field F so that $0 \leq k^0(D) - 1 \leq \beta e^{-\pi n N(D)^{-2/n}}$ whenever $N(D) < \sqrt{|\Delta|}$, the two integrals in the last expression for $wZ_f(s)$ in section 4 converge rapidly to functions that are holomorphic in $s \in \mathbf{C}$. This implies that the zeta function extends to a meromorphic function on $\mathbf{C}$. The function $|\Delta|^{s/2}Z_F(s)$ is invariant under the substitution $s \mapsto 1 - s$. It is not difficult to see that the residue of $Z_F(s)$ at the pole in $s = 1$ is given by

$$\frac{\mathrm{vol}(\mathrm{Pic}^{(0)})}{w\sqrt{|\Delta|}}.$$

Corollary 2. *Let F be a number field of degree n and let $w = \#\mu_F$ denote the number of roots of unity in F. Then there is a constant β depending only on F so that*

$$\left|\frac{\log\left(\frac{1}{w}h^0(D)\right)}{\mathrm{covol}(I)^{2/n}} + \pi\gamma(I)\right| < \beta e^{2\deg(D)/n}, \qquad \textit{for all divisors } D \textit{ with } \deg(D) < 0.$$

Here I denotes the ideal associated to D and $\gamma(I)$ denotes the Hermite constant of the lattice associated to I. In other words $\gamma(I)$ is the square of the length of the shortest non-zero vector in I divided by $\mathrm{covol}(I)^{2/n}$.

Proof. Let f_0 denote the shortest non-zero vector of I. We obviously have that $h^0(D) \geq \log(1+we^{-\pi\|f_0\|^2})$. Combining this with the inequality of Prop.2 and using the compactness of $\mathrm{Pic}^{(0)}(F)$ to bound $\|f_0\|^2$ from below, we find that

$$\beta' e^{-\pi\|f_0\|^2} \leq h^0(D) \leq \beta'' e^{-\pi\|f_0\|^2}$$

for certain $\beta', \beta'' > 0$. Dividing these quantities by w, taking the logarithm and dividing by $\mathrm{covol}(I)^{2/n}$ easily implies the corollary.

The Hermite constant only depends on the lattice modulo homothety. The corollary says that the function $-\frac{1}{\pi}\log(\frac{1}{w}h^0(D))\mathrm{covol}(I)^{-2/n}$ approaches the Hermite constant $\gamma(I)$ of I when $\deg(D)$ tends to $-\infty$. In practice the convergence is rather fast. We give a numerical example.

Example. As in section 3, we consider $F = \mathbf{Q}(\sqrt{73})$ and the function h^0 on $\mathrm{Pic}(F)$. We depict a graph of the function

$$B^0(d,x) = -\frac{\log(\frac{1}{2}h^0(D_x))}{2\pi\exp(-d)}$$

for $d = 0$ and for $d = -\frac{1}{2}\log|\Delta|$. Here D_x denotes the divisor whose integral part is O_F and whose infinite coordinate are $\frac{d}{2} + x$ and $\frac{d}{2} - x$ respectively. Its divisor class is contained in $\mathrm{Pic}^{(d)}(F)$. The lattice associated to D_x is denoted by I_x. Its covolume is equal to $\sqrt{|\Delta|}\exp(-d)$. It follows from Corollary 2 that $B^0(d,x)$ tends to $\gamma(I_x)\frac{\sqrt{73}}{2}$ as d tends to $-\infty$. We see that the graphs for $d = 0$ and $d = -\frac{1}{2}\log|\Delta|$ are extremely close. The graph for $d = -\frac{1}{2}\log|\Delta|$ visibly outdoes the one for $d = 0$ only near its local maxima.

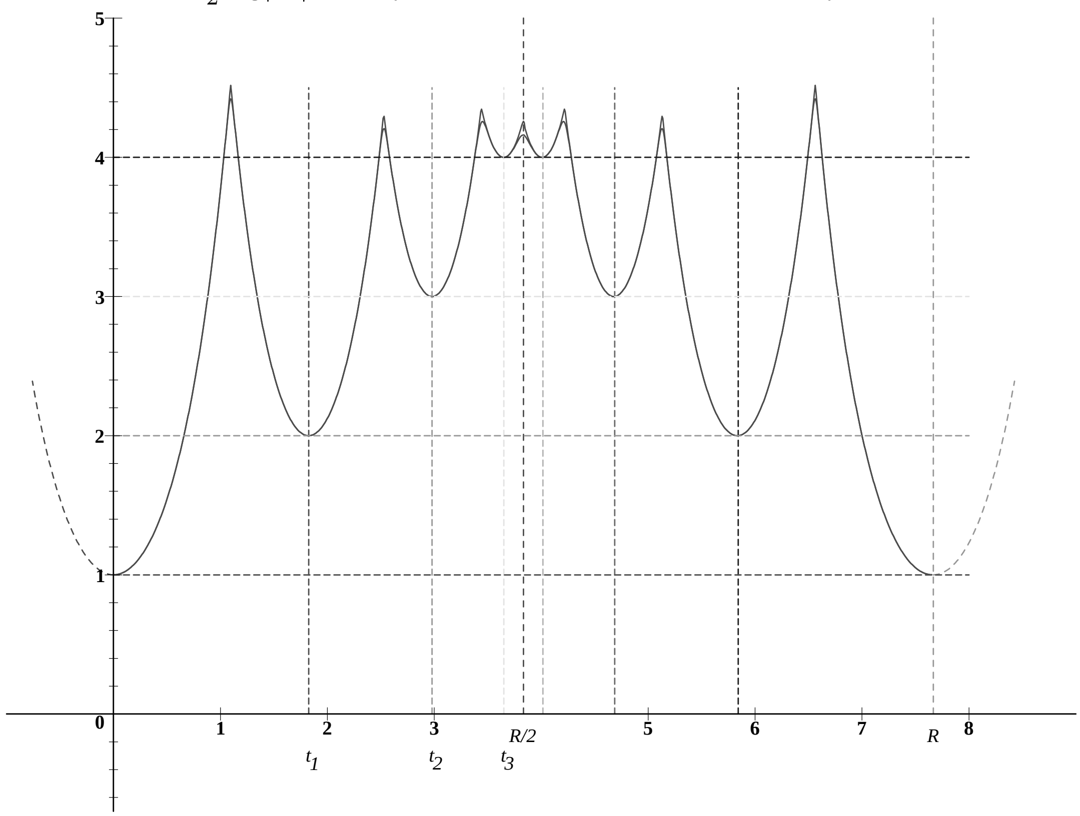

Fig.5. The function $B^0(d,x)$ for $\mathbf{Q}(\sqrt{73})$.

For every $d \in \mathbf{R}$, the function $B^0(d,x)$ is periodic modulo the regulator $R = \log(1068 + 125\sqrt{73}) = 7.66669041925828774740207570 1\ldots$ and symmetric with respect to 0. The graph reflects properties of the lattice $O_F \subset \mathbf{R} \times \mathbf{R}$. The maxima of the function $h^0(D_x)$, and hence the minima of $B^0(x)$, are attained at $\frac{1}{2}\log|f/\overline{f}|$ where f is very near one of the successive minima 1, $(9+\sqrt{73})/2$, $17+2\sqrt{73}$, $(77+9\sqrt{73})/2$ or one of their conjugates. Here $\overline{f}$ denotes the $\mathrm{Gal}(F/\mathbf{Q})$-conjugate of f. Because of our normalization, the value of $B^0(d,x)$ in these points is approximately equal to $|f\overline{f}|$.

In the geometric case the function h^0 on $\mathrm{Pic}^0(X)$ assumes the value 1 in the trivial class and is 0 elsewhere. We conjecture a similar behaviour in the arithmetic case. Roughly speaking, we expect that the function h^0 assumes a pronounced maximal value in or very close to the trivial class $[O_F]$. Computations suggest that this does indeed usually happen. However, if there is a unit in O_F^* of infinite order, all of whose absolute values are relatively close to 1, the function h^0 may assume its maximum value rather far away from $[O_F]$. In this case however it seems that $h^0[O_F]$ is nevertheless rather close to the maximum value of h^0.

If the number field F admits many automorphisms, we dare be more precise.

Conjecture. *Let F be a number field that is Galois over $\mathbf{Q}$ or over an imaginary quadratic number field. Then the function h^0 on Pic^0 assumes its maximum in the trivial class O_F.*

It is easy to see that under these conditions the function h^0 has a local maximum in the trivial class $[O_F] \in \mathrm{Pic}(F)$. The conjecture has been proved by P. Francini [Fr] for quadratic number fields.

The analogue of the geometric fact $\dim H^0(D) \leq \deg(D)+1$ if $\deg(D) \geq 0$ follows from this conjecture:

Proposition 3. *If the function $h^0(D)$ on $\mathrm{Pic}^0(F)$ assumes its maximum in the origin $[O_F]$, then for every D with $\deg(D) \geq 0$ we have $h^0(D) \leq \deg(D) + h^0(O_F)$.*

Proof. We set $u = \deg(D)/n$ and we define a divisor D' of degree 0 by

$$D' = D - \sum_{\sigma \text{ real}} u\,\sigma - \sum_{\sigma \text{ complex}} 2u\,\sigma.$$

Then $\deg(D') = 0$. By a term-by-term comparison for D and D' we get $h^0(\kappa - D) \leq h^0(\kappa - D')$. But then we have

$$\begin{aligned} h^0(D) - \deg(D) &= h^0(\kappa - D) - \log(\sqrt{|\Delta|}) \qquad \text{(by Riemann-Roch)} \\ &\leq h^0(\kappa - D') - \log(\sqrt{|\Delta|}) \\ &= h^0(D') \qquad \text{(by Riemann-Roch)} \\ &\leq h^0(O_F) \qquad \text{(by assumption).} \end{aligned}$$

This proves the Proposition.

Finally we mention Clifford's theorem. This classical result is a statement about the function $h^0(D)$ for divisors D of an algebraic curve of genus g. It says that for every divisor D with $0 \leq \deg(D) \leq 2g-2$ one has that $h^0(D) \leq \frac{1}{2}\deg(D) + 1$. We conjecture that in the arithmetic case the function $h^0(D)$ behaves in a similar way. In the proof of Clifford's Theorem one estimates the sum $h^0(D) + h^0(\kappa - D)$. This suggests that in the arithmetic case one should study the orthogonal sum of the two lattices associated to D and $\kappa - D$ and show that the "size" of this rank 2 lattice is maximal when $D = O_F$.

6. A new invariant

By evaluating the function k^0 at the trivial Arakelov divisor O_F with trivial metrics, we obtain a new invariant of a number field.

Definition. The invariant η of a number field F is defined by

$$\eta(F) := k^0(O_F) = \sum_{x \in \partial^{-1}} e^{-\pi \|x\|^2_{\mathrm{triv}}}.$$

Since, by the Riemann-Roch theorem, we have that

$$h^0(\kappa) = h^0(O_F) + \frac{1}{2}\log|\Delta| = \exp(\eta(F)\sqrt{|\Delta|}),$$

the η-invariant is directly related to $h^0(\kappa)$. The latter should be viewed as the arithmetic analogue of the *genus* of an algebraic curve.

It turns out that the invariant $\eta(F)$ has interesting properties. First we introduce a 'period':

$$\omega = \frac{\pi^{1/4}}{\Gamma(3/4)} = 1.086434811213308014575316121\ldots$$

Proposition 4. *We have $\eta(\mathbf{Q}) = \omega$. If F is a totally real number field or a CM-field of degree n then $\eta(F) = \omega^n \cdot x$ where x is an algebraic number lying in an abelian extension of $\mathbf{Q}(i)$.*

Proof. Let $\theta(\tau)$ be the theta function $\theta(\tau) = \sum_{n \in \mathbf{Z}} e^{\pi i n^2 \tau}$, ($\tau \in \mathbf{C}$, $\mathrm{Im}(\tau) > 0$). The modular function $E_4(\tau)/\theta(\tau)^8$ with E_4 the Eisenstein series of weight 4, is a rational function on the theta group Γ_ϑ, hence assumes by the theory of complex multiplication a rational value at $\tau = i$, which is $3/4$ as a calculation shows. Let ω_0 be the period of the elliptic curve $y^2 = 4x^3 - 4x$ defined by

$$\omega_0 = \int_1^\infty \frac{dx}{\sqrt{4x^3 - 4x}} = \int_0^1 \frac{dt}{\sqrt{1 - t^4}} = \frac{1}{4}B(1/4, 1/2) = \frac{\Gamma(1/4)^2}{4\sqrt{2\pi}},$$

where B is the Beta function. Since this elliptic curve has multiplication by $\mathbf{Z}[i]$, the quotient $\pi^4 E_4(i)/\omega_0^4$ is rational ($= 48$) and the first statement follows by using the distribution relation satisfied by the gamma function. For the second statement consider the quotient

$$\frac{\sum_{x \in O_F} e^{\pi i \tau \|x\|^2}}{\theta(\tau)^n}.$$

This is a modular function with rational coefficients with respect to some congruence subgroup $\Gamma_0(N)$. Hence by the theory of complex multiplication, its value at $\tau = i$ is an algebraic number lying in an abelian extension of $\mathbf{Q}(i)$.

Examples. By machine calculation one finds heuristically:

$$\eta(\mathbf{Q}(i)) = \omega^2 \cdot \frac{2+\sqrt{2}}{4},$$

$$\eta(\mathbf{Q}(\sqrt{-3})) = \omega^2 \cdot \left(\frac{2+\sqrt{3}}{4\sqrt{3}}\right)^{\frac{1}{4}},$$

$$\eta(\mathbf{Q}(\zeta_7+\zeta_7^{-1})) = \omega^3 \frac{7+3\sqrt{7}+3\sqrt{2\sqrt{7}}}{28},$$

$$\eta(\mathbf{Q}(\zeta_5)) = \omega^4 \cdot \frac{23+\sqrt{5}}{20}\sqrt{\frac{1+\sqrt{5}}{10}}.$$

7. The Two-variable Zeta Function

In this section we briefly discuss an alternative expression for the zeta function $Z_X(s)$ of a curve X over $\mathbf{F}_q$ which is suggested by the expression for $(q-1)Z_X(s)$ obtained in section 4. It is related to the two variable zeta function of Pellikaan [P]. We also describe the analogue for number fields.

Proposition 5. *The function*

$$\zeta_X(s,t) = \sum_{[D]\in \mathrm{Pic}(X)} q^{s\,h^0(D)+t\,h^1(D)}$$

converges for complex s,t *satisfying* $\mathrm{Re}(s) < 0$, $\mathrm{Re}(t) < 0$. *It can be continued to a meromorphic function on the whole* (s,t)*-plane. Its restriction to the line* $s+t=1$ *is equal to* $(q-1)q^{(g-1)s}Z_X(s)$.

Proof. By the Riemann-Roch Theorem, the function consists of a finite sum $\sum'$ and two infinite parts:

$$\sum_{[D],\deg(D)>2g-2} q^{s\,(\deg(D)+1-g)} + \sum_{[D],\deg(D)<0} q^{t\,(-\deg(D)+g-1)}.$$

Summing the two series, we find

$$\zeta_X(s,t) = \sum{}' + \#\mathrm{Pic}^{(0)}(X)\left(\frac{q^{sg}}{1-q^s} + \frac{q^{tg}}{1-q^t}\right).$$

This shows that the function admits a meromorphic continuation. Using the Riemann-Roch Theorem to eliminate the terms $h^1(D)$ in the finite sum, one finds that the restriction to the line $s+t=1$ is equal to $(q-1)q^{(g-1)s}Z_X(s)$ as required.

Note that the line $s+t=1$ lies entirely outside the domain of convergence. The following analogue of this result for a number field F can be proved in a similar way.

Proposition 6. *The function*

$$\zeta_F(s,t) = \int_{\mathrm{Pic}(F)} e^{s\,h^0(D)+t\,h^1(D)} d[D]$$

converges for complex s,t *satisfying* $\mathrm{Re}(s) < 0$, $\mathrm{Re}(t) < 0$. *It can be continued to a meromorphic function on the whole* (s,t)*-plane. Its restriction to the line* $s+t=1$ *is equal to* $w\sqrt{|\Delta|}^{s/2}\, Z_F(s)$.

For instance, for $F = \mathbf{Q}$, the function

$$\zeta_{\mathbf{Q}}(s,t) = \int_0^\infty \Theta(x)^s \Theta(1/x)^t \frac{dx}{x}$$

converges for $\mathrm{Re}(s) < 0$, $\mathrm{Re}(t) < 0$. Here $\Theta(x) = \sum_{n\in\mathbf{Z}} e^{-\pi n^2 x^2}$ is a slight modification of the usual theta function. The restriction of the analytic continuation to the line $s+t=1$ is equal to the Riemann zeta function times a gamma factor.

8. The higher rank case.

We can extend the definition of Section 3 to higher rank bundles. Let F be a number field and consider projective O_F-modules M of rank r together with hermitian metrics at the infinite places. By a theorem of Steinitz a projective O_F-module of rank r is isomorphic to $O_F^{r-1} \oplus I$ for some ideal O_F-ideal I. We define an Arakelov bundle of rank r to be a projective O_F-module M of rank r together with a hermitian metric on $M \otimes_\sigma \mathbf{C}$ at the infinite places σ. This defines a metric $\|x\|_M^2$ on $M \otimes_{\mathbf{Z}} \mathbf{R}$ as it did for fractional ideals.

For such an Arakelov bundle we can define an effectivity on the module $H^0(M) = M$ of sections by putting

$$e(x) = \exp(-\pi\|x\|_M^2) \qquad \text{for } x \in M.$$

and define the size $h^0(M)$ of $H^0(M)$ by

$$h^0(M) = \log(\sum_{x\in M} e(x)).$$

We then have as before a Riemann-Roch theorem. Let

$$\chi(M) = -\log \mathrm{covol}(M).$$

An easy induction yields $\chi(M) = \deg(M) + r\chi(O_F)$, where the degree of M is the degree of the Arakelov line bundle $\det(M) = \wedge^r M$. Then the Riemann-Roch theorem is:

$$h^0(M) - h^0(\kappa \otimes M^\vee) = \chi(M),$$

where $M^\vee$ is the dual of M with respect to the Trace map. This is again a consequence of the Poisson summation formula.

In order to have reasonable moduli we consider only *admissible* bundles. This means that the metrics at all infinite places σ are obtained by applying an element of $\mathrm{SL}(r, O_F)$ to the standard metric $\sum z_i \bar{z}_i$ on $M \otimes_\sigma \mathbf{C}$.

For simplicity we assume that the projective O_F-module $\det(M)$ is actually free. We fix a trivialization $\det(M) \cong O_F$. At the infinite places we put a (special) hermitian metric on $M \otimes_\sigma \mathbf{C}$. Since the admissable metrics are defined to be the transforms under $\mathrm{SL}(r, F_\sigma)$ of the standard metric, the data at infinity amount to a point of

$$\mathrm{SL}(M) \backslash \prod_\sigma \mathrm{SL}(M \otimes F_\sigma)/\mathcal{K},$$

where $\mathcal{K}$ is the stabilizer of a fixed allowable metric at all places σ. This quotient can be written as a quotient under $\mathrm{SL}(M)$ of

$$\prod_{\sigma \text{ real}} \mathrm{SL}(r, \mathbf{R})/\mathrm{O}(r) \times \prod_{\sigma \text{ complex}} \mathrm{SL}(r, \mathbf{C})/\mathrm{SU}(r).$$

For example, for $r = 2$ we find the upper half plane for the real σ and the hyperbolic upper half space $\mathrm{SL}(2, \mathbf{C})/\mathrm{SU}(2)$ for the complex infinite places σ. In the particular case that F is totally real and $r = 2$ we find the Hilbert modular varieties associated to F.

As in the geometric case one can now study for a fixed Arakelov line bundle L of degree d and for varying bundles M of rank r with $\chi(M) = 0$, the function $\Psi_L = h^0(M \otimes L)$ on the moduli space of Arakelov bundles of fixed rank r with $\chi(M) = 0$. This can be viewed as the analogue of the theta divisors introduced in the geometric case.

9. Some Remarks on Higher Dimensions

We finish with some remarks about generalizations. The first remark concerns the definition of $h^0(L)$ for a metrized line bundle on an arithmetic surface.

Let X be a smooth projective geometrically irreducible curve over F and assume that X extends to a semi-stable model $\mathcal{X}$ over O_F. Moreover, we assume that we are given probability measures μ_σ on all $X_\sigma(\mathbf{C})$. We consider metrized line bundles L on $\mathcal{X}$ which are provided with hermitian metrics at all primes σ such that their curvature forms are multiples of $d\mu_\sigma$. One can associate to L the cohomology modules $H^0(\mathcal{X}, L)$ and $H^1(\mathcal{X}, L)$. These are finitely generated O_F-modules. It is not possible to define good metrics on them, but Faltings defined a good metric on the determinant of the cohomology, cf. [F], p. 394. We propose to define an effectivity on $H^0(L)$ as follows. For $s \in H^0(L)$ and for each infinite prime σ the norm $\|s\|_\sigma$ is defined on $X_\sigma(\mathbf{C})$; the divisor of s is of the form $D(s) = D_f + \sum_\sigma x_\sigma F_\sigma$ with D_f a divisor on $\mathcal{X}$, with F_σ the fibre over σ and

$$x_\sigma = -\int_{X_\sigma(\mathbf{C})} \log \|s\|_\sigma^2 d\mu_\sigma.$$

We define the effectivity of s by $e(s) = e(D(s))$ with

$$e(D(s)) = \exp(-\pi \sum_{\sigma \text{ real}} e^{-2x_\sigma} - \pi \sum_{\sigma \text{ complex}} 2e^{-x_\sigma})$$

and the size of $H^0(L)$ by

$$h^0(L) = \log(\sum_{s \in H^0(X,L)} e(D(s))).$$

Note that for the trivial line bundle L we get $h^0(O_F)$. Although we do not have a definition of $h^1(L)$, and we can define $h^2(L)$ only via duality $h^2(L) = h^0(L^{-1} \otimes \omega_{\mathcal{X}})$, one could test whether this definition is reasonable for suitable very ample line bundles L. Then $h^1(L)$ and $h^2(L)$ should be exponentially small, hence our $h^0(L)$ should be close to the Faltings invariant $\chi(L)$.

As a final remark we point out that a good notion of effectivity for codimension 2 cycles in the sense of Gillet-Soulé (cf. [G-S]) on an arithmetic surface might yield a way to write the Hasse-Weil zeta function as an integral over a Chow group.

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