The Random Conductance Model: Local times large deviations, law of large numbers and effective conductance

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Louis-Ferdinand Cline, Voyage au bout de la nuit

Abstract

Reversible *random walks in random environment* are called *random walks among random conductances* (RWRC) and they naturally arise in many branches of science as models for physical phenomena. In this thesis we first introduce RWRC, highlighting the connections with electrical networks, and give a substantial background on previous literature. Then, we present a series of original results.

The first one is the proof of an annealed *large deviation principle* (LDP) for the local times of a RWRC forced to stay in a finite domain. We give an explicit expression for the rate function and obtain as a byproduct of the LDP asymptotic formulas for the non-exit probabilities from the given domain. This result has relevant applications in the *parabolic Anderson model* and in the study of *random Schrödinger operators*.

The second result deals with the *law of large numbers* for the endpoint of a RWRC. We show that whenever the α-log moments of the conductances are finite for some $\alpha > 1$, the limiting speed is zero almost surely. On the other hand, finite log moments for $\alpha < 1$ do not imply zero speed: we construct *ad hoc* counterexamples based on geometrical constructions of random trees.

Finally we analyze the fluctuations of the minimum of the *Dirichlet energy* in the random conductance model. This quantity, known as *effective conductance*, describes the total electric current flowing through an electric network and has a central role in *homogenization theory*. We establish a central limit theorem for the effective conductance under the assumptions of Dirichlet boundary conditions and conductances with small ellipticity contrast.

Zusammenfassung

Reversible Irrfahrten in zufälliger Umgebung (*random walks in random environment*, RWRE) werden Irrfahrten unter zufälligen Leitfähigkeiten (*random walks among random conductances*, RWRC) genannt und treten in vielen Teilbereichen des Wissenschaft als naturliche Modelle für physikalische Phänomene auf. In dieser Arbeit führen wir zu erst RWRC ein und heben dabei die Verbindungen zu elektronischen Netzwerken hervor. Gleichzeitig geben wir eine Übersicht über die existierende Literatur. Danach stellen wir unsere Ergebnisse vor.

Zuerst geben wir einen Beweis für ein *annealed* Prinzip der großen Abweichungen (*Large Deviation Principle*, LDP) für die Lokalzeiten einer RWRC, die sich innerhalb einer Region mit festem Durchmesser befindet. Wir geben eine explizite Darstellung der Ratenfunktion an und erhalten durch das LDP asymptotische Formeln für die Wahrscheinlichkeiten der RWRC in der gegebenen Region zu bleiben. Dieses Ergebnis findet wichtige Anwendungen bei der Betrachtung von zufälligen *Schrödinger-Operatoren* und des parabolischen Anderson-Modells (*Parabolic Anderson Model*, PAM).

Das zweite Resultat behandelt das Gesetz der grossen Zahlen für den Endpunkt eines RWRC. Wir zeigen, dass die asymptotische Geschwindigkeit fast sicher gleich Null ist, sobald die α-log Momente der Leitfähigkeiten für ein $\alpha > 1$ endlich sind. Auf der anderen Seite implizieren endliche log Momente nicht Null-Geschwindigkeit: Wir geben *ad hoc* Gegenbeispiele mit Hilfe von geometrischen Konstruktionen zufälliger Bäume.

Schließlich analysieren wir die Fluktuationen vom Minimum der Dirichlet Energie im zufälligen Leitfähigkeits Modell. Diese Quantität, bekannt als effektive Leitfähigkeit (*effective conductance*), repräsentiert den totalen elektrischen Strom, der in einem elektrischen Netzwerk fließt, und spielt eine zentrale Rolle in der Homogenisierungstheorie. Wir beweisen einen zentralen Grenzwertsatz unter den Annahmen von Dirichlet-Randbedingungen und zufällige Leitfähigkeiten mit kleinem elliptischen Kontrast.

Contents

Introduction

Two reasons for studying random walks in random environment

Random walks in random environment were first introduced in the sixties as problems coming from biology. The first track we could find in the literature is the article by Chernov [Che62] in 1962, where the random dynamics on a structure that is also random was introduced as a toy-model for the replication of DNA-chains. In 1972 an analogous mathematical model arose in the context of crystallography in a paper by Temkin (see [Tem72]). The study of these models had then a huge impact in the field of disordered physical media and energy conduction for irregular materials, see e.g. Kirkpatrick [Kir72]. This list could go on for long, but here is a first reason for being interested in random walks in random media: their applications and the request for theoretical results pop out from many different scientific areas, also very distant from each other, including biology, social sciences, theory of communications and, of course, physics.

The second aspect is genuinely mathematical: random walks in random environment proved to be an endless source of beautiful mathematical problems, where beautiful includes (at least) the meanings of interesting, challenging and deep. The tools and the applications to other mathematical subjects testify this fact: Methods from functional analysis, graph theory, theoretical physics, homogenization theory, geometry are indispensable for solving problems that may appear purely probabilistic at a first glance.

In this thesis we will analyze a particular kind of motion in random media, namely the walk associated with the random conductance model. The reason for this choice has the name of *reversibility*: when the walk is starting from the stationary measure, one cannot distinguish whether the time is flowing forward or backwards. At a technical level, this feature allows one to use results from classical harmonic analysis and carry out calculations much more explicitly than in the general case. The random conductance model covers itself a huge amount of different scenarios (the random walk on the percolation cluster is a remarkable example) and is deeply connected with physical electrical resistor networks and with stochastic homogenization theory.

We will deal in particular with three aspects of this model. The first one is the study of the local times of the walk, roughly speaking the amount of time the walk spends in each site. We will prove that, when we take the average on all possible environments, the graphic of the local times approximates a deterministic shape as the time becomes larger and larger. The second one is a law of large numbers. In all the classical examples of random walks among random conductances, the limiting speed of the walk, that is the displacement of the walker over the number of performed steps, is always zero. Is it possible to have a different behaviour? The answer is yes if we allow "very big" conductances: We will construct ingenious examples where the limiting speed is strictly positive almost surely. Finally, we will deal with the longstanding problem of the description of the effective conductance, representing the total electric current flowing through an electric network when the boundary vertices are kept at a given voltage. Under some restriction on the law of the conductances, we will prove the gaussian nature of the fluctuations of the effective conductance around its mean.

The community has spent a lot of efforts for understanding the random conductance model and thousands of pages have been written, with a particular rebirth of the interest and a peak in the production in the last five years. Nevertheless the field is still fertile, and many misteries ask to be understood.

Structure of the thesis

In order to get no one lost, here is the organization of the thesis.

Chapter 1 is divided in two main sections. The first one introduces the random conductance model (Section 1.1.1), the different types of random walks that can be defined on it (Sec. 1.1.2) and its connections with electrical networks (Sec. 1.1.3). The second part (Sec. 1.2) collects previous results in the field and introduces the original ones presented in the thesis. After a general overview of known results, the attention is focused on large deviations for random walks in random environment (Sec. 1.2.1) and a large deviation principle for the local times of a random walk among random conductances (RWRC) is stated (Sec. 1.2.2). The second topic is the law of large numbers for the end-point of an RWRC (Sec. 1.2.3) and Section 1.2.4 provides moment conditions on the conductances for having non-zero limiting speed of the walk. The final part of Section 1.2 deals with homogenization theory and the study of the effective conductance (Sec. 1.2.5). A central limit theorem for the latter is established in Section 1.2.6.

In Chapter 2 we present the complete proof of the large deviation principle for the local times, following the lines of the article [KSW12]. We also add to the article a final section (Sec. 2.4) on recent developments and possible future research on this subject.

Chapter 3 deals with the proof of the results of Section 1.2.4 as in [BS12]: Section 3.1 gives a sufficient condition for the RWRC to have zero speed, while Sections 3.2 and 3.3 give the construction of the counterexamples when such conditions are not fulfilled.

The proof of the central limit theorem for the effective conductance is the main object of Chapter 4, reporting the results obtained in the paper [BSW12].

Chapter 1

Model and results

1.1 The random conductance model and the related walks

1.1.1 The model

Take a graph $G = (V, E)$, where V is the set of its vertices and E the set of its edges. The graphs we consider will not contain double edges or loops and the edges will be undirected. In fact, the support for our conductance model will always be $\mathbb{Z}^d$ or a subset of $\mathbb{Z}^d$, unless explicitly specified otherwise.

Let $(\Omega, \mathcal{F})$ be the couple of the product space $\Omega := [0, \infty)^E$ of all possible configurations of non-negative weights assigned to the bonds of the graph and the relative Borel sigma-algebra $\mathcal{F}$. Each element $\omega \in \Omega$ is a collection of numbers $\{\omega_{xy}\}_{x \sim y}$, called *conductances*, where x, y are two elements of V and the symbol "$\sim$" means that there exists a bond $b = (x, y) \in E$ connecting x and y. The name 'conductance' has to do with the strict relation between this model and electrical networks, which we will analyze in detail in Section 1.1.3. Depending on the situations, we will use for convenience also the notations $\omega_{x,y}$, $\omega(x, y)$, ω_b or $\omega(x, x \pm e_i)$ in the lattice case, where e_i is an element of the canonical base of $\mathbb{Z}^d$, for $i = 1, ..., d$. By definition, these weights are symmetric, that is $\omega_{xy} = \omega_{yx}$ for all $(x, y) \in E$. As a convention, $\omega_{xy} = 0$ when $(x, y) \notin E$.

Let P be a probability measure on Ω. We say that P is *elliptic* if for all $(x, y) \in E$ one has $P(\omega_{xy} > 0) = 1$. We call P *strongly elliptic* if the support of each conductance is bounded away from zero and infinity, that is, there exists $\lambda > 0$ such that

$$P(\lambda \leq \omega_{xy} \leq \tfrac{1}{\lambda}) = 1. \tag{1.1}$$

In the case of discrete lattice model, we call the *shift* by a vector $z \in \mathbb{Z}^d$ of a configuration $\omega \in \Omega$ the map $\tau_z : \Omega \to \Omega$ such that for all $x \sim y \in \mathbb{Z}^d$ we have $(\tau_z \omega)_{x,y} = \omega_{x+z,y+z}$. We say that P is *shift-invariant* if for any event $A \in \mathcal{F}$ and $z \in \mathbb{Z}^d$ we have

$$P(A) = P(\tau_z A),$$

where of course $(\tau_z A) := \{\omega \in \Omega : \tau_{-z}\omega \in A\}$. Recall also that P is said *shift-ergodic* if, whenever $P(\tau_z A) = P(A)$ for every $z \in \mathbb{Z}^d$ and some event A, then $P(A) \in \{0, 1\}$.

We indicate with E the expectation with respect to P (in Chapter 2 we will make also use of the notation $\langle \cdot \rangle$ for the same object).

1.1.2 RWRC: discrete vs continuous time

From now on we will restrict, unless explicitly stated, to the d-dimensional euclidean square lattice $\mathbb{Z}^d$, where the conductances are present only on edges connecting nearest neighbours, that is, $x \sim y$ if and only if $\|x-y\|_1 = 1$, where $\|\cdot\|_1$ is the usual ℓ^1 distance.

Given a realization $\omega \in \Omega$ of conductances we can introduce many kinds of random walks exhibiting different behaviours. We illustrate here the three most studied walks:

1) the discrete-time RWRC;

2) the variable-speed random walk (VSRW);

3) the constant-speed random walk (CSRW).

1) The *discrete-time random walk among random conductances* performes one step at each interval of time and chooses its next position proportionally to the weight of the bond that brings it there. More precisely, P_z^ω is the law of the random walk starting in $z \in \mathbb{Z}^d$ and with transition probabilities given by

$$P_z^\omega(X_{n+1} = y \,|\, X_n = x) = \frac{\omega_{xy}}{\pi_\omega(x)}, \tag{1.2}$$

for $y \sim x$, $n \in \mathbb{N}_0$ and where

$$\pi_\omega(x) = \sum_{y' \in \mathbb{Z}^d \colon y' \sim x} \omega_{xy'}. \tag{1.3}$$

If $\pi_\omega(x) = 0$ than the random walk stands still forever when present at x.

2) The *VSRW* is the continuous-time process $(X_t)_{t \geq 0}$ generated by the modified discrete Laplace operator Δ^ω. This is given by

$$\Delta^\omega f(x) = \sum_{y \in \mathbb{Z}^d,\, y \sim x} \omega_{xy}(f(y) - f(x)) \qquad f : \mathbb{Z}^d \to \mathbb{R},\, x \in \mathbb{Z}^d. \tag{1.4}$$

Note that because of the symmetry of the conductances, Δ^ω is a symmetric operator. Described in words: When at point $x \in V$, the VSRW waits an exponential time with parameter $\pi_\omega(x) = \sum_{w \sim x} \omega_{xw}$ (i.e., with mean $1/\pi(x)$) and then jumps to the next point according to (1.2).

3) The *CSRW* behaves exactly like the VSRW, but the waiting times are exponential random variables with parameter 1 at each point. The generator is then

$$\tilde{\Delta}^\omega f(x) = \sum_{y \in \mathbb{Z}^d,\, y \sim x} \frac{\omega_{xy}}{\pi_\omega(x)}(f(y) - f(x)) \qquad f : \mathbb{Z}^d \to \mathbb{R},\, x \in \mathbb{Z}^d. \tag{1.5}$$

In all the three models, we call the measure on paths for a fixed environment $\omega \in \Omega$ the *quenched measure*. We will be also interested in taking the expectation with respect to the conductances of such a measure. For an event $A \in \mathcal{F}$, an event B on the space of trajectories of the random walk and a starting vertex $x \in \mathbb{Z}^d$, we define the *annealed measure* as

$$\mathbb{P}_x(A \times B) = \int_A P_x^\omega(B) \mathrm{d}P(\omega), \tag{1.6}$$

with the convention $\mathbb{P}_x(A) = \int_\Omega P_x^\omega(A) \mathrm{d}P(\omega)$.

Averaging over the conductances has the advantage to 'regularize' the environment, in the sense that the main contribution to the annealed measure is given by typical configurations of conductances. Furthermore, if P is translation invariant, then $\mathbb{P}.$ is also translation invariant. On the other hand under this measure the process loses the Markov property: information from the past can indeed specify characteristic of the discovered environment and influence the probabilities of the future steps.

One of the characteristics that causes anomalous behaviours of RWRC's is the presence of traps. The different nature of traps in the discrete, constant-speed and variable-speed cases is one of the main differences between the three models. For example, an edge with a huge conductance can trap both the discrete time walk and the CSRW: both will go back and forth on that edge for a long time with high probability. This effect becomes particularly strong if the conductances are not bounded from above. On the other hand, the VSRW will jump many times over the edge (in average as many times as the other two processes), but it will do it so fast that the total effect will be negligible from the point of view of the time spent there.

The property that makes the RWRC (in discrete or continuous time) so important among the huge family of Random Walks in Random Environment (RWRE) is reversibility. A Markov chain is said *reversible* with respect to a measure μ if, choosing the starting point according to μ, the distribution of $(X_0, X_1, ..., X_{n-1}, X_n)$ is equal to that of $(X_n, X_{n-1}, ..., X_1, X_0)$. That is, it is not possible to recognize whether the chain is running forward or backwards in time.

Calling π_ω the (not necessarily finite) measure on V described in (1.3) one can easily check that:

$$\pi_\omega(x) P^\omega(x, y) = \pi_\omega(x) \frac{\omega_{xy}}{\pi_\omega(x)} = \omega_{xy} = \omega_{yx} = \pi_\omega(y) P^\omega(y, x). \tag{1.7}$$

This is known as *detailed balance equation* and iterating gives reversibility in the discrete-time setting.

Note also that every chain on a graph that is reversible with respect to some measure μ and with transition probabilities $(p(x, y))_{x,y \in V}$ can be represented as a random walk among random conductances, setting $\omega_{xy} = \mu(x) p(x, y)$.

In continuous time the notion of reversibility translates into

$$P_x^\omega(X_t = y) = P_y^\omega(X_t = x), \qquad \forall t \geq 0, \text{ for the VSRW},$$

and

$$\frac{P_x^\omega(X_t = y)}{\pi_\omega(y)} = \frac{P_y^\omega(X_t = x)}{\pi_\omega(x)}, \qquad \forall t \geq 0, \text{ for the CSRW}.$$

1.1.3 Electrical networks

In [DS84] Doyle and Snell present in a very readable way the deep connections between our model and real-world electrical networks, justifying our use of terms borrowed from physics literature. A more rigorous, though simple, introduction to this relation is given by [LP12], Chapters 2 and 9.

For this section we go back considering a general finite connected graph $G = (V, E)$ and look at it as an electrical network in the physical sense, where edges are made of conducting wires. Two distinct sets of vertices, $A, Z \subset V$ (it is easier to think of singletons), are attached to a battery that keeps a constant difference of, say, a unit voltage between the two sets (that is, $v(a) = 1$ for all $a \in A$ and $v(z) = 0$ for $z \in Z$, where v is the voltage at a given point). Every edge $(x, y) \in E$ has a resistance r_{xy} and therefore a conductance $c_{xy} = \frac{1}{r_{xy}}$. We call $i_{xy} = \frac{v(x)-v(y)}{r_{xy}}$ the *current* flowing from x to y (Ohm's Law). Summing this quantity over all the neighbours of x must give 0, since Kirkhoff's Laws from classical Physics state that the current flowing into any point $x \in V$, $x \notin A \cup Z$, must be the same as the current flowing out of it. A little algebra shows therefore that the voltage v at x is the weighted mean of the voltage of the neighbours of x, i.e. is *harmonic* in the sense that

$$v(x) = \sum_{y \sim x} \frac{c_{xy}}{\pi(x)} v(y).$$

For the random walk among conductances $\omega_{xy} = c_{xy}$, the corresponding quatity is

$$p(x) = P_x^{\omega}(\text{the random walk reaches } A \text{ before } Z).$$

This is in fact also an harmonic function that assumes values 1 in A and 0 in Z, and by unicity $v(x) = p(x)$ for all $x \in G$.

The function $i : E \to \mathbb{R}$ is a *flow* between A and Z, that is, a function f on the directed pairs of neighbours in G satisfying $f(x, y) = f(y, x)$, and $\sum_{y \sim x} f(x, y) = 0$ for $x \notin A \cup Z$. Consider now the case $A = \{a\}$. Let p_{esc} be the probability, starting in a, of returning to a before "escaping", that is, reaching a point in Z. Then the probabilistic interpretation of the current is the following: i_{xy} is proportional to the expected number of times that a walker, starting at a and wondering around until reaching Z, will jump over the edge from x to y minus the times that he jumps from y to x. The constant of proportionality is exactly the *effective resistance* R_{eff} of the network, given by

$$R_{\text{eff}}^{-1} = p_{\text{esc}} \sum_{y \sim a} \omega_{ay} = \frac{1}{\sum_{y \sim a} i_{ay}}. \tag{1.8}$$

We will call the quantity in (1.8) also *effective conductance*, C^{eff}. The name comes from the fact that we could substitute the entire circuit with a unique wire of conductance C^{eff} between a and Z. Note that C^{eff} does not depend on the difference of voltage between a and Z, it is an intrinsic property of the network.

We can now regulate the voltage at a so that the total current coming out of a (i.e., $\sum_{y \sim a} i_{ay}$) is 1. Among all the flows with this property (called unit flows), the current is the one that minimizes the dissipation of energy, defined as

$$E_{\text{dis}}(\theta) = \tfrac{1}{2} \sum_{x,y} \theta(x,y)^2 r_{xy}. \tag{1.9}$$

This is known as *Thomson's principle* ([TT79]).

Another probabilistic interpretation of the voltage can be given via the *Green function*. Let $\mathcal{G}_Z(x,y)$ be the number of expected visits to $y \in V$ before touching set Z for the random walk started in $x \in V$. Then, setting the voltage to have a unit current flow from a to Z, the voltage at $x \in V$ is equal to

$$v(x) = \frac{\mathcal{G}_Z(a,x)}{\sum_{y \sim x} \omega_{xy}}.$$

The previous notions can be extended to infinite connected graphs, though losing a bit of their appealing realism. It is important to underline that the parallel beteen real electrical network and the random conductance model is not purely aesthetic. The techniques inherited from the physics literature can lead to very important theoretical results. One remarkable example is the theorem that says that a random walk on an infinite connected graph G with conductances $\{\omega_{xy}\}$ is transient if and only if there exists a unit flow of finite E_{dis} energy (in this setting we have called a unit flow from a point $a \in V$ a function f satisfying the previous properties with $A = \{a\}$, but $Z = \emptyset$, and with $\sum_{y \sim a} f(a,y) = 1$). Another example, closer to the content of this thesis, is [Ros12], where an "energy dissipation approach" is used for proving a result close to that of 1.2.6.

1.2 Previous and new results

The random conductance model has gained growing attention from the community in the last decade. One of the reasons is its relation to other important models in Statistical Physics, such as the gradient fields (e.g., [BS11]), reinforced random walk ([MR09, ACK12, ST12] among the others) and percolation theory (e.g., [GKZ93], [BB07], [MP07], [BDCKY11]).

RWRC offers a wide range of problems to work on, but in the following chapters we will mainly deal with three topics: large deviations for the local times of the RWRC, the law of large numbers (LLN) for the endpoint of the walk and the central limit theorem (CLT) for the effective conductance. In the next subsections we present in greater detail the previous works on these particular subjects, but now a little detour for a more general overview is incumbent. Refer to [Bis11] for a quite complete picture of the known results and of the open problems on the RCM and its random walk.

The questions that are traditionally most studied for the RWRC are those of *recurrence* and *transience*, *heat kernel estimates* and *functional central limit theorems* (FCLT). References about the first subject include [GKZ93] for the random walk on the infinite percolation cluster (see below for a precise definition of it), [Ber02] for the random walk on the infinite cluster of long-range percolation in dimensions $d = 1, 2$, [ACK12] and [ST12] for recent results on the closely related model of edge reinforced random walk. Heat kernel estimates have been addressed, e.g., in [Bar04] and [MR04] for the percolation cluster, in [Del99] for the uniformly elliptic case, in [BD10] for i.i.d. conductances bounded from below but not from above and in [BBHK08] for general distributions of conductances between 0 and 1.

Much efforts have been put in the study of FCLTs, that is, the convergence in some sense, after a space time rescaling, of the random walk to some process in continuous space (often a Brownian motion). The case of uniformly elliptic conductances is somewhat the easiest to study. In [SS04] the authors give the first complete proof of a quenched functional central limit theorem (QFCLT) for any dimension when the conductances are i.i.d.. An extension to stationary symmetric ergodic environments can be found as a particular case in Section 6 of [BD10]. In [BD10] the QFCLT is derived for i.i.d. conductances bounded away from 0 but not from infinity, i.e. they can reach arbitrarily high values. When the expectation of the conductances is infite, the CSRW rescaled in the proper way converges in law to a fractional kinetic motion ([BČ11]).

Of great interest is the case of conductances that can assume value zero. This means that the random walk is not allowed to cross certain bonds and the labyrinth-type geometry of the environment makes the analysis of the model more complicated. In order to make things meaningful in this framework, one has to assume in dimension d that $\mathbb{P}(\omega_{xy} = 0) > p_c(d)$, where $p_c(d)$ is the critical probability for bond percolation in $\mathbb{Z}^d$. This guarantees the existence of an infinite cluster and one has only to condition on the event that the starting point of the random walk lies indeed in this infinite component. In the case of the simple random walk on the supercritical percolation cluster (i.e. the conductances can assume only values 0 or 1) a QFCLT has been proven first in [SS04] for $d \geq 4$ and then in [BB07] and [MP07] at the same time for the remaining dimensions. The invariance principle can be also proven when the support of the (i.i.d.) conductances is more generally contained between 0 and 1, see [BP07] and [Mat08]. In [BBHK08] (for $d \geq 5$) and [BB12] (for $d = 4$) the authors study the probabilities of return to the origin after $2n$ steps of the walk and prove that, quite surprisingly, the usual Gaussian upper estimates for the heat kernel do not hold. As a consequence, we find ourselves in the unusual case where a CLT holds but a local CLT does not. Finally, [ABDH10] deals basically with all the previous setting at once: The quenched invariance principle is here proven for the CSRW and VSRW with general i.i.d. conductances with values in $[0, \infty)$.

A weaker or annealed version of the previous results goes back to the seminal work [DMFGW89], where the authors assume conductances which are translation invariant, ergodic and have finite mean.

In the last year a couple of articles appeared showing a uniform quenched CLT ([GP12a]) and a conditioned (the random walk is forced to stay positive) quenched CLT ([GP12b]) in the case of a RWRC whose jumps are unbounded (with polynomial bounds on the tails of the jumps) in one dimension. These results have interesting applications to Knudsen billiards (see [CP12]), modeling problems of transport and diffusion in nanotubes.

Another recent topic that is gaining growing attention is that of random walks in dynamic random environment. For dynamic random conductances, i.e. the weights on the bonds may vary in time, an invariant principle has been proven in [And12] for stationary ergodic conductances uniformly bounded from above and below, polynomially mixing in space and time.

1.2.1 Large deviations and the local times

Large deviations is the study of unlikely events, the probability of which decreases exponentially fast as time passes. [DZ10] is a prominent reference for the general theory, while [dH00] offers a smoother introduction to the topic for non-experts. We recall the general definition of a Large Deviation Principle (LDP) in a formulation similar to that of Varadhan [Var66].

Definition 1.1. *Let (E,d) a metric space and $\mathcal{B}_E$ the relative Borel-σ-algebra. Let $\{\gamma_n\}_{n\in\mathbb{N}}$ a sequence of positive numbers with $\gamma_n \to \infty$ and $I : E \to [0,\infty]$ a function such that $I \not\equiv \infty$. We say that a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ satisfies a Large Deviation Principle (LDP) with rate function I and speed γ_n if*

$$(i) \quad \liminf_{n\to\infty} \frac{1}{\gamma_n} \log \mu(O) \geq -\inf_{x\in O} I(x) \qquad \text{for all open sets } O \subset E$$

$$(ii) \quad \limsup_{n\to\infty} \frac{1}{\gamma_n} \log \mu(C) \leq -\inf_{x\in C} I(x) \qquad \text{for all closed sets } C \subset E$$

(iii) The level sets $\Phi(s) := \{x \in E : I(x) \leq s\}$ are compact.

There is a large amount of literature dedicated to large deviations statements for RWRE's, but the results are mostly dedicated to the limiting speed of the random walks (see [GdH94], [CGZ00], [Var03] and many many others). LDP's for other features of the walks, which usually are well understood in the non-random-environment case, are much less addressed.

One significant example is that of the local times, or occupation times, of the process. Define

$$\ell_t(z) := \int_0^t \delta_z(X_s)\,\mathrm{d}s \qquad z \in \mathbb{Z}^d,\ t > 0, \tag{1.10}$$

to be the time spent by a random walk $(X_s)_{s\geq 0}$ in the point $z \in \mathbb{Z}^d$ up to time t. Here $\delta_\cdot(\cdot)$ denotes the usual Dirac delta, assuming value 1 when its two arguments are the same and 0 otherwise.

Consider now $(X_s)_{s\geq 0}$ to be a random walk among *fixed* conductances $\{\omega_{xy}\}_{x,y\in\mathbb{Z}^d}$. Take a box $B_L = [-L,L]^d \cap \mathbb{Z}^d$. An interesting question is: What is the probability that the walker starting at the origin will not leave B_L up to time $T \gg 0$, that is, $\sum_{z\in B_L} \ell_T(z) = T$? But one can be much more precise and ask: What is the probability that the normalized local time is close in some sense to a given function g with support in B_L, i.e., $\frac{1}{T}\ell_T \approx g$? The answer can be found, going back to the eighties, in a series of seminal papers by Donsker and Varadhan [DV75b, DV75a, DV76, DV83] on the west side of the world and Gärtner [G77̈] on the east side, which built the basis for the theory of large deviations for the occupation time measures of various types of Markov processes.

Theorem 1.2. *The sequence $P_0^\omega(\frac{1}{t}\ell_t \in \cdot \,|\, \mathrm{supp}(\ell_t) \subset B_L)$ satisfies a large deviation principle on the space $\mathcal{M}_1(B_L)$ of the probability measures on B_L with speed t and rate function I_L given by*

$$I_L(\mu) = \big(-\Delta^\omega \sqrt{\mu}, \sqrt{\mu}\big) - C_L = \sum_{x,y\in\mathbb{Z}^d:\, x\sim y} \omega_{xy}\big(\sqrt{\mu}(y) - \sqrt{\mu}(x)\big)^2 - C_L, \tag{1.11}$$

where

$$C_L = \inf_{\mu\in\mathcal{M}_1(B_L)} \sum_{x,y\in B_L:\, x\sim y} \omega_{xy}\big(\sqrt{\mu}(y) - \sqrt{\mu}(x)\big)^2. \tag{1.12}$$

Here we have trivially extended μ to the whole $\mathbb{Z}^d$ and therefore we can include also in the sum in (1.12) *all $x, y \in \mathbb{Z}^d$, with $x \sim y$.*

A proof of the theorem similarly stated can be found in [K06̈].

It was not possible for us to find references addressing the same problem when the conductances are also random.

1.2.2 Local times large deviations for an RWRC

As pointed out in Section 1.2.1, no LDP for the local times $\ell_t(z) = \int_0^t \delta_{X_s}(z)\,\mathrm{d}s$ had been proven before in the case of random weights on the bonds. In [KSW12] we derive the annealed analogon of Theorem 1.2 in the random environment setting. As a byproduct we obtain the asymptotics of the non-exit probability from a finite set $B \subset \mathbb{Z}^d$ (not necessarily a box) and the lower tails of the principal (i.e., smallest) eigenvalue $\lambda^\omega(B)$ of $-\Delta^\omega$ in B with zero boundary condition, which can be seen as a *Schrödinger operator*.

We concentrate on the interesting case where the conductances are positive, but can assume arbitrarily small values. Here the annealed behaviour comes from a combined strategy of the conductances and the walk, and the description of their interplay is the focus of our study. Loosely speaking, the optimal joint strategy of the conductances and the walk to meet the non-exit condition $X_{[0,t]} \subset B$ for large t is that the conductances assume extremely small t-dependent values and the walker realizes very large t-dependent holding times and/or trajectories that do not leave B. We will informally describe this picture in greater detail.

Our main assumption on the i.i.d. field ω of conductances is that, for any $\{x, y\} \in E$,

$$\omega_{xy} \in (0, \infty) \qquad \text{and} \qquad \operatorname{essinf}\,(\omega_{xy}) = 0. \tag{1.13}$$

More specifically, we require some regularity of the lower tails, namely the existence of two parameters $\eta, D \in (0, \infty)$ such that

$$\log \Pr(\omega_{xy} \leq \varepsilon) \sim -D\varepsilon^{-\eta}, \qquad \varepsilon \downarrow 0. \tag{1.14}$$

That is, the edge weights can attain arbitrarily small values with prescribed probabilities.

Our main theorem is the following large deviation principle for the normalised local times before exiting B. That is, we restrict to the event $\{X_{[0,t]} \subset B\} = \{\operatorname{supp}(\ell_t) \subset B\}$. By

$$E_B := \{\{x, y\} \colon x \in B, y \in \mathbb{Z}^d, y \sim x\} \tag{1.15}$$

we denote the set of edges connecting the sites of B with their neighbours both in B and outside.

Theorem 1.3 (Annealed LDP for $\frac{1}{t}\ell_t$)**.** *Assume that ω satisfies* (1.13) *and* (1.14)*. Fix a finite connected set $B \subset \mathbb{Z}^d$ containing the origin. Then the process of normalized local times, $(\frac{1}{t}\ell_t)_{t>0}$, under the annealed sub-probability law $\langle \mathbb{P}_0^\omega(\,\cdot\, \cap \{X_{[0,t]} \subset B\}) \rangle$ satisfies an LDP on $\mathcal{M}_1(B)$, the space of probability measures on B, with speed $t^{\frac{\eta}{\eta+1}}$ and rate function J given by*

$$J(g^2) := K_{\eta,D} \sum_{\{x,y\} \in E_B} |g(y) - g(x)|^{\frac{2\eta}{\eta+1}}, \qquad g \in \ell^2(\mathbb{Z}^d), \operatorname{supp}(g) \subset B, \|g\|_2 = 1, \tag{1.16}$$

where $K_{\eta,D} = \left(1+\frac{1}{\eta}\right)(D\eta)^{\frac{1}{\eta+1}}$.

The proof of Theorem 1.3 is given in Chapter 2. More explicitly, it says

$$\liminf_{t\to\infty} t^{-\frac{\eta}{\eta+1}} \log \left\langle \mathbb{P}_0^\omega\Big(\tfrac{1}{t}\ell_t \in O, X_{[0,t]} \subset B\Big)\right\rangle \geq \; -\inf_{g^2\in O} J(g^2) \qquad \text{for } O \subset \mathcal{M}_1(B) \text{ open,} \tag{1.17}$$

$$\limsup_{t\to\infty} t^{-\frac{\eta}{\eta+1}} \log \left\langle \mathbb{P}_0^\omega\Big(\tfrac{1}{t}\ell_t \in C, X_{[0,t]} \subset B\Big)\right\rangle \leq \; -\inf_{g^2\in C} J(g^2) \qquad \text{for } C \subset \mathcal{M}_1(B) \text{ closed,} \tag{1.18}$$

and that the rate function J has compact level sets. Our convention is to extend any probability measure on B trivially to a probability measure on $\mathbb{Z}^d$; note the zero boundary condition in B that is induced in this way.

Interestingly, we can see how the boundary case $\eta = \infty$ formally reconstructs the result of Theorem 1.2.

Remark 1.4. *As can be seen from its proof, Theorem 1.3 holds literally true if* $\mathbb{Z}^d$ *is replaced by an (infinite or finite) graph and* B *by some finite subgraph.*

A heuristic explanation of the speed and rate function is given in Section 2.1. It turns out there that the conductances that give the most contribution to the LDP are of order $t^{-1/(1+\eta)}$ and assume a certain deterministic shape.

With the special choice $O = C = \mathcal{M}_1(B)$, we obtain the following corollary.

Corollary 1.5 (Non-exit probability from B)**.** *Under the assumptions of Theorem 1.3,*

$$\lim_{t\to\infty} t^{-\frac{\eta}{\eta+1}} \log \left\langle \mathbb{P}_0^\omega\big(X_{[0,t]} \subset B\big)\right\rangle = -K_{\eta,D} L_\eta(B), \tag{1.19}$$

where

$$L_\eta(B) = \inf_{g^2\in\mathcal{M}_1(B)} \sum_{\{x,y\}\in E_B} |g(y)-g(x)|^{\frac{2\eta}{\eta+1}}. \tag{1.20}$$

From Theorem 1.3, we also derive the precise logarithmic lower tails of the principal (i.e., smallest) eigenvalue $\lambda^\omega(B)$ of $-\Delta^\omega$ in B with zero boundary condition.

Corollary 1.6 (Lower tails for the bottom of the spectrum of Δ^ω)**.** *Under the assumptions of Theorem 1.3,*

$$\lim_{\varepsilon\downarrow 0} \varepsilon^\eta \log \Pr(\lambda^\omega(B) \leq \varepsilon) = -D L_\eta(B)^{\eta+1}.$$

The proof of this Corollary is postoponed to Section 2.3.

1.2.3 Law of large numbers and the point of view of the particle

While the question of the convergence of RWRC's to some continuous diffusion is pretty delicate to handle, the law of large numbers (LLN) is much more well understood.

If $(X_n)_{n\in\mathbb{N}}$ is our RWRC, call

$$\lim_{n\to\infty}\frac{X_n}{n}$$

the limiting speed, or just speed, of the walk. We ask: What are the conditions on the distribution P of the conductances in order to guarantee that such limit exists and is the same for P-almost every environment ω and P_0^ω-almost every trajectory?

Let us consider the $\mathbb{Z}^d$ grid. The following result is well known, see [Bis11], Theorem 2.4, for a more general case including the possibility of finite-mean jumps of the walk.

Theorem 1.7. *Let $\{\omega_{xy}\}_{x,y\in\mathbb{Z}^d}$ be nearest neighbor conductances sampled from a shift-ergodic elliptic distribution P with $E[\omega_{xy}]<\infty$. Then*

$$P_0^\omega\Big(\lim_{n\to\infty}\tfrac{X_n}{n}=0\Big)=1 \qquad \textit{for } P\textit{-almost every } \omega\in\Omega.$$

The proof of this fact involves a quite standard tool for RWRE's, namely the so called *motion from the point of view of the particle* (see [KV86], [Koz85], [PV81] and [PV82] for some earlier results). The technique consists in looking at the Markov chain $(\tau_{X_n}\omega)_{n\in\mathbb{N}}$ defined on the space of the environments Ω with transition probabilities

$$K(\omega,\tilde\omega)=\sum_{x\sim 0}P_0^\omega(X_1=x)\delta_{\tau_x\omega}(\tilde\omega).$$

This chain has, in the setting of the previous theorem,

$$\mathbb{Q}(\mathrm{d}\omega):=\frac{\pi_0(\omega)}{E[\pi_0(\omega)]}P(\mathrm{d}\omega)$$

as a stationary measure, which is (as the formula shows) absolutely continuous with respect to the original measure (this in general doesn't happen for non-reversible RWRE's). The price paid for moving to a much bigger state space is rewarded with the stationarity of the increments of the chain, which the original dynamics did not have. It can be shown that the chain $(\tau_{X_n}\omega)_{n\in\mathbb{N}}$ is ergodic and the result of Theorem 1.7 follows then quite easily from the usual ergodic theorem and representing $(X_n)_{n\in\mathbb{N}}$ as an additive functional of $(\tau_{X_n}\omega)_{n\in\mathbb{N}}$.

A similar result can be easily proven for the simple random walk on the percolation cluster. A much more interesting model is that of the biased random walk on the percolation cluster (in which the walker 'prefers' to go in one direction), where atipical and unexpected behaviors of the speed occur: It may happen that the model switches from a ballistic to a subballistic regime as the drift in the preferred direction increases. The literature for physical motivations and background includes [Dha84] and [DS98], while a mathematical coverage can be found in [BGP03], [Szn03], the more recent [Fri10] and [FH11], and others.

Note that the conditions of Theorem 1.7 do not require any restriction on the mixing properties of the environment, i.e. on how much far-away conductances are correlated. A complementary well known result deals with the strongest mixing condition possible: the i.i.d. case.

Theorem 1.8. *Let the environment ω be sampled with an i.i.d. law. Then*

$$\mathbb{P}\Big(\lim_{n\to\infty}\tfrac{X_n}{n}=0\Big)=1.$$

It is important to underline that no assumptions on the distribution of a single conductance were made.

We could not find any reference of such result in the literature. In Section 3.4 we will give a sketch of the proof in two dimensions (for which we thank Prof. Noam Berger).

Both Theorem 1.7 and Theorem 1.8 give sufficient conditions for the RWRC to have almost sure zero speed. To our knowledge there were no examples of walks among ergodic random environments in Z^d exhibiting non-zero speed in the literature before [BS12].

Finally we would like to mention that a lot of research has been carried out for RWRC's when the underlying graph is other than the square lattice. One of the most remarkable examples are random walks on Galton-Watson trees. For this model the speed (defined as the limit of the distance from the root divided by the number of steps performed) shows somewhat unexpected features: It is not monotonous in the strength of the bias when the tree is allowed to have leaves (see [LPP96] and [AÏ1]), while its behavious is still not fully understood in the no-leaves case ([BAFS11]).

1.2.4 Moments conditions for non-zero speed of RWRC's

For this result, we let P be a measure on Ω which satisfies the following two conditions:

(i) P is invariant and ergodic w.r.t. the group of spatial moves in $\mathbb{Z}^2$.

(ii) The marginal distribution of ω_e is the same for all choices of the edge e, i.e. vertical and horizontal edges have the same distribution.

Note that this is weaker than invariance w.r.t. rotations. Condition (ii) can be weakened significantly, but for simplicity we keep it as is.

The two Theorems 1.7 and 1.8 give as result an almost sure zero-speed. There seems to be two types of criteria involved: The first is moment conditions that control the size of the conductances, and the second is mixing conditions saying that if the environment mixes fast enough then the speed is zero. In [BS12] we only consider the first type, and show that the sharp condition is that the logarithm of the conductances has high enough moments.

Our main result is as follows.

Theorem 1.9. *Let e be an edge in $\mathbb{E}^2$.*

1. *If there exists $\alpha > 1$ such that*

$$E[\log^\alpha \omega_e] < \infty, \tag{1.21}$$

then

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{X_n}{n} = 0\right) = 1.$$

2. *For every $\alpha < 1$ there exists a distribution P on environments such that $E[\log^\alpha \omega_e] < \infty$, but*

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{X_n}{n} = 0\right) = 0.$$

Furthermore, in this case it is possible to choose P so that either

$$\mathbb{P}\left(\left\|\lim_{n\to\infty} \frac{X_n}{n}\right\|_\infty > 0\right) = 1$$

or

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{X_n}{n} \text{ does not exist}\right) = 1.$$

Remark 1.10. *Our proofs will deal with conductances bounded away from zero, but would work in the same way including the possibility of zero conductances. Note also that the choice of dimension* 2 *has been made in order to have easier and more intuitive proofs. We are confident that the same results can be proven with the very same techniques in higher dimensions, with critical* α *equal to* $d-1$*.*

1.2.5 Effective conductance and homogenization theory

As is well known, most materials, regardless how pure they may seem at the macroscopic level, have a rather complicated microscopic structure. It may then come as a surprise that physical phenomena such as heat or electric conduction are described so well using differential equations with smooth, sometimes even constant, coefficients. An explanation has been offered by homogenization theory (see the monograph by [JKO94] for an overview on the subject and its history): rapid oscillations at the microscopic level average out, or homogenize, at the macroscopic scale. However, this does not mean that the microscopic structure is simply washed out. Indeed, while it disappears from the structure of the resulting equations, it remains embedded in the values of effective material constants, e.g., the coefficients.

An illustrative example of a homogenization problem is that of effective conductance for the RCM. For any $\Lambda \subset \mathbb{Z}^d$, let $\mathbb{B}(\Lambda)$ be the edges with at least one endpoint in Λ. Given an $f\colon \mathbb{Z}^d \to \mathbb{R}$ and a finite $\Lambda \subset \mathbb{Z}^d$, let

$$Q_\Lambda(f) := \sum_{\langle x,y\rangle \in \mathbb{B}(\Lambda)} \omega_{xy}\big[f(y)-f(x)\big]^2, \tag{1.22}$$

where each pair (x,y) is counted only once. This is the electrostatic *Dirichlet energy* for the potential f with Dirichlet boundary condition on the boundary vertices of Λ (note the analogy with (1.12), where zero boundary conditions were considered instead).

For simplicity we will consider the square box $\Lambda_L := [0,L)^d \cap \mathbb{Z}^d$. A quantity of prime interest for us is the *effective conductance*, which we already defined in the context of electrical networks (compare with 1.8),

$$C_L^{\text{eff}}(t) := \inf\big\{Q_{\Lambda_L}(f)\colon f(x) = t\cdot x,\ \forall x \in \partial\Lambda_L\big\}, \tag{1.23}$$

where $t \in \mathbb{R}^d$ and where $\partial\Lambda$ are those vertices outside Λ that have an edge into Λ.

By Kirchhoff's and Ohm's laws (see, e.g., [DS84]), $C_L^{\mathrm{eff}}(t)$ is the total electric current flowing through the network when the boundary vertices are kept at voltage $t \cdot x$.

For homogeneous resistor networks, i.e., when $\omega_{xy} := a$ for all $\langle x, y\rangle$, the infimum (1.23) is achieved by $f(x) := t \cdot x$ and so $C_L^{\mathrm{eff}}(t) = a|t|^2 L^d(1 + o(1))$. A question of (reasonably) practical interest is then what happens when the conductances $\{\omega_{xy}\}$ are no longer constant, but remain close to a constant.

A comparison of Q_Λ with these ω_{xy}'s and the homogeneous case shows that $C_L^{\mathrm{eff}}(t)$ is still of the order of $|t|^2 L^d$. Moreover, thanks to the choice of the linear boundary condition, by subadditivity arguments the limit

$$c_{\mathrm{eff}}(t) := \lim_{L\to\infty} \frac{1}{L^d} C_L^{\mathrm{eff}}(t) \tag{1.24}$$

exists almost surely for any ergodic distribution of the conductances. The problem left to resolve is thus a computation of the limit value.

Although $c_{\mathrm{eff}}(t)$ can be computed only in a handful of (periodic) cases, it can be characterized in large generality: Suppose that ω is a sample from a shift-ergodic law P on the product space $\Omega := [\lambda, \frac{1}{\lambda}]^{\mathbb{B}(\mathbb{Z}^d)}$ indexed by edges of $\mathbb{Z}^d$, for some $\lambda > 0$. As is well known,

$$c_{\mathrm{eff}}(t) = \inf_{g\in L^\infty(\mathbb{P})} E\Big(\sum_{x=\hat{e}_1,\dots,\hat{e}_d} a_{0,x}(\omega)\big|t \cdot x + \nabla_x g(\omega)\big|^2\Big). \tag{1.25}$$

Here $\hat{e}_1, \dots, \hat{e}_d$ are the unit coordinate vectors in $\mathbb{R}^d$ and $\nabla_x g(\omega) := g \circ \tau_x(\omega) - g(\omega)$ is the gradient of g in direction of $x \in \mathbb{Z}^d$. The expression in (1.25) can be interpreted as the Dirichlet energy density with the spatial average naturally replaced by the ensemble average. This object has not been introduced in the form of the limit 1.24 (see [PV82], [Koz86], [Kün83] and the book [JKO94]), and proving the equivalence with the original formulation requires a bit of work (it can be deduced, e.g., through Proposition 4.3 of Chapter 4).

Once the (deterministic) leading-order of $C_L^{\mathrm{eff}}(t)$ has been identified, the next natural question is that of fluctuations. It is obvious e.g., by checking the explicitly computable $d = 1$ case — that no universal limit law can be expected for general conductance distributions, but progress could perhaps be made for the (physically most appealing) case of i.i.d. conductances. However, even here establishing just the order of magnitude of the fluctuations turned out to be an arduous task. Indeed, more than a decade ago

Wehr [Weh97] showed that $\mathrm{Var}(C_L^{\mathrm{eff}}) \geq cL^d$ for some $c > 0$ but a corresponding upper bound has been furnished only recently by Gloria and Otto [GO11]. Both of these results contain important technical caveats: Wehr requires continuously distributed ω_{xy}'s while Gloria and Otto express their results under a "massive" cutoff.

Gloria and Otto drew important ideas from an earlier unpublished note by Naddaf and Spencer [NS98] where (optimal) upper bounds on the variance are derived for certain correlated conductance laws. The main tool of [NS98] is the *Meyers estimate* (cf Meyers [Mey63]), to be used heavily in the proof of Theorem 1.11 as well. Other noteworthy earlier derivations of (suboptimal) variance upper bounds include an old paper by Yurinskii [Yur86] and a more recent paper by Benjamini and Rossignol [BR08]. Closely related to these estimates are recent derivations of quantitative central limit theorems for random walk among random conductances and approximations of the limiting diffusivity matrix, e.g., Caputo and Ioffe [CI03], Bourgeat and Piatnitski [BP04], Boivin [Boi09], Mourrat [Mou12], etc. Incidentally, the Meyers estimate is also the key tool in [CI03].

1.2.6 A central limit theorem for the effective conductance

In the article [BSW12] we prove that, for i.i.d. conductances which are (deterministically) not too far from a constant, the asymptotic law of $C_L^{\mathrm{eff}}(t)$ (defined in (1.23)) is in fact Gaussian. Let $\mathcal{N}(\mu, \sigma^2)$ denote the normal random variable with mean μ and variance σ^2. Then we have:

Theorem 1.11. *Suppose the conductances ω_{xy} are i.i.d. For each $d \geq 1$, there is $\lambda = \lambda(d) \in (0,1)$ such that the following holds: If (1.1) is satisfied $\mathbb{P}$-a.s. with this λ, then for each $t \in \mathbb{R}^d$ there is $\sigma_t^2 \in [0,\infty)$ such that*

$$\frac{C_L^{\mathrm{eff}}(t) - \mathbb{E}C_L^{\mathrm{eff}}(t)}{|\Lambda_L|^{1/2}} \underset{L\to\infty}{\overset{\mathrm{law}}{\longrightarrow}} \mathcal{N}(0, \sigma_t^2). \tag{1.26}$$

Whenever the conductance law is non-degenerate we have $\sigma_t^2 > 0$ for all $t \neq 0$.

The proof also immediately yields:

Corollary 1.12. *Under the conditions of Theorem 1.11,*

$$\frac{1}{|\Lambda_L|}\mathrm{Var}\big(C_L^{\mathrm{eff}}(t)\big) \underset{L\to\infty}{\longrightarrow} \sigma_t^2, \tag{1.27}$$

where σ_t^2 is as in (1.26)*.*

A few remarks are in order:

Remarks 1.13. *(1) Notice that* (1.26) *does not give us much information on the "order expansion" of $C_L^{\text{eff}}(t)$. Indeed, we know that $\mathbb{E}C_L^{\text{eff}}(t)$ is to the leading order equal to $c_{\text{eff}}(t)|\Lambda_L|$ but when this order is subtracted, the next-order term is (presumably) of boundary size. In $d \geq 3$, this is still larger than the typical size of the fluctuations. Notwithstanding, what* (1.26) *does tell us is the character of the leading order* random *term.*

(2) There is in fact a formula for σ_t^2, see Theorem 4.7 below, which also shows that $t \mapsto \sigma_t^2$ is a bi-quadratic (and thus smooth) function of t. However, the formula involves complicated conditioning and does not seem very useful for practical computations.

(3) There is no restriction on the single-conductance law other than (1.1)*. In particular, this law can have a non-absolutely continuous part including atoms. Certain technical problems do arise at this level of generality; see Section 4.1.5 which, we believe, is of independent interest.*

We prove Theorem 1.11 by reducing it to the Martingale Central Limit Theorem. There are two main technical ingredients: homogenization theory (which enables a stationary martingale approximation of $C_L^{\text{eff}}(t)$) and analytical estimates for finite-volume harmonic coordinates (by which we control the errors in the martingale approximation). The restrictions to rectangular boxes, linear boundary conditions and small ellipticity contrasts permit us to encapsulate the analytical input into a single step, the Meyers estimate, cf Proposition 4.4 and Theorem 4.15. These restrictions can be relaxed but not without additional arguments not all of which have been handled satisfactorily at the time [BSW12] was uploaded on the arXiv. These are deferred to a follow-up paper. We remark that two recent preprints have been brought to our attention at the time this work was first announced in conference talks. First, Nolen [Nol11] has established a normal approximation to the effective conductance defined over a periodic environment, in the limit when the period tends to infinity. Second, in a preprint that was posted at the time of writing the present note, Rossignol [Ros12] formulates and proves a central limit law for the *effective resistance* for the corresponding problem on a torus. Nolen's defines the problem over continuum, albeit with a rather strong assumption on an underlying Gaussian i.i.d. structure. Rossignol's setting is based on minimizing the electrostatic energy over *currents* (rather than potentials) subject to a restriction on the total current flowing around the torus. By a well known reciprocity relation between effective conductance and resistance, these papers appear to address similar problems (see Section 1.1.3).

Our work differs from both Nolen [Nol11] and Rossignol [Ros12] primarily in its emphasis on fixed (Dirichlet), as opposed to periodic, boundary conditions. Indeed, a majority of our technical work is aimed at controlling the resulting boundary effects. Also the way a Gaussian limit law is established is quite different: Nolen appeals to Stein's method, Rossignol uses concentration of measure while we invoke the Martingale Central Limit Theorem. A deficiency of our result compared to [Nol11] and [Ros12] is the limitation on ellipticity contrast. Nolen overcomes this by an appeal to Gloria and Otto [GO11], although this ultimately precludes the most interesting conclusion in $d = 2$. Rossignol's approach appears to work seamlessly for all elliptic product laws.

While the Gloria-Otto method can be adapted to our situation as well, just as for Nolen [Nol11] it fails to deliver the desired conclusion in $d = 2$ (the issue is that the method yields bounds on the moments of the corrector, which diverge in $d = 2$, while we need only moments of the gradients of the corrector). The moment bounds thus seem to be a separate technical matter, so for our first paper we decided to sacrifice on generality of the distribution and derived the CLT only in the simplest, albeit still physically interesting, case.

Chapter 2

Large deviations for the occupation measure

In Section 2.1 we present a heuristic derivation of Theorem 1.3. Section 2.2 is dedicated to the rigorous proof of the main result: Subsection 2.2.1 covers the lower bound (1.17) while Subsection 2.2.2 takes care of the upper bound (1.18). Finally we prove Corollary 1.6 in Section 2.3. Section 2.4 offers a brief outlook of possible future research on the subject.

2.1 Heuristic derivation

We now give a formal derivation of the LDP in Theorem 1.3. Given a fixed realisation $\varphi = \{\varphi_{xy} \colon \{x,y\} \in E_B\} \in (0,\infty)^{E_B}$ of the conductances, the probability that the normalised local time resembles some realisation $g^2 \in \mathcal{M}_1(B)$ is roughly

$$\mathbb{P}_0^{\varphi}\Big(\tfrac{1}{t}\ell_t \approx g^2\Big) \approx \exp\big\{-tI_{\varphi}(g^2)\big\}, \tag{2.1}$$

where the corresponding Donsker-Varadhan rate function is given by

$$I_{\varphi}(g^2) = \big(-\Delta^{\varphi} g, g\big) = \sum_{\{x,y\}\in E_B} \varphi_{xy}|g(x)-g(y)|^2. \tag{2.2}$$

This is a formal application of the LDP for the normalized occupation times of a Markov process with symmetric generator Δ^{φ} as in [DV75b] and [G77̈] (see Theorem 1.2); by

$(\cdot,\cdot)$ we denote the standard inner product on $\ell^2(\mathbb{Z}^d)$. Note that the event $\{X_{[0,t]}\subset B\}$ is contained in $\{\frac{1}{t}\ell_t\approx g^2\}$, therefore we drop it from the notation.

Taking random conductances into account, we expect an LDP on a slower scale than t, as small t-dependent values of the conductances lead to a slower decay of the annealed probability of the event $\{\frac{1}{t}\ell_t\approx g^2\}$. Therefore, we rescale ω by a factor t^r with some $r>0$ to be determined later, and approximate

$$\begin{aligned}\Pr\big(t^r\omega\approx\varphi\big)=\Pr\big(\forall\{x,y\}\in E_B\colon \omega_{xy}\approx t^{-r}\varphi_{xy}\big)&=\prod_{\{x,y\}\in E_B}\Pr\big(\omega_{xy}\approx t^{-r}\varphi_{xy}\big)\\ &\approx\exp\big\{-t^{r\eta}H(\varphi)\big\},\end{aligned}\tag{2.3}$$

where the rate function for the conductances is given by

$$H(\varphi):=D\sum_{\{x,y\}\in E_B}\varphi_{xy}^{-\eta}.\tag{2.4}$$

Here we made use of the tail assumptions in (1.14). Hence, combining (2.1) and (2.3),

$$\begin{aligned}\Big\langle\mathbb{P}_0^\omega\big(\tfrac{1}{t}\ell_t\approx g^2\big)\mathbb{1}_{\{t^r\omega\approx\varphi\}}\Big\rangle&\approx\mathbb{P}_0^{t^{-r}\varphi}\big(\tfrac{1}{t}\ell_t\approx g^2\big)\Pr\big(\omega\approx t^{-r}\varphi\big)\\ &\approx\exp\Big\{-tI_{t^{-r}\varphi}(g^2)-t^{r\eta}H(\varphi)\Big\}\\ &\approx\exp\Big\{-\sum_{\{x,y\}\in E_B}\Big(t^{1-r}\varphi_{xy}\big(g(x)-g(y)\big)^2+t^{r\eta}D\varphi_{xy}^{-\eta}\Big)\Big\}.\end{aligned}\tag{2.5}$$

We obtain the slowest decay by choosing r such that $t^{1-r}=t^{r\eta}$, which means $r=(1+\eta)^{-1}$. Then the right-hand side has scale $t^{\frac{\eta}{\eta+1}}$, which is the scale of the desired LDP. In order to find the rate function, we optimize over φ and obtain that the choice $\varphi=\varphi^{(g)}$ with

$$\varphi_{xy}^{(g)}=(D\eta)^{\frac{1}{\eta+1}}|g(y)-g(x)|^{-\frac{2}{\eta+1}},\qquad\{x,y\}\in E_B,\tag{2.6}$$

contributes most to the joint probability. Therefore, we have the result

$$\Big\langle\mathbb{P}_0^\omega\big(\tfrac{1}{t}\ell_t\approx g^2\big)\Big\rangle\approx\exp\Big\{-t^{\frac{\eta}{\eta+1}}J(g^2)\Big\},$$

where the rate function is identified as

$$J(g^2)=\inf_\varphi\big[I_\varphi(g^2)+H(\varphi)\big]=I_{\varphi^{(g)}}(g^2)+H(\varphi^{(g)})=K_{\eta,D}\sum_{\{x,y\}\in E_B}|g(y)-g(x)|^{\frac{2\eta}{\eta+1}}.\tag{2.7}$$

The tail assumptions we have made on the environment distribution lead to a fairly remarkable interaction between the random influences of the environment on the one hand and the random walk on the other. Under more general assumptions, e.g.,

$$\log \Pr(\omega_{xy} \le \varepsilon) \sim -\alpha(\varepsilon), \qquad \varepsilon \to 0$$

for some sufficiently regular nonincreasing function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$, we would expect an analogous result to hold. However, if $\alpha(\varepsilon)$ is not a polynomial in ε, the scale and rate function of a corresponding LDP certainly would not have such an explicit form.

2.2 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. This amounts to showing the two inequalities in (1.17) and (1.18), since the compactness of the level sets follows immediately from the continuity of J and compactness of the space $\mathcal{M}_1(B)$. The two inequalities are proven in the next two sections.

2.2.1 Proof of the lower bound

In order to prove (1.17), we need to control the transition from one realization of the environment to another. To this end, we first identify the density of this transition on process level. We feel that this should be generally known, but could not find a suitable reference. For $\varphi\colon E \to (0,\infty)$ we abbreviate $\bar\varphi(x) := \sum_{y\sim x} \varphi(x,y)$. We also write φ_{xy} instead of $\varphi(x,y)$.

Lemma 2.1. *Assume that $\varphi, \psi\colon E \to (0,\infty)$ are bounded both from above and away from zero. Denote by $S(t)$ the number of jumps the process $X = (X_s)_{s\in[0,t]}$ makes up to time t and by $0 < \tau_1 < \ldots < \tau_{S(t)}$ the corresponding jump times. Fix some starting point $x \in \mathbb{Z}^d$ and put $\tau_0 = 0$. Then, for all $t \in [0,\infty)$,*

$$\Phi_t(X) := \prod_{i=1}^{S(t)} \left(\frac{\varphi(X_{\tau_{i-1}}, X_{\tau_i})}{\psi(X_{\tau_{i-1}}, X_{\tau_i})} \mathrm{e}^{-(\tau_i - \tau_{i-1})\left[\bar\varphi(X_{\tau_{i-1}}) - \bar\psi(X_{\tau_{i-1}})\right]} \right) \mathrm{e}^{-(t-\tau_{S(t)})\left[\bar\varphi(X_t) - \bar\psi(X_t)\right]}$$

is the Radon-Nikodym density of $\mathbb{P}_x^\varphi$ with respect to $\mathbb{P}_x^\psi$ with time horizon t.

Proof. We will write Φ_t instead of $\Phi_t(X)$. Obviously, $\Phi_t > 0$ almost surely. We start showing that, for all $t \geq 0$, the expectation of Φ_t under $\mathbb{P}_x^\psi$ is one. Then, we use Kolmogorov's extension theorem to show the existence of a measure $\mathbb{P}_x$ such that $\mathbb{P}_x(A) = \mathbb{E}_x^\psi(\Phi_t \mathbb{1}_A)$ for all $A \in \mathcal{F}_t$, where $(\mathcal{F}_t)_{t\in[0,\infty)}$ is the natural filtration generated by X. It remains to show that the process X under $\mathbb{P}_x$ is a Markov process and that it is generated by Δ^φ, which implies $\mathbb{P}_x = \mathbb{P}_x^\varphi$.

Let us start by showing that the expectation of Φ_t under $\mathbb{P}_x^\psi$ is one. Consider the discrete-time process

$$Z_n := \prod_{i=1}^n \left(\frac{\varphi(X_{\tau_{i-1}}, X_{\tau_i})}{\psi(X_{\tau_{i-1}}, X_{\tau_i})} \mathrm{e}^{-(\tau_i - \tau_{i-1})\left[\bar\varphi(X_{\tau_{i-1}}) - \bar\psi(X_{\tau_{i-1}})\right]} \right).$$

We have, for $x \in \mathbb{Z}^d$,

$$\mathbb{E}_x^\psi[Z_1] = \sum_{y\sim x} \frac{\psi_{xy}}{\bar\psi(x)} \frac{\varphi_{xy}}{\psi_{xy}} \int_0^\infty \bar\psi(x) \mathrm{e}^{-\bar\psi(x)s - (\bar\varphi(x) - \bar\psi(x))s} \, \mathrm{d}s = \sum_{y\sim x} \frac{\varphi_{xy}}{\bar\varphi(x)} = 1.$$

Combining this equation with the strong Markov property, we see that $(Z_n)_n$ is a martingale with respect to the filtration $(\mathcal{F}_{\tau_n})_{n\in\mathbb{N}}$ generated by the jumping times and that

$$\mathbb{E}_x^\psi \left[\frac{\varphi(X_t, X_{\tau_{S(t)+1}})}{\psi(X_t, X_{\tau_{S(t)+1}})} \mathrm{e}^{-(\tau_{S(t)+1} - t)\left[\bar\varphi(X_t) - \bar\psi(X_t)\right]} \middle| \mathcal{F}_t \right] = \mathbb{E}_{X_t}^\psi [Z_1] = 1 \tag{2.8}$$

$\mathbb{P}_x^\psi$-almost surely for all $x \in \mathbb{Z}^d$. Then, we obtain

$$\mathbb{E}_x^\psi[\Phi_t] = \mathbb{E}_x^\psi[Z_{S(t)+1}], \qquad x \in \mathbb{Z}^d,$$

by inserting the first term of (2.8) under the expectation and using that Φ_t is $\mathcal{F}_t$-measurable. Consequently, it remains to show that $\mathbb{E}_x^\psi[Z_{S(t)+1}] = 1$. As $S(t)+1$ is an unbounded, but almost surely finite stopping time with respect to the filtration $(\mathcal{F}_{\tau_n})_{n\in\mathbb{N}}$, the optional sampling theorem yields that $\mathbb{E}_x^\psi[Z_{S(t)+1}] \leq 1$. On the other hand, for all integers $k > 0$,

$$\mathbb{E}_x^\psi[Z_{S(t)+1}] \geq \mathbb{E}_x^\psi[Z_{S(t)+1} \mathbb{1}_{S(t)+1\leq k}] = \mathbb{E}_x^\psi[Z_{S(t)+1\wedge k}] - \mathbb{E}_x^\psi[Z_k \mathbb{1}_{S(t)\geq k}] = 1 - \mathbb{E}_x^\psi[Z_k \mathbb{1}_{S(t)\geq k}]. \tag{2.9}$$

To show that the last term is arbitrarily close to one for large k, we recall that on $\{S(t) \geq k\}$

$$Z_k \leq \left(\frac{\max_{x\in\mathbb{Z}^d,\, y\sim x} \varphi_{xy}}{\min_{x\in\mathbb{Z}^d,\, y\sim x} \psi_{xy}} \right)^k \mathrm{e}^{t \max\{|\varphi_{xy} - \psi_{xy}| :\, \{x,y\}\in E\}} =: \alpha_k,$$

so $\mathbb{E}_x^\psi[Z_k \mathbb{1}_{S(t)\geq k}]$ is bounded from above by $\alpha_k \mathbb{P}_x^\psi(S(t) \geq k)$. As all jumping times are exponentially distributed with a parameter smaller than $\gamma := \max_{x\in\mathbb{Z}^d} \bar{\psi}(x)$, we may estimate

$$\mathbb{P}_x^\psi(S(t) \geq k) \leq \mathrm{e}^{\gamma t} \sum_{n=k}^{\infty} \frac{(\gamma t)^n}{n!}.$$

The tail of an exponential series is super-exponentially small, which means $\alpha_k \mathbb{P}_x^\psi(S(t) \geq k) \to 0$ for $k \to \infty$. Since (2.9) was true for all k, we see that $\mathbb{E}_x^\psi[Z_{S(t)+1}] = 1$.

For arbitrary $k \in \mathbb{N}$ and $t_1, \dots, t_k \geq 0$ define $\hat{t} = \max_{i\in\{1,\dots,k\}} t_i$ and a measure $Q_{t_1,\dots,t_k}$ on $(\mathbb{Z}^d)^k$ by

$$Q_{t_1,\dots,t_k}(x_1, \dots, x_k) = \mathbb{E}_x^\psi[\Phi_{\hat{t}} \mathbb{1}_{\{X_{t_1}=x_1,\dots,X_{t_k}=x_k\}}], \qquad x_1, \dots, x_k \in \mathbb{Z}^d.$$

We verify without much effort that $\mathbb{E}_x^\psi[\Phi_{t+s}\mathbb{1}_A] = \mathbb{E}_x^\psi[\Phi_t \mathbb{1}_A]$ for all $A \in \mathcal{F}_t$ and $t, s > 0$, which implies consistency of the family of measures above. Thus, by Kolmogorov's extension theorem, there exists a measure $\mathbb{P}_x$ with finite-dimensional distributions as above, and we have $\mathbb{P}_x(A) = \mathbb{E}_x^\psi[\Phi_t \mathbb{1}_A]$ for all $t > 0$ and $A \in \mathcal{F}_t$. We show that the process X under $\mathbb{P}_x$ satisfies the Markov property, i.e.,

$$\mathbb{E}_x[\mathbb{1}_{\{X_{t+s}=y\}}|\mathcal{F}_t] = \mathbb{P}_{X_t}(X_s = y) \quad \mathbb{P}_x\text{-a.s. for all } y \in \mathbb{Z}^d, s, t > 0 \tag{2.10}$$

where $\mathbb{E}_x$ denotes expectation with regard to $\mathbb{P}_x$. Note that $\mathbb{P}_{X_t}$ is defined as we have considered an arbitrary starting point x in what we have shown so far. Indeed, for all $A \in \mathcal{F}_t$

$$\begin{aligned}
\mathbb{E}_x\big[\mathbb{E}_x[\mathbb{1}_{\{X_{t+s}=y\}}|\mathcal{F}_t]\mathbb{1}_A\big] &= \mathbb{E}_x[\mathbb{1}_{\{X_{t+s}=y\}}\mathbb{1}_A] = \mathbb{E}_x^\psi[\Phi_{t+s}\mathbb{1}_{\{X_{t+s}=y\}}\mathbb{1}_A] \\
&= \mathbb{E}_x^\psi\big[\mathbb{E}_x^\psi[\Phi_{t+s}\mathbb{1}_{\{X_{t+s}=y\}}|\mathcal{F}_t]\mathbb{1}_A\big] \\
&\overset{(*)}{=} \mathbb{E}_x^\psi\big[\Phi_t \mathbb{E}_{X_t}^\psi[\Phi_s \mathbb{1}_{\{X_s=y\}}]\mathbb{1}_A\big] \\
&= \mathbb{E}_x\big[\mathbb{E}_{X_t}[\mathbb{1}_{\{X_s=y\}}]\mathbb{1}_A\big],
\end{aligned}$$

where equation $(*)$ is due to the fact that X satisfies the Markov property under $\mathbb{P}_x^\psi$ and $\Phi_{t+s}\Phi_t^{-1}\mathbb{1}_{\{X_{t+s}=y\}}$ depends only on $X_{[t,t+s]}$. Consequently, we have shown (2.10) and X is a Markov process under $\mathbb{P}_x$ with a unique infinitesimal generator. Elementary calculations show that

$$\frac{1}{t}\Big(\mathbb{E}_x^\psi\left[f(X_t)\Phi_t\right] - f(x)\Big) \xrightarrow{t\to 0} \Delta^\varphi f(x)$$

for arbitrary $x \in \mathbb{Z}^d$ and $f\colon \mathbb{Z}^d \to \mathbb{R}$. This implies $\mathbb{P}_x = \mathbb{P}_x^\varphi$ and the proof is complete. □

Now we use Lemma 2.1 to compare probabilities for two environments that are close to each other.

Corollary 2.2. *Let $\varphi, \psi\colon E \to (0,\infty)$ with $0 < \psi_{xy} - \varepsilon \le \varphi_{xy} \le \psi_{xy} + \varepsilon$ for some $\varepsilon > 0$ and all $\{x,y\} \in E$. Moreover, let F be some event that depends on the process $(X_s)_{s\in[0,t]}$ up to time t only. Then*

$$\mathbb{P}_0^\varphi\big(F\big) \ge \mathrm{e}^{-4d\varepsilon t}\mathbb{P}_0^{\psi-\varepsilon}\big(F\big).$$

Proof. Let Φ_t denote the Radon-Nikodym density of $\mathbb{P}_0^\varphi$ with respect to $\mathbb{P}_0^{\psi-\varepsilon}$ up to time t. Employing the representation given in Lemma 2.1, we have

$$\begin{aligned}\Phi_t &\ge \prod_{i=1}^{S(t)} \Big(\mathrm{e}^{-(\tau_i-\tau_{i-1})\left[\bar\varphi(X_{\tau_{i-1}})-\bar\psi(X_{\tau_{i-1}})+2d\varepsilon\right]}\Big)\, \mathrm{e}^{-(t-\tau_{S(t)})\left[\bar\varphi(X_t)-\bar\psi(X_t)+2d\varepsilon\right]}\\ &\ge \prod_{i=1}^{S(t)} \Big(\mathrm{e}^{-(\tau_i-\tau_{i-1})4d\varepsilon}\Big)\, \mathrm{e}^{-(t-\tau_{S(t)})4d\varepsilon} \ge \mathrm{e}^{-4d\varepsilon t}.\end{aligned}$$

The desired inequality follows immediately. □

Remark 2.3. *If the event A is contained in $\{\mathrm{supp}(\ell_t) \subset B\}$, it suffices to require $0 < \psi_{xy} - \varepsilon \le \varphi_{xy} \le \psi_{xy} + \varepsilon$ for some $\varepsilon > 0$ and all $\{x,y\} \in E_B$.*

Let us now show (1.17). Fix an open set $O \subset \mathcal{M}_1(B)$. As the event $\{X_{[0,t]} \subset B\}$ is contained in $\{\frac{1}{t}\ell_t \in O\}$, we omit it in the notation. Observe that the distributions of $\frac{1}{t}\ell_t$ under $\mathbb{P}_0^\omega$ and $\frac{1}{t^{1-r}}\ell_{t^{1-r}}$ under $\mathbb{P}_0^{t^r\omega}$ coincide for all $0 < r < 1$. Hence

$$\liminf_{t\to\infty} \frac{1}{t^{\frac{\eta}{\eta+1}}} \log \Big\langle \mathbb{P}_0^\omega\Big(\tfrac{1}{t}\ell_t \in O\Big)\Big\rangle = \liminf_{t\to\infty} \frac{1}{t} \log \Big\langle \mathbb{P}_0^{t^{\frac{1}{\eta}}\omega}\Big(\tfrac{1}{t}\ell_t \in O\Big)\Big\rangle,$$

which will simplify the application of a classical Donsker-Varadhan LDP for random walks in fixed environment later. Choose an element $g^2 \in O$ arbitrarily. For $M > 0$ define $\varphi_M^{(g)}\colon E_B \to (0,\infty)$ by

$$\varphi_M^{(g)}(x,y) = \begin{cases} (D\eta)^{\frac{1}{\eta+1}}|g(y)-g(x)|^{-\frac{2}{\eta+1}} & \text{if } |g(y)-g(x)| > 0,\\ M & \text{otherwise.}\end{cases}$$

Next, we introduce the set

$$A = \{\varphi\colon E_B \to (0,\infty) \,\big|\, \varphi_M^{(g)} - \varepsilon \le \varphi \le \varphi_M^{(g)}\}, \tag{2.11}$$

where $\varepsilon > 0$ is picked smaller than $\frac{1}{2}\min_{E_B}\varphi_M^{(g)}$. By dint of Corollary 2.2,

$$\begin{aligned}\Big\langle \mathbb{P}_0^{t^{\frac{1}{\eta}}\omega}\Big(\tfrac{1}{t}\ell_t \in O\Big)\Big\rangle &\ge \Big\langle \mathbb{P}_0^{t^{\frac{1}{\eta}}\omega}\Big(\tfrac{1}{t}\ell_t \in O\Big)\mathbb{1}_{\{t^{\frac{1}{\eta}}\omega\in A\}}\Big\rangle \\ &\ge \inf_{\varphi\in A}\mathbb{P}_0^{\varphi}\Big(\tfrac{1}{t}\ell_t \in O\Big)\,\mathrm{Pr}\,\big(t^{\frac{1}{\eta}}\omega \in A\big) \\ &\ge \mathrm{e}^{-4d\varepsilon t}\mathbb{P}_0^{\varphi_M^{(g)}-\varepsilon}\Big(\tfrac{1}{t}\ell_t \in O\Big)\,\mathrm{Pr}\,\big(t^{\frac{1}{\eta}}\omega \in A\big).\end{aligned} \tag{2.12}$$

Using the tail assumption in (1.14), we see that

$$\lim_{t\to\infty}\frac{1}{t}\log\mathrm{Pr}\,\big(t^{\frac{1}{\eta}}\omega \in A\big) = -H(\varphi_M^{(g)}),$$

where H is given in (2.4). Furthermore, we apply the lower bound of the classical Donsker-Varadhan LDP (see [DV75b] or [G$\ddot{7}$7]) to get

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}_0^{\varphi_M^{(g)}-\varepsilon}\Big(\tfrac{1}{t}\ell_t \in O\Big) \ge -\inf_O I_{\varphi_M^{(g)}-\varepsilon},$$

where I_φ is given in (2.2). Hence, from (2.12) we obtain

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\log\Big\langle \mathbb{P}_0^{t^{\frac{1}{\eta}}\omega}\Big(\tfrac{1}{t}\ell_t \in O\Big)\Big\rangle &\ge -4d\varepsilon - \inf_O I_{\varphi_M^{(g)}-\varepsilon} - H(\varphi_M^{(g)}) \\ &\ge -4d\varepsilon - \inf_O I_{\varphi_M^{(g)}} - H(\varphi_M^{(g)}) \\ &\ge -4d\varepsilon - I_{\varphi_M^{(g)}}(g^2) - H(\varphi_M^{(g)}),\end{aligned}$$

since $I_{\varphi_M^{(g)}-\varepsilon} \le I_{\varphi_M^{(g)}}$ and $g^2 \in O$. Now we send ε to zero and M to ∞, to obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log\Big\langle \mathbb{P}_0^{t^{\frac{1}{\eta}}\omega}\Big(\tfrac{1}{t}\ell_t \in O\Big)\Big\rangle \ge -I_{\varphi^{(g)}}(g^2) - H(\varphi^{(g)}) = -J(g^2),$$

where $\varphi^{(g)} = \lim_{M\to\infty}\varphi_M^{(g)}$ is given in (2.6), and we used (2.7). The desired lower bound follows by passing to the infimum over all $g^2 \in O$.

2.2.2 Proof of the upper bound

In this section we prove (1.18). Let us first fix some configuration $\varphi \in (0,\infty)^E$ and start with an estimate for the probability $\mathbb{P}_0^{\varphi}(\frac{1}{t}\ell_t \in \cdot)$. This approach has actually been used by other authors before, but we provide an independent proof for the sake of completeness.

Lemma 2.4. *Fix an arbitrary set $A \subset \mathcal{M}_1(B)$. Then*

$$\mathbb{P}_0^\varphi\Big(\tfrac{1}{t}\ell_t \in A\Big) \le \frac{f(0)}{\min_B f} \exp\Big\{t \sup_{h^2\in A} \sum_{x\in B} \frac{\Delta^\varphi f(x)}{f(x)} h^2(x)\Big\} \tag{2.13}$$

for arbitrary $f\colon \mathbb{Z}^d \to [0,\infty)$ with $\operatorname{supp}(f) = B$ and $t > 0$.

Proof. We consider the Cauchy problem

$$\begin{cases} \partial_t u(x,t) = \Delta^\varphi u(x,t) + V(x) u(x,t), & x \in \mathbb{Z}^d,\ t>0,\\ u(x,0) = f(x), & x\in\mathbb{Z}^d, \end{cases} \tag{2.14}$$

with

$$V = -\frac{\Delta^\varphi f}{f} \mathbb{1}_B.$$

Obviously, $u(\cdot,t) \equiv f(\cdot)$ solves (2.14). On the other hand, by the Feynman-Kac formula, any nonnegative solution u satisfies

$$u(x,t) = \mathbb{E}_x^\varphi\Big[\mathrm{e}^{\int_0^t V(X_s)\mathrm{d}s} u(X_t,t)\Big], \qquad x\in\mathbb{Z}^d,\ t\ge 0. \tag{2.15}$$

Therefore, we may estimate

$$\begin{aligned} f(0) &= \mathbb{E}_0^\varphi\Big[\mathrm{e}^{-\int_0^t \frac{\Delta^\varphi f(X_s)}{f(X_s)}\mathrm{d}s} f(X_t)\Big]\\ &\ge \mathbb{E}_0^\varphi\Big[\mathrm{e}^{-\sum_{x\in B} \frac{\Delta^\varphi f(x)}{f(x)}\ell_t(x)} f(X_t)\mathbb{1}_{\{\frac{1}{t}\ell_t\in A\}}\Big]\\ &\ge \min_B f\ \exp\Big\{-t\sup_{h^2\in A}\sum_{x\in B}\frac{\Delta^\varphi f(x)}{f(x)}h^2(x)\Big\}\mathbb{P}_0^\varphi\Big(\tfrac{1}{t}\ell_t\in A\Big), \end{aligned}$$

which is a rearrangement of the assertion. □

Now fix some closed set $C \subset \mathcal{M}_1(B)$. As a closed subset of a finite-dimensional space, C is compact with respect to the Euclidean topology. We are going to apply a standard compactness argument, which is in the spirit of the proof of the upper bound in Varadhan's lemma [DZ98, Thm. 4.3.1]. The idea is to cover C with certain open balls, where 'open' refers to the Euclidean topology.

Fix $\delta > 0$. For $g^2 \in C$ define

$$d_g = \min\big\{|g(y)-g(x)|\colon \{x,y\}\in E,\ g(x)\ne g(y)\big\} \in (0,\infty),$$

where we recall that g^2 is defined on the entire $\mathbb{Z}^d$ and is zero outside B. Consider the open ball in $\mathcal{M}_1(B)$ of radius $\delta_g := \min\{d_g^4, \delta\}$ centered at g^2. Fixing a configuration $\varphi \in (0,\infty)^E$, we can apply Lemma 2.4 with $f(\cdot) := g(\cdot) + \sqrt{\delta_g}\mathbb{1}_B$ and obtain

$$\mathbb{P}_0^\varphi\Big(\tfrac{1}{t}\ell_t \in B_{\delta_g}(g^2)\Big) \le \frac{1+\sqrt{\delta_g}}{\sqrt{\delta_g}} \exp\Big\{t \sup_{h^2 \in B_{\delta_g}(g^2)} \sum_{x\in B} \frac{\Delta^\varphi(g+\sqrt{\delta_g}\mathbb{1}_B)(x)}{g(x)+\sqrt{\delta_g}} h^2(x)\Big\}. \quad (2.16)$$

In what follows, we show

$$\sup_{h^2 \in B_{\delta_g}(g^2)} \sum_{x\in B} \frac{\Delta^\varphi(g+\sqrt{\delta}_g\mathbb{1}_B)(x)}{g(x)+\sqrt{\delta}_g} h^2(x) \le -I_\varphi(g^2)(1-7\delta^{\frac{1}{4}}), \quad (2.17)$$

where we recall from (2.2) that $I_\varphi(g^2) = \sum_{\{x,y\}\in E} \varphi_{xy}|g(x)-g(y)|^2 = -(\Delta^\varphi g, g)$. To that end, we replace h^2 by $(g+\sqrt{\delta_g}\mathbb{1}_B)^2$ and control the error terms.

$$\begin{aligned} &\sup_{h^2 \in B_{\delta_g}(g^2)} \sum_{x\in B} \frac{\Delta^\varphi(g+\sqrt{\delta_g}\mathbb{1}_B)(x)}{g(x)+\sqrt{\delta_g}} h^2(x) \\ &\quad = \sum_{x\in B} \frac{\Delta^\varphi(g+\sqrt{\delta_g}\mathbb{1}_B)(x)}{g(x)+\sqrt{\delta_g}} (g(x)+\sqrt{\delta_g})^2 \\ &\qquad + \sup_{h^2 \in B_{\delta_g}(g^2)} \sum_{x\in B} \frac{\Delta^\varphi(g+\sqrt{\delta_g}\mathbb{1}_B)(x)}{g(x)+\sqrt{\delta_g}} \big[(h^2(x)-g^2(x)) - 2\sqrt{\delta_g}g(x) - \delta_g\big]. \end{aligned} \quad (2.18)$$

The first sum is easily estimated against the standard Donsker-Varadan rate function:

$$\begin{aligned} \sum_{x\in B} \frac{\Delta^\varphi(g+\sqrt{\delta_g}\mathbb{1}_B)(x)}{g(x)+\sqrt{\delta_g}} (g(x)+\sqrt{\delta_g})^2 &= \big(\Delta^\varphi(g+\sqrt{\delta_g}\mathbb{1}_B), g+\sqrt{\delta_g}\mathbb{1}_B\big) \\ &\le \big(\Delta^\varphi g, g\big) = -I_\varphi(g^2), \end{aligned}$$

where we have used the symmetry of the operator Δ^φ and that $g = 0$ outside B. In order to estimate the last term in (2.18), we treat the contribution of every summand within the square brackets separately. We begin with the first part and observe that

$|h^2(x)-g^2(x)| = |h(x)-g(x)|\,|h(x)+g(x)| \le 2\delta_g$ for all $h^2 \in B_{\delta_g}(g^2)$ and $x \in B$. Thus

$$\begin{aligned}
&\sum_{x\in B} \frac{\Delta^\varphi(g+\sqrt{\delta_g}\mathbb{1}_B)(x)}{g(x)+\sqrt{\delta_g}}(h^2(x)-g^2(x)) \\
&\quad = \sum_{\substack{\{x,y\}\in E:\\ x,y\in B}} \varphi_{xy}\frac{g(y)-g(x)}{g(x)+\sqrt{\delta_g}}(h^2(x)-g^2(x)) - \sum_{\substack{\{x,y\}\in E:\\ x\in B, y\notin B}} \varphi_{xy}(h^2(x)-g^2(x)) \\
&\quad \le \sum_{\substack{\{x,y\}\in E\\ x,y\in B}} \varphi_{xy}\frac{|g(x)-g(y)|}{\sqrt{\delta_g}}2\delta_g + \sum_{\substack{\{x,y\}\in E:\\ x\in B, y\notin B}} \varphi_{xy}2\delta_g \\
&\quad \le 4\delta^{\frac14} I_\varphi(g^2).
\end{aligned}$$

The last step is due to the fact that $\delta_g^{\frac14} \le g(x)-g(y)$ whenever $g(x)-g(y)>0$. Secondly,

$$\begin{aligned}
&\sum_{x\in B} \frac{\Delta^\varphi(g+\sqrt{\delta_g}\mathbb{1}_B)(x)}{g(x)+\sqrt{\delta_g}}(-2\sqrt{\delta_g}g(x)) \\
&\quad \le \sum_{\substack{\{x,y\}\in E:\\ x,y\in B}} \varphi_{xy}|g(x)-g(y)|\Big|\frac{2\sqrt{\delta_g}g(x)}{g(x)+\sqrt{\delta_g}} - \frac{2\sqrt{\delta_g}g(y)}{g(y)+\sqrt{\delta_g}}\Big| + \sum_{\substack{\{x,y\}\in E:\\ x\in B, y\notin B}} \varphi_{xy}2\sqrt{\delta_g}g(x) \\
&\quad \le \sum_{\substack{\{x,y\}\in E:\\ x,y\in B}} \varphi_{xy}|g(x)-g(y)|^2\frac{2\delta_g}{\sqrt{\delta_g}d_g} + \sum_{\substack{\{x,y\}\in E:\\ x\in B, y\notin B}} \varphi_{xy}2\sqrt{\delta_g}|g(x)-g(y)| \\
&\quad \le 2\delta^{\frac14} I_\varphi(g^2).
\end{aligned}$$

Here, we have used $\delta_g^{\frac14} \le d_g$. The only part left is

$$\begin{aligned}
&\sum_{x\in B} \frac{\Delta^\varphi(g+\sqrt{\delta_g}\mathbb{1}_B)(x)}{g(x)+\sqrt{\delta_g}}(-\delta_g) \\
&\quad \le \sum_{\substack{\{x,y\}\in E:\\ x,y\in B}} \varphi_{xy}|g(x)-g(y)|\Big|\frac{1}{g(x)+\sqrt{\delta_g}} - \frac{1}{g(y)+\sqrt{\delta_g}}\Big|\delta_g + \sum_{\substack{\{x,y\}\in E:\\ x\in B, y\notin B}} \varphi_{xy}\delta_g \\
&\quad \le \sum_{\substack{\{x,y\}\in E:\\ x,y\in B}} \varphi_{xy}|g(x)-g(y)|^2\frac{1}{\sqrt{\delta_g}d_g}\delta_g + \sum_{\substack{\{x,y\}\in E:\\ x\in B, y\notin B}} \varphi_{xy}\delta_g \\
&\quad \le \delta^{\frac14} I_\varphi(g^2).
\end{aligned}$$

Combining (2.18) with the last three estimates, we obtain (2.17) and in particular

$$\mathbb{P}_0^\varphi\Big(\tfrac1t \ell_t \in B_\delta(g^2)\Big) \le \frac{1+\sqrt{\delta_g}}{\sqrt{\delta_g}} \prod_{\{x,y\}\in E} \exp\big\{-t\,\varphi_{xy}|g(x)-g(y)|^2(1-7\delta^{\frac14})\big\}. \tag{2.19}$$

The balls $B_{\delta_g}(g^2)$ with $g^2 \in C$ cover C and since this set is compact, we may extract a finite subcovering of C. Denote by $(g_i^2)_{i=1,\dots,N}$ the centers of the balls in this subcovering. Then, applying (2.19) for $\varphi = t^{\frac{1}{\eta}}\omega$, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\Big\langle \mathbb{P}_0^{t^{\frac{1}{\eta}}\omega}\Big(\tfrac{1}{t}\ell_t \in C\Big)\Big\rangle \\
&\qquad \le \max_{i=1,\dots,N}\limsup_{t\to\infty}\frac{1}{t}\log\Big\langle \mathbb{P}_0^{t^{\frac{1}{\eta}}\omega}\Big(\tfrac{1}{t}\ell_t \in B_{\delta_{g_i}}(g_i^2)\Big)\Big\rangle \\
&\qquad \le \max_{i=1,\dots,N}\sum_{\{x,y\}\in E_B}\limsup_{t\to\infty}\frac{1}{t}\log\Big\langle \exp\big\{-t^{\frac{1+\eta}{\eta}}\omega_{xy}|g_i(y)-g_i(x)|^2(1-7\delta^{\frac{1}{4}})\big\}\Big\rangle.
\end{aligned}$$

According to de Bruijn's exponential Tauberian theorem [BGT89, Theorem 4.12.9], the tail assumption (1.14) is equivalent to the condition that, for any $M > 0$ and $\{x, y\} \in E$,

$$\lim_{t\to\infty}\frac{1}{t}\log\Big\langle \exp\big\{-t^{\frac{1+\eta}{\eta}}\omega_{xy}M\big\}\Big\rangle = -K_{\eta,D}M^{\frac{\eta}{1+\eta}}, \tag{2.20}$$

where we recall $K_{\eta,D} = \big(1+\frac{1}{\eta}\big)(D\eta)^{\frac{1}{\eta+1}}$ from Theorem 1.3. Thus, with δ so small that $1-7\delta^{\frac{1}{4}} > 0$, we obtain

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{t}\log\Big\langle \mathbb{P}^{t^{\frac{1}{\eta}}\omega}\Big(\tfrac{1}{t}\ell_t \in C\Big)\Big\rangle &\le \max_{i=1,\dots,N}\sum_{\{x,y\}\in E_B} -K_{\eta,D}|g_i(y)-g_i(x)|^{\frac{2\eta}{1+\eta}}(1-7\delta^{\frac{1}{4}})^{\frac{\eta}{1+\eta}} \\
&\le -(1-7\delta^{\frac{1}{4}})^{\frac{\eta}{1+\eta}}\inf_{g^2\in C} J(g^2)
\end{aligned}$$

with J as in (2.7). Since we may choose δ arbitrarily small, the proof of (1.18) is complete.

2.3 Proof of Corollary 1.6

Proof. A Fourier expansion shows that, Pr-almost surely,

$$\mathbb{P}_0^\omega(X_{[0,t]}\subset B) = \sum_{i=1}^{|B|} \mathrm{e}^{-t\lambda_i^\omega} v_i^\omega(0)(v_i^\omega, 1\!\!1) \le \sum_{i=1}^{|B|}\mathrm{e}^{-t\lambda_i^\omega}|B| \le |B|^2\mathrm{e}^{-t\lambda^\omega(B)},$$

where $0 < \lambda^\omega(B) = \lambda_1^\omega \le \cdots \le \lambda_{|B|}^\omega$ are the eigenvalues of Δ^ω with zero boundary condition in B and $(v_i^\omega)_{i=1,\dots,|B|}$ a corresponding orthonormal base of eigenvectors. We also have, Pr-almost surely,

$$\mathrm{e}^{-t\lambda^\omega(B)} \le \sum_{i=1}^{|B|}\mathrm{e}^{-t\lambda_i^\omega}(v_i^\omega, 1\!\!1)^2 \le \sum_{z\in B}\mathbb{P}_z^\omega(X_{[0,t]}\subset B).$$

Applying Theorem 1.3 to $B - z$ and using the shift-invariance of ω, we see that the expectation of the right-hand side has the same logarithmic asymptotics as $\langle \mathbb{P}_0^\omega(X_{[0,t]} \subset B)\rangle$. Therefore, the two above inequalities show that

$$\log \left\langle \mathrm{e}^{-t\lambda^\omega(B)} \right\rangle \sim \log \left\langle \mathbb{P}_0^\omega(X_{[0,t]} \subset B) \right\rangle, \qquad t \to \infty. \tag{2.21}$$

Now de Bruijn's exponential Tauberian theorem [BGT89, Theorem 4.12.9], together with (1.19) yields the desired asymptotics. □

2.4 Outlook: growing domains

The LDP in Theorem 1.3 is formulated for the simplest domain possible, that is a finite set of points. What happens if we let the domain grow with time?

Imagine for simplicity to have a box B and to blow it up by a factor α_t. We require of course that α_t grows in time, but also that $\alpha_t \ll t^{1/2}$ in order to give to the random walk the time to fill the whole blown up box $\alpha_t B$. The problem emerging from this new scenario is the meaning of an LDP on a sequence of different spaces (the box changes its size as the time passes). In order to make sense out of it, we must rescale the boxes mapping them to a common space, say the initial box B. This brings to highly non-trivial analytical problems, as we pass from a discrete to a contionuous setting, wishing to replace discrete gradients with a derivatives.

An interesting choice for the rate of growth could be $\alpha_t = t^{\frac{1}{d+2}}$. In fact, this guarantees that the exponential rate of decay of the probability for the random walk to stay in the box $\alpha_t B$ (of the order $\frac{t}{\alpha_t^2}$) matches the exponential rate of decay for the probability of "controlling" the value of the $d \cdot \alpha_t^d$ conductances in the box.

König and Wolff have already investigated the case when the i.i.d. conductances have a double stretched exponential tail near zero as in Theorem 1.3 and found a variety of very interesting possible behaviours.

Theorem 2.5. *Assume that ω satisfies* (1.13) *and* (1.14) *with $\eta > \frac{d}{2}$ ($\eta > 1$ if $d = 1$), and in addition that $\omega_{xy} 1\!\!1_{\{\omega_{xy} < \varepsilon\}}$ has for some $\varepsilon > 0$ a continuous increasing density. Suppose $1 \ll \alpha_t \ll t^{\frac{1}{d+2}}$ is increasing and fix a set $G \subset \mathbb{R}^d$ that is open, connected, bounded, with sufficiently regular boundary and containing the origin. Let $\mathcal{F} := \{f^2 : f \in L^2(G), ||f||_2 = 1\}$ equipped with the weak topology of integrals against bounded* *continuous functions. Then the process of (properly rescaled and normalized) local times* $L_t(x) := \frac{\alpha_t^d}{t}\ell_t(\lfloor \alpha_t x \rfloor)$ *satisfies*

$$\liminf_{t\to\infty} \frac{1}{\gamma_t} \log \left\langle \mathbb{P}_0^\omega (L_t \in O | \operatorname{supp}\ell_t \subset \alpha_t G \right\rangle \geq - \inf_{f^2 \in O} J_0(f^2) \qquad \text{for all } O \subset \mathcal{F} \text{ open,}$$

where the speed is $\gamma_t = t^{\frac{\eta}{1+\eta}} \alpha_t^{\frac{d-2\eta}{1+\eta}}$, $J_0 = J - \inf_{g\in\mathcal{F}} J$ *and*

$$J(f^2) := \begin{cases} K_{\eta,D} \sum_{i=1}^d \int_G |\partial_i f(y)|^{\frac{2\eta}{1+\eta}} \mathrm{d}y & \text{if } f \in H_0^1(G) \\ \infty & \text{otherwise} \end{cases}. \tag{2.22}$$

$K_{\eta,D}$ *is the same constant as in Theorem 1.3. Furthermore,* J_0 *has compact level sets and for the non-exit probabilities also the corresponding upper bound holds:*

$$\liminf_{t\to\infty} \frac{1}{\gamma_t} \log \left\langle \mathbb{P}_0^\omega (\operatorname{supp}\ell_t \subset \alpha_t G \right\rangle \leq - \inf_{f^2 \in \mathcal{F}} J(f^2). \tag{2.23}$$

The proof of this theorem can be found in the Ph.D. thesis of Tilman Wolff [Wol13]. In fact, it is expected that a full LDP with the rate function described in (2.22) holds. For the non-exit probability it is also shown therein that the exponent $\eta = \frac{1}{2}$ is critical: below this threshold the speed of the LDP is no longer the same. This is due to the fact that, thanks to the fat tails, it is possible for the conductances to assume extremely small values in bounded regions, trapping the random walk.

While in the finite-box case the assumption of conductances that can attain arbitrarily small values was fundamental in order to have interesting results, this is not true anymore for the growing-box case. The limiting shape of the local times is hard to predict even in the simplest settings, for example when the conductances can assume only two values, say 1 and 2. Homogenization Theory may be the key ingredient (see e.g. the harmonic coordinate technique in Chapter 4) to be combined with the tools provided in the previous sections.

It is also worth noticing that the growing-domain case brings as a natural question the quenched behaviour of the local times, which was trivial in the finite-domain setting. The simplest case, i.e. strongly elliptic conductances, has also been treated in [Wol13].

Chapter 3

The speed of the RWRC

In Section 3.1 we show Part 1 of Theorem 1.9, which ends up being a simple application of the Varopoulos-Carne Theorem. In Sections 3.2 and 3.3 we show Part 2 of Theorem 1.9. The construction builds upon the example constructed by Bramson, Zeitouni and Zerner in [BZZ06].

Finally, in Section 3.4 we give a proof of Theorem 1.8.

3.1 Moment conditions for speed zero

In this section we prove Part 1 of Theorem 1.9.

In order to prove it, we will use the well known Varopoulos-Carne bound. For proof see, e.g., [Car85].

Lemma 3.1 (Varopoulos-Carne)**.** *Let L be an irreducible Markov transition kernel with reversible measure π. For states x and y, denote $d(x,y) = \min\{n : L^n(x,y) > 0\}$. Then for every x, y and n,*

$$L^n(x,y) \leq 2\sqrt{\frac{\pi(y)}{\pi(x)}} \cdot e^{-\frac{d(x,y)^2}{2n}}. \tag{3.1}$$

Proof of Part 1 of Theorem 1.9. The measure π on $\mathbb{Z}^2$, defined by $\pi(x) = \sum_{y\sim x} \omega_{\{x,y\}}$ is a reversible measure for our random walk. As in (1.21), let

$$D = E\left[\log^{\alpha} \omega_e\right] < \infty.$$

For $n \in \mathbb{N}$, consider the points $x \in \mathbb{Z}^2$ such that $||x||_\infty = n$, and call E_n the set of edges having at least one end in these points. Note that $|E_n| = 24n$.

Then by Markov's inequality, for every $n \in \mathbb{N}$ and $K > 0$ we get

$$\mathbb{P}\big(\exists e \in E_n \text{ s.t. } \omega_e > \tfrac{K}{4}\big) \leq 24n \frac{D}{\log^{\alpha}(\frac{K}{4})}.$$

In particular, if $K = e^{n^{\beta}}$ with $1/\alpha < \beta < 1$, then

$$\mathbb{P}\left(\exists e \in E_n \text{ s.t. } \omega_e > \tfrac{K}{4}\right) \leq C n^{1-\alpha\beta},$$

for some constant $C > 0$.

Observe that $1 - \alpha\beta < 0$. Therefore, by Borel-Cantelli lemma, for an integer $\kappa > (\alpha\beta - 1)^{-1}$, a.s. for all n large enough and every edge $e \in E_{n^{\kappa}}$, we have

$$\omega_e \leq \tfrac{1}{4} e^{n^{\kappa\beta}}.$$

Therefore, for every x s.t. $\|x\|_{\infty} = n^{\kappa}$, we have that $\pi(x) \leq e^{n^{\kappa\beta}}$.

Now fix $M \in \mathbb{N}$ and assume that M is large. For every n large enough,

$$\begin{aligned}
P^{\omega}\big(\|X_{Mn^{\kappa}}\|_{\infty} > n^{\kappa}\big) &\leq P^{\omega}\big(\exists k \leq Mn^{\kappa} : \|X_{Mn^{\kappa}}\|_{\infty} = n^{\kappa}\big) \\
&\leq \sum_{k=1}^{Mn^{\kappa}} \sum_{x:\, \|x\|_{\infty} = n^{\kappa}} P^{\omega}(X_k = x) \\
&\leq \sum_{k=1}^{Mn^{\kappa}} \sum_{x:\, \|x\|_{\infty} = n^{\kappa}} 2\sqrt{\tfrac{\pi(x)}{\pi(0)}} e^{-\frac{n^{2\kappa}}{2k}} \\
&\leq C' \pi(0)^{-1/2} \exp\left\{\tfrac{n^{\kappa\beta}}{2} - \tfrac{n^{\kappa}}{2M}\right\},
\end{aligned}$$

for some constant $C' > 0$.

Therefore, again by Borel-Cantelli, almost surely for all n large enough,

$$\|X_{Mn^{\kappa}}\|_{\infty} \leq n^{\kappa}.$$

From here we immediately get that almost surely

$$\limsup_{n\to\infty} \frac{\|X_n\|_{\infty}}{n} \leq \frac{2}{M}$$

and in fact, since M is arbitrary,

$$\mathbb{P}\Big(\lim_{n\to\infty} \frac{X_n}{n} = 0\Big) = 1.$$

□

3.2 Trees

In this section and in the next one we prove Part 2 of Theorem 1.9. The section is divided into two different subsections. In Subsection 3.2.1 we create the structure for the random environment where, with probability one, the sequence $\left(\frac{X_n}{n}\right)$ does not converge, and in Subsection 3.2.2 we create another example where with probability one the sequence $\left(\frac{X_n}{n}\right)$ converges to a speed which is not zero. In both cases $E[\log^\alpha \omega_e] < \infty$ for arbitrary $\alpha < 1$. The example in Subsection 3.2.1 is a direct application of the tree construction of Bramson, Zeitouni and Zerner [BZZ06]. For the construction in Subsection 3.2.2, we need to modify the tree of [BZZ06]. The construction is inspired by the construction in [BZZ06], but we need to change quite a few details in order for the speed to converge.

In both cases, we adapt trees into environments for the random walk in the exact same fashion. This is done in Section 3.3. Now, we give a short introduction with the necessary terms from [BZZ06], and then, in Subsection 3.2.1 and 3.2.2, we create the actual trees.

An *ancestral function* is a (in our case random) function $a : x \in \mathbb{Z}^2 \to a(x) \in \mathbb{Z}^2$ with the following properties:

- x and $a(x)$ are nearest neighbours;
- $a(a(x)) \neq x$;
- the set of edges $F_a := \{\{x, a(x)\} : x \in \mathbb{Z}^2\}$ is a forest (i.e. the graph $(\mathbb{Z}^2, F_a)$ contains no cycles).

Every connected component of F_a is an infinite tree. $a(x)$ can be seen as the parent of x and we denote by $a^n(x)$ the n-th generation ancestor of x, for $n \geq 0$ (with the convention $a^0(x) = x$).

We also say that an ancestral function is directed if for some $i, j \in \{+1, -1\}$ and for every $x \in Z^2$, $a(x) - x \in \{(0, i), (j, 0)\}$.

The length of the longest branch starting in x (or the distance from x of its farthest descendant, if one prefers the genealogical metaphore) is

$$h(x) := \sup\{n \geq 0 : \exists y \in \mathbb{Z}^2 \text{ such that } a^n(y) = x\}. \qquad (3.2)$$

We are interested in the distribution of $h(0)$ in the case of a random translation invariant ancestral function.

Theorem 1 in [BZZ06] says that for any stationary ancestral function there exists a constant $c \geq 0$ such that

$$\liminf_{n\to\infty} \tfrac{1}{n}\mathbb{P}(h(0) \geq n) \geq c. \tag{3.3}$$

In the same article the authors show that this is in fact the best lower bound achievable. We give the 2-dimensional version of Theorem 2 in that paper:

Theorem 3.2 ([BZZ06], Theorem 2)**.** *There exists a stationary directed ancestral function $(a(x))_{x\in\mathbb{Z}^2}$ that is polinomially mixing of order 1 and for which*

$$\limsup_{n\to\infty} n\mathbb{P}(h(0) \geq n) < \infty. \tag{3.4}$$

We now describe the BZZ tree, as appearing in [BZZ06].

3.2.1 The BZZ tree

We provide now the construction of the ancestral function used in [BZZ06], restricted to the 2 dimensional case. We will make use of the same notations as [BZZ06] with an additional tilde.

Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{Z}^2$, with e_1 parallel to the x-axis. Fix two constants $\tilde{\theta}$ and $\tilde{n}_0 \in \mathbb{N}$ such that $2\sqrt{2} \leq \tilde{\theta} \leq \tilde{n}_0^2$. For every $x \in \mathbb{Z}^2$ let $\tilde{L}(x)$ be i.i.d. random variables with atomless distribution and satisfying

$$\tilde{\mathrm{P}}(\tilde{L}(x) > t) = \frac{\tilde{\theta}}{t^2} \qquad \text{for } t \geq \tilde{n}_0. \tag{3.5}$$

We define an *umbrella* of intesity t to be

$$\tilde{U}_t = \bigcup_{i=1,2} \tilde{U}_{i,t} \tag{3.6}$$

where

$$\tilde{U}_{i,t} = \left\{y = (y_1, y_2) \in \mathbb{Z}^2 : y_i = 0, y_j \in (0,t], j \neq i\right\} \tag{3.7}$$

are the sides of the umbrella. The strength of the umbrella is also defined to be equal to its intensity.

For every $x \in \mathbb{Z}^2$ we will open the umbrella $x + \tilde{U}_{\tilde{L}(x)}$. Informally, one can think of the ancestral function as a drop of rain trying to fall towards the up-right direction of the plane and sliding on the sides of the umbrellas. Whenever two or more umbrellas overlap, the water will consider only the strongest of them and penetrate the perpendicular ones.

Formally, one defines for every $x \in \mathbb{Z}^2$ the strongest umbrella passing through that point perpendicular to direction e_i, for $i \in \{1, 2\}$, as

$$\tilde{\lambda}_i(x) = \sup_{y \in \mathbb{Z}^2 :\, x \in y + \tilde{U}_{i,\tilde{L}(y)}} \tilde{L}(y). \tag{3.8}$$

Note that the sup is taken over a non-empty set and it is easy to show that $\tilde{\lambda}_i(x)$ is also a.s. finite.

Since the distributions of the $\tilde{L}(x)$'s are atomless, the direction $I(x) \in \{1, 2\}$ such that

$$\tilde{\lambda}_{I(x)}(x) = \min\{\tilde{\lambda}_i(x), i = 1, 2\}$$

is well defined. The ancestral function we are looking for is

$$\tilde{a}(x) = x + e_{I(x)}. \tag{3.9}$$

The set of edges $\big\{\{x, \tilde{a}(x)\},\, x \in \mathbb{Z}^2\big\}$ through which the drops of rain have flown forms a random forest (which can be shown to be in fact a random tree spanning the whole $\mathbb{Z}^2$). This is the ancestral function used to prove Theorem 3.2, and we will call the graph obtained with it the BZZ tree.

3.2.2 The Diagonal tree

We will now slightly modify the example seen in the previous subsection. Our aim is to build a new tree for which the behaviour of $h(0)$ is essentially the same as in the BZZ tree, but with a different shape of the graph. Roughly speaking, it will not allow to have long strips that are "too horizontal" or "too vertical". This feature and its importance will become more clear when we will describe the dynamics on these trees.

Fix suitable constants θ and $n_0 \in \mathbb{N}$ such that $10 \le \theta \le n_0^2$ and so that following equation (3.10) makes sense. For every $x \in \mathbb{Z}^2$ consider i.i.d. random variables $L(x) > 1$ with atomless distributions fulfilling

$$\mathrm{P}(L(0) > t) = \frac{\theta \log t}{t^2} \qquad \text{for all } t \ge n_0. \tag{3.10}$$

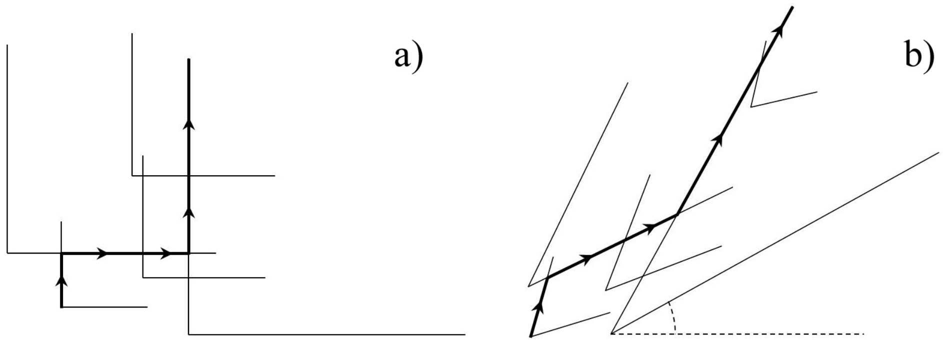

Figure 3.1: *Both in the straight umbrellas case* a) *and in the narrow umbrellas case* b) *the drop of water follows the side of the biggest umbrella met. Note that in* b) *the longest umbrellas are also the ones that are the narrowest.*

The new umbrellas we want to open are a bit different from the tilde-umbrellas of the previous section.

Define an umbrella of intensity t as

$$U_t = \bigcup_{i=1,2} U_{i,t} \tag{3.11}$$

where $U_{2,t}$ is the best $\mathbb{Z}^2$-grid lower approximation of the open segment of length t that makes an angle of $\frac{\pi}{4} - \frac{1}{\log t}$ with the x-axis, living in the first quadrant and starting in the origin. $U_{1,t}$ is the reflection of $U_{2,t}$ with respect to the bisecting line of the first quadrant. $U_{1,t}$ and $U_{2,t}$ are the sides of the umbrella. Note that this time the intensity gives us the strength, the length but also the width of the umbrella. In particular, the longer the umbrella, the more narrow it is.

We can think once more that drops of rain pouring from every point of the lattice try to fall towards the up-right direction and that every time they reach a new vertex, they are deflected by the strongest umbrella that passes through that vertex (see Figure 3.1).

In analogy with the straight-umbrellas case we define the strongest umbrella through x perpendicular to direction e_i, for $i, j \in \{1, 2\}$ and $i \neq j$, as

$$\lambda_i(x) = \sup_{y\in\mathbb{Z}^2 \colon [x,x+e_j]\in y+U_{L(y)}} L(y). \tag{3.12}$$

Note that since $L(0) > 1$ and since we are taking the lower (for the first component) and upper (for the second) approximations of the segments described above, $[x, x+e_1] \in U_{2,L(x-e_1)}$ and $[x, x+e_2] \in U_{1,L(x-e_2)}$, so that the sup on the right hand side of (3.12) is taken over a non-empty set. It requires slightly more work compared to the straight-umbrellas case to prove that it is also a.s. finite and therefore well defined.

We need some more notations. Similarly to [BZZ06], for $m, n \in \mathbb{Z}$ call S_m^n the slab

$$S_m^n = \big\{x = (x_1, x_2) \in \mathbb{Z}^2 : m \le x_1 + x_2 \le n\big\}.$$

The protecting area G (see Figure 3.2) is defined as

$$\begin{aligned} G := \Big\{ & x = (x_1, x_2) \in -\mathbb{N}^2 \big| \, \exists n \in \mathbb{N} : \\ & x \in S_{-n}^{-n} \text{ and } -x_1 \in \Big[y_n \cdot \cos\left(\tfrac{\pi}{4} - \alpha_n\right), y_n \cdot \cos\left(\tfrac{\pi}{4} + \alpha_n\right) \Big] \Big\}, \end{aligned} \tag{3.13}$$

where $\alpha_n = \arctan \frac{\sqrt{2}}{3 \log n}$ and $y_n = \frac{n}{3\sqrt{2}\log n}\sqrt{2 + 9\log^2 n}$. These values guarantee that every segment $S_{-m}^{-m} \cap G$ is $\frac{2m}{3\log m}$ long, and therefore contains $\frac{\sqrt{2}m}{3\log m}$ points of $\mathbb{Z}^2$ (up to one unit, at most).

Note that every umbrella $x + U_s$ with $x \in G$, $-(x_1 + x_2) = n$ and $s \in [n, n^2]$, "protects" the origin O, meaning that O lies inside the "$\mathbb{Z}^2$-triangle" generated by the sides $x + U_{1,s}$ and $x + U_{2,s}$.

Lemma 3.3. *There is a constant c such that for $i = 1, 2$ and $t > n_0$,*

$$\mathbb{P}(\lambda_i(0) > t) \le c \frac{\log t}{t}. \tag{3.14}$$

Proof. This is a straightforward calculation.

$$\begin{aligned} \mathbb{P}(\lambda_i(0) > t) \quad &\le \quad C \int_t^\infty [s] \frac{\log s}{s^3} ds \\ \le C \sum_{k=0}^\infty \int_{2^k t}^{2^{k+1} t} s \frac{\log s}{s^3} ds \quad &\le \quad C \sum_{k=0}^\infty \frac{\log 2^k t}{2^k t} \\ = C \frac{1}{t} \sum_{k=0}^\infty \frac{1}{2^k} [\log t + \log 2^k] \quad &\le \quad c \frac{\log t}{t}. \end{aligned}$$

□

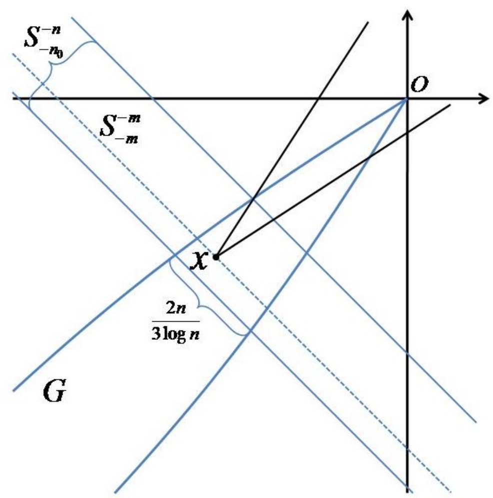

Figure 3.2: *The protecting area G is the region of the plane from which we can have umbrellas that protect the origin. In particular, having a suitably strong umbrella starting in the part of G delimited by the slab $S^{-n}_{-n_0}$ will ensure $h(0) < n$ with high probability.*

Also in this case, the fact that the distributions of the $L(x)$'s are atomless guarantees the uniqueness of a direction $I(x) \in \{1, 2\}$ such that

$$\lambda_{I(x)}(x) = \min\{\lambda_i(x), i = 1, 2\}.$$

For example, if $I(x) = 1$, it means that the strongest vertical umbrella through x is weaker than the strongest horizontal one. $I(x)$ is the direction which the drop of water will follow.

We can therefore define the new ancestral function

$$a(x) = x + e_{I(x)}. \tag{3.15}$$

By its construction, it follows automatically that $a : \mathbb{Z}^2 \to \mathbb{Z}^2$ is stationary and directed.

Theorem 3.4. *The random ancestral function described in* (3.15) *is such that*

$$\limsup_{n\to\infty} \frac{n}{\log^2 n} \mathbb{P}(h(0) > n) < \infty. \tag{3.16}$$

3.2.3 Proof of Theorem 3.4

We closely follow the proof of Theorem 2 in [BZZ06].

We say that an umbrella U penetrates a weaker umbrella V in point $x \in \mathbb{Z}^2$ if one side of U intersect one side of V and x is the upper-right point of their intersection. The following lemma bounds the probability that an umbrella of intensity t starting in the origin gets penetrated by another umbrella in a given point z.

Lemma 3.5. *Fix any $t > n_0$. Let $z \in \mathbb{Z}^2$ such that $[z, z + e_i] \in U_{j,t}$, for some $i, j \in \{1, 2\}$. Then there exists a constant $c > 0$ independent of t such that*

$$\mathbb{P}\big(I(z) \neq i | L(0) = t\big) \leq c\,\frac{\log t}{t}. \tag{3.17}$$

Proof. For convenience, we shift the umbrella so that z is translated to the origin. We look first at the event E_k that the umbrella gets penetrated in the origin by an umbrella of intensity $s \in [k, k+1]$, for $k+1 > t$. Note that such a penetrating umbrella can come only from S^{-1}_{-k-1}. Furthermore, on every S^m_m, $m \in \{-k-1, ..., -1\}$, there are almost surely at most four points that can generate it, since for all the others the slope of the sides would prevent them from penetrating the original umbrella in the origin. Hence

$$\mathbb{P}(E_k) \leq \sum_{m=1}^{k+1} 4\left(\frac{\theta \log k}{k^2} - \frac{\theta \log(k+1)}{(k+1)^2}\right) \leq c'\,\frac{\log k}{k^2},$$

for some constant c'.

It is now easy to see that

$$\mathbb{P}\big(I(0) \neq i | L(-z) = t\big) \leq \sum_{k=\lfloor t \rfloor}^{\infty} \mathbb{P}(E_k) \leq c\,\frac{\log t}{t}.$$

□

For $n \geq n_0$, define now the random variables

$$M_n := \max\big\{m \in \{n_0, ..., n\} : \exists x \in S^{-m}_{-m} \cap G \text{ with } m < L(x) < m^2\big\}, \tag{3.18}$$

with the convention $M_n = n_0 - 1$ whenever the set on the right hand side is empty.

Proving that, for some constant c,

$$\mathbb{P}(h(0) > m,\, M_n = m) \le c\frac{\log^2 n}{n^2} \qquad \forall m = n_0, ..., n, \tag{3.19}$$

would imply

$$\mathbb{P}(h(0) > n) \le \sum_{m=n_0-1}^{n} \mathbb{P}(h(0) > m,\, M_n = m) \le c\frac{\log^2 n}{n}, \tag{3.20}$$

that is the statement of the theorem.

We first prove (3.19) in the easy case $m = n_0 - 1$.

$$\begin{aligned}
\mathbb{P}(h(0) > m, M_n = n_0 - 1) &\le \mathbb{P}(M_n = n_0 - 1) \\
&= \mathbb{P}\big(\text{for all } m = n_0, ..., n \text{ and } x \in S_{-m}^{-m} \cap G,\, L(y) \notin (m, m+1]\big) \\
&= \prod_{m=n_0}^{n} \big(1 - \mathbb{P}(L(0) \ge m) + \mathbb{P}(L(0) > m^2)\big)^{\#(S_{-m}^{-m} \cap G)} \\
&= \prod_{m=n_0}^{n} \Big(1 - \frac{\theta \log m}{m^2} + \frac{\theta \log m^2}{m^4}\Big)^{\frac{\sqrt{2}}{3}\frac{m}{\log m}} \\
&\le \prod_{m=n_0}^{n} \Big(1 - \theta\big(1 - \tfrac{2}{n_0^2}\big)\frac{\log m}{m^2}\Big)^{\frac{\sqrt{2}}{3}\frac{m}{\log m}} \\
&\le \mathrm{e}^{-\theta\left(1 - \frac{2}{n_0^2}\right)\frac{\sqrt{2}}{3}\sum_{m=n_0}^{n}\frac{1}{m}} \le c\, n^{-2}
\end{aligned} \tag{3.21}$$

by the choice of θ, for some $c > 0$.

For the more complex cases $m = n_0, ..., n$ we faithfully follow [BZZ06] once again. For $m, n, r \in \mathbb{Z}$, $x \in \mathbb{Z}^2$, define the events

$$A_m^n(x, r) = \big\{L(y) \notin \big(-y \cdot \vec{1} + r, (-y \cdot \vec{1} + r)^2\big) \text{ for all } y \in S_m^n \cap (x + G)\big\}. \tag{3.22}$$

Firstly note that

$$\begin{aligned}
\mathbb{P}\big(h(0) > m, M_n = m\big) &\le \sum_{x \in S_{-m}^{-m} \cap G} \mathbb{P}\big(h(0) > m,\, L(x) \in (m, m^2), A_{m-n}^{-1}(0, 0)\big) \\
&= \sum_{x \in S_{-m}^{-m} \cap G} \mathbb{P}\big(h(-x) > m,\, L(0) \in (m, m^2), A_{m-n}^{-1}(-x, m)\big) \\
&= \sum_{x \in S_m^m \cap -G} \mathbb{P}\big(h(x) > m,\, L(0) \in (m, m^2), A_{m-n}^{-1}(x, m)\big),
\end{aligned}$$

where we have used stationarity to obtain the second line and we write $-G = \big\{x = (x_1, x_2) : (-x_1, -x_2) \in G\big\}$.

Take now the segment $S_m^m \cap -G$ and divide it in eight parts of the same length and call them $I_1, ..., I_8$. For every $j \in \{1, ..., 8\}$, consider $\hat{x}^j$ and $\check{x}^j$, the points with respectively the highest and the lowest y-coordinate on I_j. Draw the cones $\hat{C}^j$ and $\check{C}^j$ with amplitude $\frac{2}{\log m^2}$ whose bisector makes an angle of $\frac{5}{4}\pi$ with the x-axis and with vertices $\hat{x}^j$ and $\check{x}^j$ respectively. Observe that the points in the area $\hat{C}^j \cap \check{C}^j \cap S^{-1}_{-n+m}$ are contained in $S^{-m}_{-n+m} \cap (x + G)$ for every $x \in I^j$. Therefore the event

$$E_j(m, n) := \big\{L(y) \notin \big(- (y_1 + y_2) + m, (-(y_1 + y_2) + m)^2\big) \quad \text{for all } y = (y_1, y_2) \in \hat{C}^j \cap \check{C}^j \cap S^{-1}_{-n+m}\big\}$$

is contained in the event $A^{-1}_{m-n}(x, m)$ for all $x \in I_j$. Hence

$$\begin{aligned}
\sum_{x \in S_m^m \cap -G} & \mathbb{P}\big(h(x) > m,\ L(0) \in (m, m^2), A^{-1}_{m-n}(x, m)\big) \\
&\leq \sum_{j=1}^{8} \sum_{x \in I_j} \mathbb{P}(h(x) > m,\ L(0) \in (m, m^2), A^{-1}_{m-n}(x, m)) \\
&\leq \sum_{j=1}^{8} \sum_{x \in I_j} \mathbb{P}(h(x) > m,\ L(0) \in (m, m^2), E_j(m, n)) \\
&= \sum_{j=1}^{8} \mathbb{E}\Big[\#\{x \in I_j : h(x) > m\};\ L(0) \in (m, m^2);\ E_j(m, n)\Big]. \qquad (3.23)
\end{aligned}$$

The interval (m, m^2) can be divided in a finite number of disjoint subintervals such that the $\mathbb{Z}^2$ approximation of every umbrella with intensity in a given subinterval looks the same at least up to the first m edges. More precisely, there exists $M \in \mathbb{N}$ and there exist $\{m_1 = m < m_2 < ... < m_M = m^2\}$ such that, for any $k \in \{1, 2, ..., M\}$, $\forall h, l \in (m_k, m_{k+1})$, one has $U_h|_m = U_l|_m$, where $U_h|_m$ is the umbrella of intensity h whose sides are restricted to the first m edges (going from bottom-left towards up-right). Therefore, we can rewrite (3.23) as

$$\sum_{j=1}^{8} \sum_{l=1}^{M} \mathbb{E}\Big[\#\{x \in I_j : h(x) > m\};\ L(0) \in (m_l, m_{l+1});\ E_j(m, n)\Big]. \qquad (3.24)$$

For any point $x \in S_m^m \cap -G$ to have $h(x) > m$, there must be a branch coming out of x that perforates the protecting umbrella generated by the origin (since $L(0) \in (m, m^2)$).

That is, at least one point z on $U_{L(0)}|_m$ must be penetrated by another umbrella. On the other hand, every penetrated z can give rise to at most one of such x's. Hence, for any $l = 1, ..., M$, given $L(0) \in (m_l, m_{l+1})$,

$$\#\{x \in S_m^m \cap -G : \, h(x) > m\} \leq \sum_{i=1,2} \sum_{[z,z+e_i] \in U_{L(0)}|_m} \mathbb{1}_{\{I(z) \neq i\}}. \tag{3.25}$$

Plugging this in (3.24) gives

$$\begin{aligned}
&\mathbb{P}\big(h(0) > m, \, M_n = m\big) \\
&\quad \leq \sum_{j=1}^{8} \sum_{l=1}^{M} \sum_{i=1,2} \sum_{[z,z+e_i] \in U_{L(0)}|_m} \mathbb{P}\big(I(z) \neq i, \, L(0) \in (m_l, m_{l+1}), \, E_j(m,n)\big).
\end{aligned}$$

The intersection of the first two events inside the last probability is not independent of $E_j(m,n)$, but there is a negative correlation between them. We obtain therefore the upper bound

$$\begin{aligned}
&\mathbb{P}\big(h(0) > m, \, M_n = m\big) \\
&\quad \leq \sum_{j=1}^{8} \sum_{l=1}^{M} \sum_{i=1,2} \sum_{[z,z+e_i] \in U_{L(0)}|_m} \mathbb{P}\big(I(z) \neq i, \, L(0) \in (m_l, m_{l+1})\big) \mathbb{P}\big(E_j(m,n)\big).
\end{aligned}$$

We can now directly compute the right hand side of last expression. For $[z, z+e_i] \in U_{L(0)}|_m$ we have, by Lemma 3.5,

$$\begin{aligned}
\mathbb{P}\big(I(z) = i; \, L(0) \in (m_l, m_{l+1})\big) &= \int_{m_l}^{m_{l+1}} \mathbb{P}\big(I(z) = i | L(0) = t\big) \Big(\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{P}(L(0) \leq t)\Big) \mathrm{d}t \\
&\leq \int_{m_l}^{m_{l+1}} c \, \frac{\log t}{t} \frac{\theta}{t^3} (2 \log t - 1) \mathrm{d}t \\
&\leq K \frac{\log^2 m}{m^4} (m_{l+1} - m_l),
\end{aligned} \tag{3.26}$$

for some constant K.

Summing over the directions $i = 1, 2$ and over all the $z \in \mathbb{Z}^2$ such that $[z, z+e_i] \in U_{L(0)}|_m$ and then summing over $l = 1, ..., M$, one is left with a factor of order $\frac{\log^2 m}{m^2}$.

In order to evaluate the probability of any $E_j(m,n)$, note that every $E_j \cap S_{-k}^{-k}$ contains more than $\frac{1}{4} \frac{k}{\log k}$ points of the lattice. In fact, each cone $\hat{C}^j$ and $\check{C}^j$ intersects $S(k)$, the hyperplane containing S_{-k}^{-k}, on the segments $\hat{H}_k^j$ and $\check{H}_k^j$, each of length bigger than $\frac{2}{3}\frac{k}{\log k}$ (it is the double of the length of the cathetus of a right triangle, whose opposite angle measures $\frac{1}{\log(m^2)}$ degrees and with the other cathetus $\frac{m}{\sqrt{2}}$ long). The intersection of $\hat{H}_k^j$ and $\check{H}_k^j$ is therefore longer than $\left(\frac{1}{3}+\frac{1}{3}-\frac{3}{16}\right)\frac{k}{\log k}=\frac{23}{48}\frac{k}{\log k}$. Since the distance between close points on $E_j\cap S_{-k}^{-k}$ is $\sqrt{2}$, the total number of points is bigger than $\frac{1}{\sqrt{2}}\frac{23}{48}\frac{k}{\log k}\geq\frac{1}{4}\frac{k}{\log k}$. By the independence of the $(L(x))_{x\in\mathbb{Z}^2}$

$$
\begin{aligned}
\mathbb{P}\big(E_j(m,n)\big) &\leq \prod_{k=m+1}^{n}\Big(1-\frac{\theta\log k}{k^2}\Big)^{\frac{1}{4}\frac{k}{\log k}}\\
&\leq \exp\Big\{-\frac{\theta}{4}\sum_{k=m+1}^{n}\frac{1}{k}\Big\}\\
&\leq \exp\Big\{-\frac{\theta}{4}\int_{m+1}^{n}\frac{1}{s}\mathrm{d}s\Big\}\\
&=\Big(\frac{n}{m+1}\Big)^{-\frac{\theta}{4}}.
\end{aligned}
\tag{3.27}
$$

Putting all together we finally obtain, for some constant c,

$$
\begin{aligned}
\mathbb{P}\big(h(0)>m,\,M_n=m\big) &\leq c\frac{\log^2 m}{m^2}\Big(\frac{n}{m+1}\Big)^{-\frac{\theta}{4}}\\
&\leq c(m+1)^{\frac{\theta}{4}-2}n^{-\frac{\theta}{4}}\log^2 m\\
&\leq c\,n^{-2}\log^2 n.
\end{aligned}
\tag{3.28}
$$

3.3 The environment

The two random trees constructed in the previous sections will provide, in some sense, the support for our random environments. In both cases, the ω's are constructed in the following way.

Sample a realization of the tree as described above. For every $z\in\mathbb{Z}^2$, the edge $\{z,a(z)\}$ will have a conductance value of $\omega_{\{z,a(z)\}}=\mathrm{e}^{(h(z)+1)^A}$, where $a:\mathbb{Z}^2\to\mathbb{Z}^2$ is the ancestral function used for constructing the sampled tree and $A>1$ is a constant. We set all the other conductances to be equal to one.

For both the BZZ and the Diagonal tree, the conductances have infinite α-logmoments for any $\alpha>1$. On the other hand, choosing appropriately the constant $A>1$, we can obtain conductances with finite α-logmoments for α arbitrarily close to 1 from below.

Proposition 3.6. *Take $\bar{\alpha} < 1$. Then, the conductances of the random environments described above with $1 < A < \frac{1}{\bar{\alpha}}$ are such that*

$$E\big[\log^{\alpha}\omega_e\big] \leq \infty \qquad \forall \alpha \leq \bar{\alpha} \tag{3.29}$$

and

$$E\big[\log^{\alpha}\omega_e\big] = \infty \qquad \forall \alpha \geq 1. \tag{3.30}$$

Proof. We first prove it for the random environment built on the BZZ-tree support.

$$\begin{aligned} E\big[\log^{\alpha}\omega_e\big] &= \int_0^{\infty} P(\log^{\alpha}\omega_e > t)\mathrm{d}t \\ &= \sum_{k=0}^{\infty}\int_{k^{\alpha A}}^{(k+1)^{\alpha A}} P(\log^{\alpha}\omega_e > t)\mathrm{d}t \\ &\leq \sum_{k=0}^{\infty} P(\log^{\alpha}\omega_e > k^{\alpha A})((k+1)^{\alpha A} - k^{\alpha A}) \\ &= \sum_{k=0}^{\infty} P(h(0) > k)((k+1)^{\alpha A} - k^{\alpha A}) \end{aligned} \tag{3.31}$$

By equations (3.3) and (3.4) we know that for all sufficiently large $k \in \mathbb{N}$, say $k \geq K$,

$$\frac{c}{k} \leq P(h(0) > k) \leq \frac{c'}{k}.$$

Therefore, on the one hand, taking $\alpha \leq \bar{\alpha}$,

$$E\big[\log^{\alpha}\omega_e\big] \leq C + \sum_{k=K}^{\infty}\frac{c'}{k}\alpha A(k+1)^{\alpha A-1} < \infty,$$

where $C > 0$ is the finite contribution of the first $K-1$ terms of the sum, while, on the other hand, when $\alpha \geq 1$ we obtain, with a minor modification of (3.31),

$$E\big[\log^{\alpha}\omega_e\big] > \sum_{k=K}^{\infty}\frac{c}{k}\alpha A k^{\alpha A-1} = \infty.$$

Note that the very same proof is valid for the random environment built over the Diagonal tree structure, since the $\log^2$-correction in Theorem 3.4 doesn't change the behaviour of the series (3.31). $\square$

Proposition 3.7. *For almost every environment ω sampled from the constructions of the previous section, the random walk among the conductances ω will eventually follow the tree. This means that almost surely there exists $\bar{n} < \infty$ such that for all $n \geq \bar{n}$, if $X_n = x$ then $X_{n+1} = a(x)$, where $a : \mathbb{Z}^2 \to \mathbb{Z}^2$ is the ancestral function used to construct the tree underlying the environment.*

Proof. The probability that, starting in a point $x \in \mathbb{Z}^2$, the random walk will follow the tree forever is, by the independece of the jumps, bigger than

$$\prod_{k=1}^{\infty} \frac{\mathrm{e}^{k^A}}{2\mathrm{e}^{(k-1)^A} + \mathrm{e}^{k^A} + 1}. \tag{3.32}$$

It is easy, in fact, to get convinced that this is a very pessimistic estimate. It represents the case in which we start from a leaf of the tree (that is, a vertex that is ancestor of no other vertices) and where every time $\omega_{\{X_n, a(X_n)\}}$ is of order k (that is, equal to e^{k^A}), then the two edges under and at the left of X_n are of order $k-1$.

Call $T_1, T_2, \ldots$ the times in which the random walk doesn't go in the direction of the ancestral function. After each of these times, a new attempt to follow the tree is performed. Therefore if we show that the product (3.32) is a constant strictly bigger than zero, than the sum of the probabilities of succeeding in following the tree in one of the attempts is infinite. By Borel-Cantelli lemma, this means that almost surely there will be a finite time from which we will always follow the tree.

We are left to show that (3.32) is bigger than zero, or, equivalently, that its log is bigger than $-\infty$:

$$\begin{aligned} \log\left(\prod_{k=1}^{\infty} \frac{\mathrm{e}^{k^A}}{2\mathrm{e}^{(k-1)^A} + \mathrm{e}^{k^A}}\right) &= -\sum_{k=1}^{\infty} \log\left(1 + 2\frac{\mathrm{e}^{(k-1)^A}}{\mathrm{e}^{k^A}}\right) \\ &> -2\sum_{k=1}^{\infty} \mathrm{e}^{(k-1)^A - k^A} \\ &> -2A\sum_{k=1}^{\infty} \mathrm{e}^{-(k-1)^{A-1}} > -\infty, \end{aligned} \tag{3.33}$$

where we have used the mean value theorem for the bound $k^A - (k-1)^A \geq A(k-1)^{A-1}$. $\square$

Proposition 3.8. *The random walk among random conductances with environment built on the BZZ tree, as described above, has almost surely no limiting speed.*

Proposition 3.9. *The random walk among random conductances with environment built on the diagonal tree, as described above, has almost surely a limiting speed which is not zero.*

Proof of Proposition 3.8. From Proposition 3.7 we know that with probability 1 there exists a finite time from which the random walk will use only edges pointing the right or up direction with respect to its current position. Without loss of generality we can think this time to be time 0. In order to study the limiting speed of the process, we have to go back to the underlying structure of the tree on which we have built the environment. Note that every time the random walk makes a step in the direction of the ancestral function, it finds several new umbrellas perpendicular to its previous step and a new parallel one. If the strongest perpendicular umbrella is stronger than any other umbrella on the direction of the previous step, the branch of tree changes orientation and the next step of the random walk will follow it; otherwise, it will perform another step in the same direction as before.

The distribution of the length $\bar{L}$ of the new perpendicular umbrellas met at each step is easy to calculate:

$$\begin{aligned}\mathbb{P}(\bar{L}(0) > t) &= \mathbb{P}\big(\exists j \in \mathbb{N} \text{ such that } \tilde{L}((0,-j)) > \max\{t, j\}\big) \\ &= 1 - \prod_{j=1}^{\lfloor t \rfloor} \mathbb{P}(\tilde{L}((0,-j)) \le t) \prod_{j=\lfloor t \rfloor+1}^{\infty} \mathbb{P}(\tilde{L}((0,-j)) \le j) \\ &= 1 - \Big(1 - \frac{\tilde{\theta}}{t^2}\Big)^{\lfloor t \rfloor} \prod_{j=\lfloor t \rfloor+1}^{\infty} \Big(1 - \frac{\tilde{\theta}}{j^2}\Big) \end{aligned} \tag{3.34}$$

so that, by straightforward calculations,

$$\frac{c'}{t} \le \mathbb{P}(\bar{L}(0) > t) \le \frac{c''}{t}, \tag{3.35}$$

for some $c', c'' > 0$.

Now, being on a branch of the tree, what is the probability of passing from the umbrella that has generated that part of the branch to a stronger one before the umbrella itself ends? Suppose that that the random walk is on a branch of the tree generated by an umbrella of length $k > 2n_0^2$. Then the probability of not meeting a stronger perpendicular umbrella or a stronger umbrella on the same direction of the current one before leaving the present umbrella is bigger than

$$\left(1-\frac{c''}{k}\right)^k\left(1-\frac{\tilde{\theta}}{k^2}\right)^k > \mathrm{e}^{-2c''-1},$$

that is a constant strictly smaller than 1 and independent of k.

Considering only the strongest umbrellas through each point, call *rush* a sequence of intersecting umbrellas each bigger of the previous one that determines a part of the final tree.

Starting on any rush, the probability of leaving it (that is, of travelling the whole length of one of the umbrellas without meeting a stronger one) after having visited $N \in \mathbb{N}$ different umbrellas is

$$\mathbb{P}(\text{Leave the rush after more than } N \text{ umbrellas}) < \left(1-c\mathrm{e}^{-2c''}\right)^N, \tag{3.36}$$

for some $c > 0$.

This means, by Borel-Cantelli lemma, that with probability 1 the random walk will leave any rush in finite time. Given a realization of the walk, call $\tau(1) \in \mathbb{N}$ the time in which the random walk leaves the first rush, $\tau(2)$ the time in which it leaves the second one and so on. $\tau(1) < \tau(2) < \ldots$ is a sequence of (almost surely finite) integer stopping times that goes to infinity.

Fix $T > 1$ and define the times $T_1 = T$, $\tau_1 = \min_{i=1,2,\ldots}\{\tau(i) : \tau(i) > T_1\}$ and recursively

$$\begin{aligned} T_k &= \tau_{k-1} + \tau_{k-1}T^k & \forall k > 1, \\ \tau_k &= \min_{i=1,2,\ldots}\{\tau(i) : \tau(i) > T_{k-1}\} & \forall k > 1. \end{aligned} \tag{3.37}$$

Our aim is now to show that in the intervals of the form (T_{k-1}, T_k), the longest umbrella met is of length of the order $\tau_{k-1}T^k$. We don't want the longest umbrella to be much longer than this, otherwise it could "interfere" with the next intervals: consider the event

$$\begin{aligned} E_k = \{&\text{In the interval } (T_{k-1}, T_k) \text{ the longest umbrella met} \\ &\text{is stronger than } T^{k+1}\tau_k\}. \end{aligned}$$

Its probability can be bounded from above by

$$\begin{aligned}\mathbb{P}(E_k) &< 1 - \Big(1 - \frac{\tilde{\theta}}{\tau_k T^{k+1}}\Big)^{T_k} \\ &< 1 - \mathrm{e}^{-2\frac{T_k}{\tau_k} T_k} \\ &< \frac{c}{T^{k+1}},\end{aligned}$$

for some constant $c > 0$, since $T_k \leq \tau_k$. By Borel-Cantelli lemma, $\mathbb{P}(E_k \text{ i.o.}) = 0$.

On the other hand, we don't want the longest umbrella to be shorter than that. This is because we want it to be long a positive fraction of the entire time interval (T_{k-1}, T_k). In fact, the interval (T_{k-1}, T_k) is long about $\tau_{k-1}T^k$. Furthermore, we want the random walk to follow this umbrella for a positive fraction (say an $\varepsilon > 0$ fraction) of its length before leaving the time interval. This two events guarantee a relevant contribution to the speed up to time T_k. Therefore take, for a fixed $\varepsilon > 0$ small,

$$\begin{aligned}F_k = \big\{&\text{In the interval } (T_{k-1}, T_k(1-\varepsilon)) \text{ the longest umbrella met is stronger} \\ &\text{than } \tau_{k-1}T^k \text{ and is bigger than the biggest umbrella in } (T_k(1-\varepsilon), T_k)\big\}.\end{aligned}$$

By the independence of the new umbrellas discovered at each step, we have, for all $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}(F_k) &> (1-\varepsilon)\mathbb{P}(\text{one of the } T_k - T_{k-1} \text{ umbrellas is longer than } \tau_{k-1}T^k) \\ &= (1-\varepsilon)\Big(1 - \Big(1 - \frac{c'}{\tau_{k-1}T^k}\Big)^{T_k - T_{k-1}}\Big(1 - \frac{\tilde{\theta}}{\tau_{k-1}^2 T^{2k}}\Big)^{T_k - T_{k-1}}\Big) \\ &> (1-\varepsilon)\Big(1 - \Big(1 - \frac{c'}{\tau_{k-1}T^k}\Big)^{T_k - T_{k-1}}\Big) \\ &> (1-\varepsilon)\Big(1 - \mathrm{e}^{-\frac{c'}{2}\frac{1}{\tau_{k-1}T^k}(\tau_{k-1} + \tau_{k-1}T^k)}\Big) \\ &= C \qquad (3.38)\end{aligned}$$

where $C > 0$ is a constant not depending on k. By the second Borel-Cantelli lemma, there are almost surely infinitely many intervals (T_{k-1}, T_k) for which F_k happens.

Hence, almost surely there exists a $\bar{k} \in \mathbb{N}$ (depending eventually on the realization of the environment and of the random walk) such that E_k does not happen for every $k > \bar{k}$ while F_k holds infinitely many times. Take now the strongest umbrella met up to time $T_{\bar{k}}$. Its length $L > 0$ is almost surely finite, so that $\kappa := \min\{k : T_k > T_{\bar{k}} + L\}$ is well defined. Note that $\forall k > \kappa + 1$, in the interval (T_k, T_{k+1}) there is no umbrella longer than $T^{k+1}\tau_k$ met in the past.

Take the infinite subsequence $\kappa < k_1 < k_2 < ...$ such that F_{k_i} holds true for every $i \in \mathbb{N}$ and such that the longest umbrella met in the k_i'th interval (T_{k_i-1}, T_{k_i}) is followed by the random walk at least for a positive fraction $0 < \eta < \epsilon$ of its length. Note that since there is no longer umbrella coming from a previous interval, once the random walk meets this umbrella it follows it until its end or at least until the end of the interval itself, and the probability of meeting the umbrella before the last η fraction of its length is strictly positive. This implies that we have such a sequence $(k_i)_{i=1,2,...}$ almost surely.

Suppose now that a limiting speed $v = (v[1], v[2])$ existed. We want to show that in each of those intervals there is at least one time t at which the ratio $\sum_{j=1}^{t} X_j/t$ is far from v, bringing to a contradiction. Call $t_i \in \mathbb{N}$ the time at which the longest umbrella of the interval (T_{k_i-1}, T_{k_i}) is met and $t^i = \{\text{"Time of the last point of the umbrella"} \wedge T_{k_i}\}$. By definition, this umbrella is longer than $\tau_{k_i-1}T^{k_i}$, it is met before time $T_{k_i}(1-\varepsilon)$ and before the last η-fraction of its length. Call

$$\frac{1}{t_i}\sum_{j=1}^{t_i} X_j = (v_i[1], v_i[2]) =: v_i$$

and

$$\frac{1}{t^i}\sum_{j=1}^{t^i} X_j = (v^i[1], v^i[2]) =: v^i$$

the partial speeds up to time t_i and t^i respectively. Without loss of generality suppose that we met the longest umbrella on its horizontal side. Note that

$$v_i[1] = \frac{1}{t_i}\big(v^i[1]t^i - t^i + t_i\big)$$

and that

$$\frac{t^i}{t_i} > 1 + \frac{\eta\tau_{k_i-1}T^{k_i}}{(1-\varepsilon)\tau_{k_i-1}(T^{k_i}+1)} > 1 + \frac{\eta}{2(1+\varepsilon)} =: \beta > 0.$$

Further suppose $v[1], v[2] \notin \{0, 1\}$. Then if $v[1] > v^i[1]$

$$|v[1] - v_i[1]| = v[1] - v^i[1]\tfrac{t^i}{t_i} + \tfrac{t^i}{t_i} - 1 > (\beta - 1)(1 - v[1]) > 0, \qquad (3.39)$$

while if $v[1] \leq v^i[1]$

$$\begin{aligned}\max\left\{|v[1]-v^i[1]|,|v[1]-v_i[1]|\right\} &\geq \tfrac{1}{2}(v^i[1]-v_i[1]) \\ &= \tfrac{1}{2}\left(v^i[1]-v^i[1]\tfrac{t^i}{t_i}+\tfrac{t^i}{t_i}-1\right) \\ &> \tfrac{1}{2}(\beta-1)(1-v[1]). \end{aligned} \tag{3.40}$$

In both cases the distance from the limiting speed is bigger than a constant that is independent of k_i and strictly bigger than zero.

The cases $v[1] = 1$ and $v[1] = 0$ have probability 0. In fact, the probability of meeting in any interval $(T_{k_{i-1}}, (1-\varepsilon)T_{k_i})$ a vertical (respectively, horizontal) umbrella of order $\tau_{k_i-1}T^{k_i}$ that is stronger of any other horizontal (vertical) umbrellas met before (and of following it for a time of $O(t)$) is strictly positive, for the reasons mentioned above.

□

Proof of Proposition 3.9. Let $v = (0.5, 0.5)$. We claim that, almost surely,

$$\lim_{n\to\infty}\frac{X_n}{n} = v.$$

As in the previous proof, let N_0 be such that for every $n > N_0$, we have $X_{n+1} = a(X_n)$, where a is the ancestral function. By a minor modification of Proposition 3.7, N_0 is almost surely finite. We need to prove that for every $\varepsilon > 0$ there exists a (random) finite M such that for every $n > M$, we have $\|X_n/n - v\| < \varepsilon$, where we write $\|\cdot\|$ for, e.g., the usual ℓ^1-norm. To this end, we need to understand the various umbrellas that the random walk traverses. By the construction of the diagonal tree, there exists $K > 0$ such that for every umbrella which is stronger than K, for every two points x and y on the umbrella whose distance is larger than some $U = U(\varepsilon)$, we have

$$\left\|\frac{y-x}{\|y-x\|} - v\right\| < \varepsilon.$$

For umbrellas which are not stronger than K, their distribution is symmetric w.r.t. the diagonal, and their directions are i.i.d. and therefore they give an average of v.

Therefore,

$$\lim_{n\to\infty}\frac{X_n}{n} = v.$$

□

3.4 Proof of Theorem 1.8

Here we give a sketch of the proof of Theorem 1.8 for the two dimensional case.

Fix a threshold $M > 0$ such that $P(\omega_{xy} \leq M)$ is bigger than the critical probability for bond percolation. For any configuration of conductances ω, an edge is *good* if its conductance is smaller or equal to M and *bad* otherwise. Standard percolation estimates tell us that all the connected components of bad edges are finite and that for any bad edge e one has $P(|U_e| > n) \leq \mathrm{e}^{-cn}$ for some constant $c > 0$, where $|U_e|$ is the number of bad edges in the component of e. Now we define a new set of conductances ω' such that

$$\omega'_e = \begin{cases} 2M(|U_e| + |\partial U_e|) & \text{if } e \text{ is bad or a boundary edge} \\ \omega_e & \text{otherwise} \end{cases},$$

where a good edge is a boundary edge if it shares an endpoint with a bad edge (i.e. it is its neighbour) and ∂U_e is the set of all (good) neighbours of the bad edges in the component of e.

Note that the new environment ω' is ergodic and that

$$E[\omega'_{xy}] \leq \sum_{i=1}^{\infty} 2Mn^2 P(|U_e| + |\partial U_e| = n) + M < \infty.$$

Hence, the well known Nash-Williams criterion (see, e.g., [LP12]) implies easily the recurrence of the random walk on ω'. This is equivalent to say (compare Section 1.1.3) that for every flow θ the dissipation energy $E^{\omega'}_{\text{dis}}(\theta)$ in 1.9 relative to ω' must be infinite. If we show that for every flow θ we have $E^{\omega}_{\text{dis}}(\theta) \geq E^{\omega'}_{\text{dis}}(\theta)$, then we are done.

For $\nu = \omega, \omega'$ we have

$$\begin{aligned} E^{\nu}_{\text{dis}}(\theta) &= \sum_{e \text{ good not boundary}} \frac{\theta^2(e)}{\nu_e} + \sum_{e \text{ bad or boundary}} \frac{\theta^2(e)}{\nu_e} \\ &= \sum_{e \text{ good not boundary}} \frac{\theta^2(e)}{\nu_e} + \sum_{U \text{ bad connected component}} \sum_{e \in U \cup \partial U} \frac{\theta^2(e)}{\nu_e}, \end{aligned}$$

where one should be careful not to count more than once boundary edges in order to have exact equality (in any case, counting them more than once does not change the finiteness of E^{ν}_{dis}).

We just need to show that for every bad component U one has

$$\sum_{e\in U\cup\partial U}\frac{\theta^2(e)}{\omega_e} =: E_{\text{dis}}^{\omega,U} \geq E_{\text{dis}}^{\omega',U}.$$

Note that for every $e \in U\cup\partial U$ one has $|\theta(e)| \leq \sum_{e'\in\partial U}|\theta(e')|$ by the definition of dissipation energy and therefore $\theta^2(e) \leq |\partial U|\sum_{e'\in\partial U}\theta^2(e')$. One finally obtains

$$\begin{aligned}
E_{\text{dis}}^{\omega',U} &= \sum_{e\in U\cup\partial U}\frac{\theta^2(e)}{\omega'_e} \\
&\leq \frac{\sum_{e\in U\cup\partial U}\theta^2(e)}{2M(|U|+|\partial U|)^2} \\
&\leq \frac{|U\cup\partial U||\partial U|\sum_{e\in\partial U}\theta^2(e)}{2M(|U|+|\partial U|)^2} \\
&\leq \frac{1}{M}\sum_{e\in\partial U}\theta^2(e) \\
&\leq \sum_{e\in U\cup\partial U}\frac{\theta^2(e)}{\omega_e} = E_{\text{dis}}^{\omega,U}. \qquad (3.41)
\end{aligned}$$

Chapter 4

A Central Limit Theorem for the effective conductance

In Section 4.1 we discuss the strategy of the proof of Theorem 1.11 and state its principal ingredients in the form of suitable propositions. In Subsection 4.1.6 we describe the organization of the rest of the chapter.

Note that in this chapter we will make use of the sign a_{xy} for the conductances rather than ω_{xy}, being the first notation more used in the Homogenization Theory lterature.

4.1 Key ingredients

4.1.1 Martingale approximation

A standard way to control fluctuations of a function of i.i.d. random variables is by way of a *martingale approximation*. Let us order the random variables $\{a_{xy}\colon \langle x,y\rangle \in \mathbb{B}(\Lambda_L)\}$ in any (for now) convenient way and let $\mathcal{F}_k$ to be the σ-algebra generated by the first k of them. (Since we only aim at a distributional convergence, the σ-algebras may depend on L.) Then

$$C_L^{\text{eff}}(t) - \mathbb{E}C_L^{\text{eff}}(t) = \sum_{k=1}^{|\mathbb{B}(\Lambda_L)|} Z_k, \tag{4.1}$$

where

$$Z_k := \mathbb{E}\big(C_L^{\text{eff}}(t)\big|\mathscr{F}_k\big) - \mathbb{E}\big(C_L^{\text{eff}}(t)\big|\mathscr{F}_{k-1}\big). \tag{4.2}$$

Obviously, the quantity Z_k is a martingale increment. In order to show distributional convergence to $\mathcal{N}(0,\sigma^2)$, it suffices to verify the (Lindenberg-Feller-type of) conditions of the Martingale Central Limit Theorem due to Brown [Bro71]:

(1) There exists $\sigma^2 \in [0,\infty)$ such that

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|} \mathbb{E}(Z_k^2|\mathcal{F}_{k-1}) \underset{L\to\infty}{\longrightarrow} \sigma^2 \tag{4.3}$$

in probability, and

(2) for each $\epsilon > 0$,

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|} \mathbb{E}\big(\, Z_k^2 \mathbb{1}_{\{|Z_k|>\epsilon|\Lambda_L|^{1/2}\}}\big|\mathcal{F}_{k-1}\big) \underset{L\to\infty}{\longrightarrow} 0 \tag{4.4}$$

in probability.

The sums on the left suggest invoking the Spatial Ergodic Theorem, but for that we would need to ensure that the individual terms in the sum are (at least approximated by) functions that are stationary with respect to shifts of $\mathbb{Z}^d$. This necessitates the following additional input:

(i) a specific choice of the ordering of the edges, and

(ii) a more explicit representation for Z_k.

We will now discuss various aspects of these in more detail.

4.1.2 Stationary edge ordering

Recall that $\mathbb{B}(\mathbb{Z}^d)$ denotes the set of all (unordered) edges in $\mathbb{Z}^d$. We will order $\mathbb{B}(\mathbb{Z}^d)$ as follows: Let $\preceq$ denote the lexicographic ordering of the vertices of $\mathbb{Z}^d$. Explicitly, for $x = (x_1,\dots,x_d)$ and $y = (y_1,\dots,y_d)$ we have $x \preceq y$ if either $x = y$ or $x \neq y$ and there exists $i \in \{1,\dots,d\}$ such that $x_j = y_j$ for all $j < i$ and $x_i < y_i$. We will write $x \prec y$ if $x \neq y$ and $x \preceq y$.

For the purpose of defining a stationary ordering of the edges, and also easier notation in some calculations that are to follow, we now identify $\mathbb{B}(\mathbb{Z}^d)$ with the set of pairs (x,i), where $x \in \mathbb{Z}^d$ and $i \in \{1,\dots,d\}$, so that (x,i) corresponds to the edge between the vertices x and $x+\hat{e}_i$. We will then write

$$(x,i) \preceq (y,j) \quad \text{if} \quad \begin{cases} \text{either } x \prec y \\ \text{or} \quad\; x = y \text{ and } i \le j. \end{cases} \tag{4.5}$$

Again, $(x,i) \prec (y,j)$ if $(x,i) \preceq (y,j)$ but $(x,i) \neq (y,j)$. It is easy to check that $\preceq$ is a complete order on $\mathbb{B}(\mathbb{Z}^d)$. A key fact about this ordering is its stationarity with respect to shifts:

Lemma 4.1. *If $(x,i) \preceq (y,j)$ then also $(x+z,i) \preceq (y+z,j)$ for all $z \in \mathbb{Z}^d$.*

Proof. This is a trivial consequence of the definition. □

Now we proceed to identify the sigma algebras $\{\mathscr{F}_k\}$ in the martingale representation above. Recall that $\Omega := [\lambda, {}^1\!/\!{}_\lambda]^{\mathbb{B}(\mathbb{Z}^d)}$ denotes the set of conductance configurations satisfying (1.1). Writing ω for elements of Ω we use $a_{xy} = a_{xy}(\omega)$, for $\langle x,y\rangle \in \mathbb{B}(\mathbb{Z}^d)$, to denote the coordinate projection corresponding to edge $\langle x,y\rangle$. Given $L \ge 1$, set $N := |\mathbb{B}(\Lambda_L)|$ and let $b_1,\dots,b_N$ be the enumeration of $\mathbb{B}(\Lambda_L)$ induced by the ordering of edges $\preceq$ defined above. Then we set

$$\mathscr{F}_k := \sigma(\omega_b\colon b \preceq b_k), \qquad k = 1,\dots,N, \tag{4.6}$$

with

$$\mathscr{F}_0 := \sigma(\omega_b\colon b \prec b_1). \tag{4.7}$$

By definition $\mathscr{F}_0$ is independent of the edges in $\mathbb{B}(\Lambda_L)$ while $\mathscr{F}_N$ determines the entire configuration in $\mathbb{B}(\Lambda_L)$. Note also that $\mathscr{F}_k$ includes information about edges that are not in $\mathbb{B}(\Lambda_L)$. This will be of importance once we replace Z_k by a random variable that depends on all of ω.

4.1.3 An explicit form of martingale increment

Having addressed the ordering of the edges, and thus the definition of the σ-algebras $\mathscr{F}_k$, we now proceed to derive a more explicit form of the quantity Z_k from (4.2). Given

$\omega \in \Omega$, define the operator $\mathcal{L}_\omega$ on ($\mathbb{R}$ or $\mathbb{R}^d$-valued) functions on the lattice via

$$(\mathcal{L}_\omega f)(x) := \sum_{y\colon \langle x,y\rangle \in \mathbb{B}(\mathbb{Z}^d)} a_{xy}(\omega)\,\bigl[f(y) - f(x)\bigr]. \tag{4.8}$$

This is an elliptic finite-difference operator — a random Laplacian — that arises as the generator of the random walk among random conductances $\{a_{xy}(\omega)\}$ (see, e.g., Biskup [Bis11] for a review of these connections). The existence/uniqueness for the associated Dirichlet problem implies that for any finite $\Lambda \subset \mathbb{Z}^d$ there is a unique $\Psi_\Lambda\colon \Omega \times (\Lambda \cup \partial\Lambda) \to \mathbb{R}^d$ such that $x \mapsto \Psi_\Lambda(\omega, x)$ obeys

$$\begin{cases} \mathcal{L}_\omega\, \Psi_\Lambda(\omega, x) = 0, & x \in \Lambda, \\ \Psi_\Lambda(\omega, x) = x, & x \in \partial\Lambda. \end{cases} \tag{4.9}$$

It is then easily checked that $f(x) := t \cdot \Psi_\Lambda(\omega, x)$ is the unique minimizer of $f \mapsto Q_\Lambda(f)$ over all functions f with the boundary values $f(x) = t \cdot x$ for $x \in \partial\Lambda$. In particular, we have

$$C_L^{\text{eff}}(t) = Q_{\Lambda_L}\bigl(\, t \cdot \Psi_{\Lambda_L}\bigr) \tag{4.10}$$

for all $t \in \mathbb{R}^d$. The function $x \mapsto \Psi_\Lambda(\omega, x)$ will sometimes be referred to as a *finite-volume harmonic coordinate*. (The first line in (4.9) justifies this term.)

The minimum value $Q_\Lambda(t \cdot \Psi_\Lambda)$ is a non-decreasing, continuous and concave function of $\{a_{xy}\colon \langle x, y\rangle \in \mathbb{B}(\Lambda)\}$. Thanks to the uniqueness of the solution to (4.9), $Q_\Lambda(t \cdot \Psi_\Lambda)$ is also continuously differentiable in a_{xy}'s with

$$\frac{\partial}{\partial a_{xy}} Q_\Lambda(t \cdot \Psi_\Lambda) = \bigl[t \cdot \Psi_\Lambda(\omega, y) - t \cdot \Psi_\Lambda(\omega, x)\bigr]^2, \qquad \langle x, y\rangle \in \mathbb{B}(\Lambda). \tag{4.11}$$

This relation is of fundamental importance for what is to come.

Abusing the notation slightly, let $\omega_1, \dots, \omega_N$, with $N := |\mathbb{B}(\Lambda)|$, denote the components of the configuration ω over $\mathbb{B}(\Lambda)$ labeled in the order induced by $\preceq$ defined above. Let

$$q(\omega_1, \dots, \omega_N) := Q_\Lambda(t \cdot \Psi_\Lambda) \tag{4.12}$$

mark explicitly the dependence of the right-hand side on these variables. The product structure of the underlying probability measure then allows us to give a more explicit expression for the increment $Z_k = Z_k(\omega_1,\dots,\omega_k)$:

$$
\begin{aligned}
Z_k = \int \mathbb{P}(\mathrm{d}\omega_k')\dots\mathbb{P}(\mathrm{d}\omega_N')\bigl[q(\omega_1,\dots,\omega_k,\omega_{k+1}',\dots,\omega_N')\\
- q(\omega_1,\dots,\omega_{k-1},\omega_k',\dots,\omega_N')\bigr] \qquad (4.13)\\
= \int \mathbb{P}(\mathrm{d}\omega_k')\dots\mathbb{P}(\mathrm{d}\omega_N')\int_{\omega_k'}^{\omega_k}\mathrm{d}\tilde\omega_k\,\frac{\partial}{\partial\tilde\omega_k}q(\omega_1,\dots,\omega_{k-1},\tilde\omega_k,\omega_{k+1}',\dots,\omega_N'),
\end{aligned}
$$

with the inner integral in Riemann sense. A key point is that the last partial derivative is (modulo notational changes) given by (4.11), i.e., Z_k is the modulus-squared of the gradient of $t\cdot\Psi_\Lambda$ over the k-th edge in $\mathbb{B}(\Lambda)$ integrated over part of the variables. To see that Z_k is a martingale increment note that the Riemann integral changes sign when its limits are interchanged.

4.1.4 Input from homogenization theory

In order to apply the Spatial Ergodic Theorem to the sums on the left of (4.3) (4.4), we will substitute for Z_k a quantity that is stationary with respect to the shifts of $\mathbb{Z}^d$. This will be achieved by replacing the discrete gradient of Ψ_Λ — which by (4.11) enters as the partial derivative of q in the formula for Z_k — by the gradient of its infinite-volume counterpart, to be denoted by ψ. The existence and properties of the latter object are standard:

Proposition 4.2 (Infinite-volume harmonic coordinate)**.** *Suppose the law of the conductances is ergodic with respect to the shifts of* $\mathbb{Z}^d$ *and assume* (1.1) *for some* $\lambda\in(0,1)$*. Then there is a function* $\psi\colon\Omega\times\mathbb{Z}^d\to\mathbb{R}^d$ *such that*

(1) (ψ is $\mathcal{L}_\omega$-harmonic) $\mathcal{L}_\omega\psi(\omega,x)=0$ for all x and $\mathbb{P}$-a.e. ω.

(2) (ψ is shift covariant) For $\mathbb{P}$-a.e. ω we have $\psi(\omega,0):=0$ and

$$
\psi(\omega,y)-\psi(\omega,x)=\psi(\tau_x\omega,y-x),\qquad x,y\in\mathbb{Z}^d. \tag{4.14}
$$

(3) (ψ is square integrable)

$$
\mathbb{E}\Biggl(\sum_{x=\hat e_1,\dots,\hat e_d} a_{0,x}(\omega)\bigl|\psi(\omega,x)\bigr|^2\Biggr)<\infty. \tag{4.15}
$$

(4) (ψ is approximately linear) The corrector $\chi(\omega,x) := \psi(\omega,x) - x$ satisfies

$$\lim_{|x|\to\infty} \frac{\mathbb{E}\bigl(|\chi(\omega,x)|^2\bigr)}{|x|^2} = 0. \tag{4.16}$$

Proof. Properties (1-3) are standard and follow directly from the construction of ψ (which is done, essentially, by showing that a minimizing sequence in (1.25) converges in a suitable L^2-sense; see, e.g., Biskup [Bis11, Section 3.2] for a recent account of this). As to (4), a moment's thought reveals that it suffices to show this for x of the form $n\hat{e}_i$, where $n \to \pm\infty$. This follows from the Mean Ergodic Theorem, similarly as in [Bis11, Lemma 4.8]. □

The replacement of (the gradients of) Ψ_Λ by ψ necessitates developing means to quantify the resulting error. For this we introduce an L^p-norm on functions $f\colon \Omega\times(\Lambda\cup\partial\Lambda)\to\mathbb{R}^d$ by the usual formula

$$\|\nabla f\|_{\Lambda,p} := \biggl(\frac{1}{|\Lambda|}\sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda)} \mathbb{E}\bigl|f(\omega,y)-f(\omega,x)\bigr|^p\biggr)^{1/p}. \tag{4.17}$$

Analogously, we also introduce a norm on functions $\varphi\colon \Omega\times\mathbb{Z}^d\to\mathbb{R}^d$ by

$$\|\nabla\varphi\|_p := \Bigl(\sum_{x=\hat{e}_1,\dots,\hat{e}_d} \mathbb{E}\bigl|\varphi(\omega,x)-\varphi(\omega,0)\bigr|^p\Bigr)^{1/p}. \tag{4.18}$$

Here we introduced the symbol ∇f for an $\mathbb{R}^d$-valued function whose i-th component at x is given by $\nabla_i f(x) := f(x+\hat{e}_i) - f(x)$ — abusing our earlier use of this notation. It is reasonably well known, albeit perhaps not written down explicitly anywhere, that the gradients of Ψ_Λ and ψ are close in $\|\cdot\|_{\Lambda,2}$-norm (see, however, Proposition 3.1 of Caputo and Ioffe [CI03] for a torus version of this statement).

Proposition 4.3. *Suppose the law $\mathbb{P}$ on conductances $\{a_{xy}\}$ is ergodic with respect to shifts of $\mathbb{Z}^d$ and obeys* (1.1) *for some $\lambda\in(0,1)$. Then*

$$\bigl\|\nabla(\Psi_{\Lambda_L}-\psi)\bigr\|_{\Lambda_L,2} \underset{L\to\infty}{\longrightarrow} 0. \tag{4.19}$$

As we will elaborate on later (see Remark 4.14), this is exactly what is needed to establish the representation (1.25) for the limit value $c_{\text{eff}}(t)$ of the sequence $L^{-d}C_L^{\text{eff}}(t)$. However, in order to validate the conditions (4.3–4.4) of the Martingale Central Limit Theorem, more than just square integrability is required. For this we state and prove:

Proposition 4.4 (Meyers' estimate)**.** *Suppose $\mathbb{P}$ is ergodic with respect to shifts. For each $d \geq 1$, there is $\lambda = \lambda(d) \in (0,1)$ such that if* (1.1) *holds $\mathbb{P}$-a.s. with this λ, then for some $p > 4$,*

$$\|\nabla\psi\|_p < \infty \tag{4.20}$$

and

$$\sup_{L\geq 1} \big\|\nabla(\Psi_{\Lambda_L} - \psi)\big\|_{\Lambda_L,p} < \infty. \tag{4.21}$$

Proposition 4.4 is the sole reason for our restriction on ellipticity contrast. We believe that, on the basis of the technology put forward in Gloria and Otto [GO11], no such restriction should be needed. To attest this we note that versions of the above bounds actually hold pointwise for a.e. $\omega \in \Omega$ satisfying (1.1); i.e., for norms without the expectation $\mathbb{E}$. In addition, from [GO11, Proposition 2.1] we in fact know (4.20) for all $p \in (1,\infty)$ when $d \geq 3$. A torus version of Proposition 4.4 appeared in Theorem 4.1 of Caputo and Ioffe [CI03].

4.1.5 Perturbed corrector and variance formula

Unfortunately, a direct attempt at the substitution of (the gradients of) Ψ_Λ by ψ in (4.13) reveals another technical obstacle: As (4.13) relies on the Fundamental Theorem of Calculus, the replacement of Ψ_Λ by ψ requires the latter function to be defined for ω that may lie outside of the support of $\mathbb{P}$. This is a problem because ψ is generally determined by conditions (1-4) in Proposition 4.2 only on a set of full $\mathbb{P}$-measure. Imposing additional assumptions on $\mathbb{P}$ — namely, that the single-conductance distribution is supported on an interval with a bounded and non-vanishing density — would allow us to replace the Lebesgue integral in (4.13) by an integral with respect to $\mathbb{P}(\mathrm{d}\tilde{\omega}_k)$ and thus eliminate this problem. Notwithstanding, we can do much better by invoking a rank-one perturbation argument which we describe next.

Fix an index $i \in \{1,\dots,d\}$ and recall the notation $\nabla_i f(x) := f(x+\hat{e}_i) - f(x)$. For a vertex $x \in \mathbb{Z}^d$ and a finite set $\Lambda \subset \mathbb{Z}^d$ satisfying $x \in \Lambda$ or $x + \hat{e}_i \in \Lambda$, let $\mathfrak{g}_\Lambda^{(i)}(\omega, x)$ be defined by

$$\mathfrak{g}_\Lambda^{(i)}(\omega,x)^{-1} := \inf\big\{Q_\Lambda(f)\colon f(x+\hat{e}_i) - f(x) = 1,\, f_{\partial\Lambda} = 0\big\}, \tag{4.22}$$

where $0^{-1} := \infty$. (In Section 4.4 we will see that $\mathfrak{g}_\Lambda^{(i)}$ is also a double gradient of the Green function for operator $\mathcal{L}_\omega$.) Note that (4.13) and (4.11) ask us to understand how

$\nabla_i\Psi_\Lambda(\omega,x)$ changes when the coordinate of ω over $\langle x,x+\hat{e}_i\rangle$ is perturbed. Somewhat surprisingly, this change takes a multiplicative form:

Proposition 4.5 (Rank-one perturbation)**.** *Let $\Lambda\subset\mathbb{Z}^d$ be finite and $x,y\in\Lambda$ be nearest neighbors; $y=x+\hat{e}_i$ for some $i\in\{1,\dots,d\}$. For any ω,ω' that agree everywhere except at edge $b:=\langle x,y\rangle$,*

$$\nabla_i\Psi_\Lambda(\omega',x)=\bigl[1-(\omega_b'-\omega_b)\mathfrak{g}_\Lambda^{(i)}(\omega',x)\bigr]\nabla_i\Psi_\Lambda(\omega,x). \tag{4.23}$$

For the prefactor we alternatively get

$$1-(\omega_b'-\omega_b)\mathfrak{g}_\Lambda^{(i)}(\omega',x)=\exp\Bigl\{-\int_{\omega_b}^{\omega_b'}\mathrm{d}\tilde{\omega}_b\ \mathfrak{g}_\Lambda^{(i)}(\tilde{\omega},x)\Bigr\}, \tag{4.24}$$

where $\tilde{\omega}$ coincides with ω except at b, where it equals $\tilde{\omega}_b$. In particular, $1-(\omega_b'-\omega_b)\mathfrak{g}_\Lambda^{(i)}(\omega',x)$ is bounded away from 0 and ∞ uniformly in $\omega\in\Omega$ and $\Lambda\subset\mathbb{Z}^d$.

It is worthy a note that (4.23) is a special case of a more general rank-one perturbation formula; cf Lemma 4.22, which may be of independent interest. Incidentally, such formulas have proved extremely useful in the analysis of random Schrödinger operators. The $\Lambda\uparrow\mathbb{Z}^d$-limit of the right-hand side can now be controlled uniformly in $\omega\in\Omega$:

Proposition 4.6. *Suppose* (1.1) *holds for some $\lambda\in(0,1)$. Then $\Lambda\mapsto\mathfrak{g}_\Lambda^{(i)}(\omega,x)$ is non-decreasing and bounded away from zero and infinity uniformly in $\Lambda\subset\mathbb{Z}^d$ and $\omega\in\Omega$. In particular, for all $\omega\in\Omega$ and all $x\in\mathbb{Z}^d$ the limit*

$$\mathfrak{g}^{(i)}(\omega,x):=\lim_{\Lambda\uparrow\mathbb{Z}^d}\mathfrak{g}_\Lambda^{(i)}(\omega,x) \tag{4.25}$$

exists and satisfies

$$\mathfrak{g}^{(i)}(\omega,x)^{-1}=\inf\bigl\{Q_{\mathbb{Z}^d}(f)\colon f(x+\hat{e}_i)-f(x)=1,\ |\mathrm{supp}(f)|<\infty\bigr\}, \tag{4.26}$$

where $\mathrm{supp}(f):=\{x\in\mathbb{Z}^d\colon f(x)\neq0\}$*. In particular, $(\omega,x)\mapsto\mathfrak{g}^{(i)}(\omega,x)$ is stationary in the sense that $\mathfrak{g}^{(i)}(\tau_z\omega,x+z)=\mathfrak{g}^{(i)}(\omega,x)$ holds for all $\omega\in\Omega$ and all $x,z\in\mathbb{Z}^d$.*

Before we wrap up the outline of the proof of Theorem 1.11, let us formulate a representation for the limiting variance σ_t^2 from Theorem 1.11: For $x\in\mathbb{Z}^d$ and $i\in\{1,\dots,d\}$, let b denote the edge corresponding to the pair (x,i) and let

$$h(\omega,x,i):=\int\mathbb{P}(\mathrm{d}\omega_b')\int_{\omega_b'}^{\omega_b}\mathrm{d}\tilde{\omega}_b\ \bigl[1-(\tilde{\omega}_b-\omega_b)\mathfrak{g}^{(i)}(\tilde{\omega},x)\bigr]^2, \tag{4.27}$$

where $\tilde{\omega}$ is the configuration equal to ω except at b, where it equals $\tilde{\omega}_b$. Define the matrix $\hat{Z}(x,i) := \{\hat{Z}_{jk}(x,i)\}_{j,k=1,\dots,d}$ by the quadratic form

$$\bigl(t, \hat{Z}(x,i)t\bigr) := \mathbb{E}\Bigl(h(\cdot,x,i)\bigl|\nabla_i(t\cdot\psi)(\cdot,x)\bigr|^2 \,\Big|\, \sigma\bigl(\omega_{b'}\colon b' \preceq (x,i)\bigr)\Bigr), \tag{4.28}$$

where (x,i) represents the edge $\langle x, x+\hat{e}_i\rangle$ and $t\in\mathbb{R}^d$. Then we have:

Theorem 4.7 (Limiting variance)**.** *Under the assumptions of Theorem 1.11, the matrix elements of $\hat{Z}(x,i)$ are square integrable. In particular, σ_t^2 from Theorem 1.11 is given by*

$$\sigma_t^2 = \sum_{i=1}^d \mathbb{E}\Bigl(\bigl(t, \hat{Z}(0,i)t\bigr)^2\Bigr), \qquad t\in\mathbb{R}^d. \tag{4.29}$$

As an inspection of (4.28) reveals, the limiting variance is thus a bi-quadratic form in t. Although concisely written, the expression is not very useful from the practical point of view; particularly, due to the unwieldy conditioning in (4.28). The representation using the h-function also adds to this; it is no longer obvious, albeit still true, that

$$\mathbb{E}\Bigl((t, \hat{Z}(x,i)t) \,\Big|\, \sigma\bigl(\omega_{b'}\colon b' \prec (x,i)\bigr)\Bigr) = 0, \tag{4.30}$$

i.e., that $(t,\hat{Z}(x,i)t)$ is a martingale increment. A question of interest is whether an expression can be found for σ_t^2 that is more amenable to computations.

Remark 4.8. *Since $t\mapsto C_L^{\text{eff}}(t)$ is quadratic in t, the above actually implies that, as $L\to\infty$, the joint law of the random variables*

$$\biggl\{\frac{C_L^{\text{eff}}(t) - \mathbb{E}C_L^{\text{eff}}(t)}{|\Lambda_L|^{1/2}}\colon t\in\mathbb{R}^d\biggr\} \tag{4.31}$$

tends to the law of multivariate Gaussian $\{G_t\colon t\in\mathbb{R}^d\}$ with

$$E(G_t) = 0 \quad \textit{and} \quad E\bigl(G_tG_s\bigr) = \sum_{i=1}^d \mathbb{E}\Bigl(\bigl(t,\hat{Z}(0,i)t\bigr)\bigl(s,\hat{Z}(0,i)s\bigr)\Bigr), \tag{4.32}$$

where $\hat{Z}(0,i)$ is as in (4.28). Naturally, $t\mapsto G_t$ is a quadratic form as well.

4.1.6 Organization

The proofs (and the rest of the paper) are organized as follows. In Section 4.2 we assemble the ingredients — following the steps outlined in the present section — into the proofs of Theorems 1.11 and 4.7 and Corollary 1.12. In Section 4.3 we then show that the finite-volume harmonic coordinate approximates its full lattice counterpart in an L^2-sense as stated in Proposition 4.3 and establish the Meyers estimate from Proposition 4.4. A key technical tool is the Calderón-Zygmund regularity theory and a uniform bound on the triple gradient of the Green function of the simple random walk in finite boxes. Finally, in Section 4.4, we prove Propositions 4.5 and 4.6 dealing with the harmonic coordinate over environments perturbed at a single edge.

4.2 Proof of the CLT

In this section we verify the conditions (4.3) (4.4) of the Martingale Central Limit Theorem and thus prove Theorems 1.11 and 4.7. All derivations are conditional on Propositions 4.3–4.6 the proofs of which are postponed to later sections. Throughout we will make use of the following simple but useful consequence of Hölder's inequality:

Lemma 4.9. *For any $p' > p > 2$, $\alpha := \frac{2}{p}\frac{p'-p}{p'-2}$ and $\beta := \frac{p'}{p}\frac{p-2}{p'-2}$,*

$$\big\|\nabla(\Psi_{\Lambda_L} - \psi)\big\|_{\Lambda_L,p} \leq \big\|\nabla(\Psi_{\Lambda_L} - \psi)\big\|^{\alpha}_{\Lambda_L,2} \big\|\nabla(\Psi_{\Lambda_L} - \psi)\big\|^{\beta}_{\Lambda_L,p'}.$$

Proof. Apply Hölder's inequality to the function $f := |\nabla(\Psi_{\Lambda_L} - \psi)|$. □

Assume now the setting developed in Section 4.1; in particular, the ordering of edges and sigma-algebras $\mathscr{F}_k$ from Section 4.1.2 and the martingale increment Z_k from (4.2) and its representation (4.13) from Section 4.1.3. In analogy with equation (4.27), we also define

$$h_\Lambda(\omega, x, i) := \int \mathbb{P}(\mathrm{d}\omega'_b) \int_{\omega'_b}^{\omega_b} \mathrm{d}\tilde{\omega}_b \big[1 - (\tilde{\omega}_b - \omega_b)\mathfrak{g}^{(i)}_\Lambda(\tilde{\omega}, x)\big]^2, \tag{4.33}$$

where $b := \langle x, x + \hat{e}_i\rangle$ and $\tilde{\omega}$ is the configuration equal to ω except at b, where it equals $\tilde{\omega}_b$. By Proposition 4.5, we may write the martingale increment Z_k as

$$Z_k = \mathbb{E}\Big(h_{\Lambda_L}(\cdot, x_k, i_k)\big|\nabla_{i_k}(t \cdot \Psi_{\Lambda_L})(\cdot, x_k)\big|^2 \,\Big|\mathcal{F}_k\Big), \tag{4.34}$$

where x_k and i_k are the vertex and the edge direction corresponding to b_k, i.e., $b_k = \langle x_k, x_k + \hat{e}_{i_k}\rangle$. Recall the notation for $\hat{Z}(x,i)$ from (4.28) and note that this is well defined and finite $\mathbb{P}$-a.s. thanks to the estimates (4.20–4.21) as well as boundedness of h. Note the dependence of Z_k on L.

Proposition 4.10 (Martingale CLT — first condition)**.** *Assume that the premises (and thus conclusions) of Propositions 4.3–4.6 hold. Then $Z_k \in L^2(\mathbb{P})$ for all k and*

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}(Z_k^2|\mathcal{F}_{k-1}) \underset{L\to\infty}{\longrightarrow} \sum_{i=1}^{d}\mathbb{E}\Bigl(\bigl(t,\hat{Z}(0,i)t\bigr)^2\Bigr) \tag{4.35}$$

in $\mathbb{P}$-probability and $L^1(\mathbb{P})$.

Proof. Fix $t \in \mathbb{R}^d$. Thanks to Lemma 4.1 and Proposition 4.2(2), for each $i \in \{1,\dots,d\}$, the collection of conditional expectations

$$\Bigl\{\mathbb{E}\Bigl(\,\bigl(\,t,\hat{Z}(x,i)t\,\bigr)^2\,\Big|\,\sigma\bigl(\omega_b\colon b \prec (x,i)\bigr)\,\Bigr)\colon x\in\mathbb{Z}^d\Bigr\} \tag{4.36}$$

is stationary with respect to the shifts on $\mathbb{Z}^d$ and, by Proposition 4.4, uniformly bounded in $L^1(\mathbb{P})$. Labeling the edges in $\mathbb{B}(\Lambda_L)$ according to the order $\preceq$, the Spatial Ergodic Theorem yields

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\bigl((t,\hat{Z}(x_k,i_k)t)^2\big|\mathcal{F}_{k-1}\bigr) \underset{L\to\infty}{\longrightarrow} \sum_{i=1}^{d}\mathbb{E}\Bigl(\bigl(t,\hat{Z}(0,i)t\bigr)^2\Bigr) \tag{4.37}$$

with the limit $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$. To see how this relates to our claim, abbreviate

$$A_k := h_{\Lambda_L}(\cdot,x_k,i_k)\bigl|\nabla_{i_k}(t\cdot\Psi_{\Lambda_L})(\cdot,x_k)\bigr|^2, \tag{4.38}$$

$$B_k := h(\cdot,x_k,i_k)\bigl|\nabla_{i_k}(t\cdot\psi)(\cdot,x_k)\bigr|^2, \tag{4.39}$$

and denote

$$R_{L,k} := \mathbb{E}\Bigl[\mathbb{E}\bigl[A_k\,\big|\mathcal{F}_k\bigr]^2 - \mathbb{E}\bigl[B_k\,\big|\mathcal{F}_k\bigr]^2\Big|\mathcal{F}_{k-1}\Bigr]. \tag{4.40}$$

By (4.34) we have $Z_k = \mathbb{E}(A_k\,|\mathcal{F}_k)$, while (4.28) reads $(t,\hat{Z}(x_k,i_k)t) = \mathbb{E}(B_k\,|\mathcal{F}_k)$. Hence, as soon as we show that

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\bigl(|R_{L,k}|\bigr) \underset{L\to\infty}{\longrightarrow} 0, \tag{4.41}$$

the claim (4.35) will follow.

The proof of (4.41) will proceed by estimating $\mathbb{E}|R_{L,k}|$ which will involve applications of the Cauchy-Schwarz inequality (in order to separate terms) and Jensen's inequality (in order to eliminate conditional expectations). First we note

$$\mathbb{E}|R_{L,k}| \le \big(\mathbb{E}\big[(A_k - B_k)^2\big]\big)^{1/2}\big(\mathbb{E}\big[(A_k + B_k)^2\big]\big)^{1/2}. \tag{4.42}$$

Writing $A_k = B_k + (A_k - B_k)$ and noting $(a+b)^2 \le 2a^2 + 2b^2$ tells us

$$\mathbb{E}\big[(A_k + B_k)^2\big] \le 2\mathbb{E}\big[(A_k - B_k)^2\big] + 8\mathbb{E}\big(B_k^2\big). \tag{4.43}$$

Summing over k and applying Cauchy-Schwarz, we find that

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\big(|R_{L,k}|\big) \le \sqrt{\alpha\big(2\alpha + 8\beta\big)}, \tag{4.44}$$

where

$$\alpha := \frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\big[(A_k - B_k)^2\big] \quad \text{and} \quad \beta := \frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}(B_k^2). \tag{4.45}$$

By inspection of (4.44) we now observe that it suffices to show that β stays bounded while α tends to zero in the limit $L \to \infty$.

The boundedness of β follows from (4.20) and the fact that $h(\cdot, x, i)$ is bounded; indeed, these yield $\mathbb{E}(|B_k|^2) \le \|h\|_\infty^2 |t|^4 \|\nabla\psi\|_4^4$ uniformly in k and L. Concerning the terms constituting α, using $(a+b)^2 \le 2a^2 + 2b^2$ we first separate terms as

$$\begin{aligned}\mathbb{E}\big[(A_k - B_k)^2\big] \le {} & 2\mathbb{E}\Big(\big|h_{\Lambda_L}(\cdot, x_k, i_k)\big|^2 \big|\,|\nabla_{i_k}(t\cdot\Psi_{\Lambda_L})(\cdot, x_k)|^2 - |\nabla_{i_k}(t\cdot\psi)(\cdot, x_k))|^2\big|^2\Big) \\ & + 2\mathbb{E}\Big(\big|h_{\Lambda_L}(\cdot, x_k, i_k) - h(\cdot, x_k, i_k)\big|^2\big|\nabla_{i_k}(t\cdot\psi)(\cdot, x_k)\big|^4\Big).\end{aligned} \tag{4.46}$$

Since h_Λ is uniformly bounded, the average over k of the first term is bounded by a constant times the product of $(\|\nabla\Psi_{\Lambda_L}\|_{\Lambda_L,4} + \|\nabla\psi\|_{\Lambda_L,4})^2$ and $\|\nabla(\Psi_{\Lambda_L} - \psi)\|_{\Lambda_L,4}^2$. The latter tends to zero as $L \to \infty$ by Proposition 4.4, Proposition 4.3 and Lemma 4.9 (with the choices $p := 4$ and $p' > 4$ but sufficiently close to 4).

For the second term in (4.46) we pick $p > 4$ and use Hölder's inequality to get

$$\begin{aligned}&\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\Big(\big|h_{\Lambda_L}(\cdot, x_k, i_k) - h(\cdot, x_k, i_k)\big|^2\big|\nabla_{i_k}(t\cdot\psi)(\cdot, x_k)\big|^4\Big) \\ &\quad \le |t|^4\,\|\nabla\psi\|_{\Lambda_L,p}^4\Bigg(\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}\Big(\big|h_{\Lambda_L}(\cdot, x_k, i_k) - h(\cdot, x_k, i_k)\big|^{2q}\Big)\Bigg)^{1/q},\end{aligned} \tag{4.47}$$

where q satisfies ${}^4/_p + {}^1/_q = 1$. The norm of $\|\nabla\psi\|_{\Lambda_L,p}$ is again bounded by Proposition 4.4 as long as p is sufficiently close to 4; to apply (4.20), we need to invoke the stationarity of $\nabla\psi$ to realize $\|\nabla\psi\|_{\Lambda_L,p} = \|\nabla\psi\|_p$.

For the second term in (4.47) we first need to show that for each $\epsilon > 0$ there is $N \geq 1$ so that for all $\omega \in \Omega$,

$$\mathrm{dist}_{\ell^1(\mathbb{Z}^d)}(x, \Lambda_L^{\mathrm{c}}) \geq N \quad \Rightarrow \quad \bigl| h_{\Lambda_L}(\omega, x, i) - h(\omega, x, i) \bigr| < \epsilon. \tag{4.48}$$

For this we use that, thanks to (4.27), (4.33) and (1.1),

$$\bigl| h_\Lambda(\omega, x, i) - h(\omega, x, i) \bigr| \leq C \int_\lambda^{1/\lambda} \mathrm{d}\tilde{\omega}_b \, \bigl| \mathfrak{g}_\Lambda^{(i)}(\tilde{\omega}, x) - \mathfrak{g}^{(i)}(\tilde{\omega}, x) \bigr| \tag{4.49}$$

for some constant $C = C(\lambda) < \infty$. To estimate the right-hand side, by the monotonicity of $\Lambda \mapsto \mathfrak{g}_\Lambda^{(i)}(\tilde{\omega}, x)$ and its stationarity with respect to shifts, we have

$$\bigl| \mathfrak{g}_\Lambda^{(i)}(\omega, x) - \mathfrak{g}^{(i)}(\omega, x) \bigr| \leq \bigl| \mathfrak{g}_{\Lambda_N}^{(i)}(\tau_x\omega, 0) - \mathfrak{g}^{(i)}(\tau_x\omega, 0) \bigr|, \qquad \omega \in \Omega, \tag{4.50}$$

as soon as $x + \Lambda_N \subset \Lambda$. Then (4.48) follows from (4.49) and the fact that the difference on the right-hand side of (4.50) converges to zero uniformly in $\omega \in \Omega$.

We now bound the last term in (4.47) as follows. The terms for which x_k is at least N steps away from Λ_L are bounded by ϵ thanks to (4.49); the sum over the remaining terms is of order NL^{d-1} thanks to the uniform boundedness of $h_\Lambda - h$. Hence, in the limit $L \to \infty$, the second term in (4.47) is of order $\epsilon^{1/q}$; taking $\epsilon \downarrow 0$ shows that α tends to zero as $L \to \infty$. Invoking (4.44), this finishes the proof of (4.41) and the whole claim. □

Proposition 4.11 (Martingale CLT — second condition)**.** *Assume that the premises (and thus conclusions) of Propositions 4.3–4.6 hold. Then for each* $\epsilon > 0$*,*

$$\frac{1}{|\Lambda_L|} \sum_{k=1}^{|\mathbb{B}(\Lambda_L)|} \mathbb{E}\Bigl(Z_k^2 \mathbb{1}_{\{|Z_k| > \epsilon |\Lambda_L|^{1/2}\}} \Bigm| \mathcal{F}_{k-1} \Bigr) \underset{L\to\infty}{\longrightarrow} 0, \tag{4.51}$$

in $\mathbb{P}$*-probability.*

Proof. This could be proved by strengthening a bit the statement of Proposition 4.10 (from squares of the Z's to a slightly higher power), but a direct argument is actually easier.

First we note that it suffices to show convergence in expectation. Let $p > 4$ be such that the statements in Proposition 4.4 hold. By Hölder's and Chebyshev's inequalities we have

$$\mathbb{E}\Big(Z_k^2 \mathbb{1}_{\{|Z_k|>\epsilon|\Lambda_L|^{1/2}\}}\Big) \le \Big(\frac{1}{\epsilon|\Lambda_L|^{1/2}}\Big)^{\frac{p-4}{2}} \mathbb{E}\big(|Z_k|^{p/2}\big). \tag{4.52}$$

Since h_{Λ_L} is bounded, Jensen's inequality yields

$$\mathbb{E}\big(|Z_k|^{p/2}\big) \le C\mathbb{E}\bigg(\Big[\mathbb{E}\Big(\big|\nabla_{i_k}(t\cdot\Psi_\Lambda)(\cdot,x_k)\big|^2\,\Big|\mathcal{F}_k\Big)\Big]^{p/2}\bigg) \le C\mathbb{E}\Big(\big|\nabla_{i_k}(t\cdot\Psi_\Lambda)(\cdot,x_k)\big|^p\Big). \tag{4.53}$$

It follows that

$$\frac{1}{|\Lambda_L|}\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|} \mathbb{E}\big(|Z_k|^{p/2}\big) \le C|t|^p \|\nabla\Psi_{\Lambda_L}\|^p_{\Lambda_L,p}. \tag{4.54}$$

The right-hand side is bounded uniformly in L. Using this in (4.52), the claim follows. □

We can now finish the proof of our main results:

Proof of Theorems 1.11 and 4.7 from Propositions 4.3–4.6. The distributional convergence in (1.26) is a direct consequence of the Martingale Central Limit Theorem whose conditions (4.3–4.4) are established in Propositions 4.10 and 4.11. The limiting variance σ_t^2 is given by the right-hand side of (4.35), in agreement with (4.29). It remains to prove that $\sigma_t^2 > 0$ whenever $t \neq 0$ and the law $\mathbb{P}$ is non-degenerate.

Suppose on the contrary that $\sigma_t^2 = 0$. Then for each i we would have $\mathbb{E}((t,\hat{Z}(0,i)t)^2) = 0$ and thus $(t,\hat{Z}(0,i)t) = 0$ $\mathbb{P}$-a.s. Denoting $b := \langle 0,\hat{e}_i\rangle$, (4.27–4.28) imply that, for $\mathbb{P}$-a.e. ω_b,

$$\int \mathbb{P}(\mathrm{d}\omega_b')\int_{\omega_b'}^{\omega_b} \mathrm{d}\tilde{\omega}_b\; \mathbb{E}\Big(\big[1-(\tilde{\omega}_b-\omega_b)\mathfrak{g}_\Lambda^{(i)}(\tilde{\omega},0)\big]\big|\nabla_i(t\cdot\psi)(\omega,0)\big|^2\Big|\mathcal{F}_{(0,i)}\Big) = 0, \tag{4.55}$$

where $\mathcal{F}_{(0,i)} := \sigma(\omega_b)$. Let $\Omega_1 \subset [\lambda, 1/\lambda]$ be the set of ω_b where this holds. Then $\mathbb{P}(\Omega_1) = 1$ and, since $\mathbb{P}$ is non-degenerate, Ω_1 contains at least two points. The expectation in (4.55) is independent of ω_b'; subtracting the expression for two (generic) choices of ω_b in Ω_1 then shows that the inner integral must vanish for all $\omega_b, \omega_b' \in \Omega_1$. But (4.24) tells us that the prefactor in square brackets, and thus the conditional expectation, is non-negative (in fact, it is bounded away from zero). Hence, this can only happen when

$$\nabla_i(t\cdot\psi)(\cdot,0) = 0, \qquad \mathbb{P}\text{-a.s. for all } i = 1,\dots,d. \tag{4.56}$$

But then $c_{\rm eff}(t)=0$, which cannot hold for $t\neq 0$ when (1.1) is in force. □

Proof of Corollary 1.12 from Propositions 4.3–4.6. Thanks to (4.1–4.2) and Proposition 4.10, $C_L^{\rm eff}(t)$ is a martingale whose increments, Z_k are square integrable. Therefore,

$$\operatorname{Var}\bigl(C_L^{\rm eff}(t)\bigr)=\sum_{k=1}^{|\mathbb{B}(\Lambda_L)|}\mathbb{E}(Z_k^2). \tag{4.57}$$

But the right-hand side is the expectation of the quantity on the left of (4.35). Since the convergence in (4.35) occurs in $L^1(\mathbb{P})$, the claim follows. □

4.3 The Meyers estimate

The goal of this section is to give proofs of Propositions 4.3 and 4.4. The former is a simple consequence of the Hilbert-space structure underlying the definition of a harmonic coordinate; the latter (to which this section owes its name) is a far less immediate consequence of the Calderón-Zygmund regularity theory for singular integral operators.

4.3.1 L2 bounds and convergence

Recall our notation $\mathcal{L}_\omega$ for the operator in (4.8). We begin by noting an explicit representation of the minimum of $f\mapsto Q_\lambda(f)$ as a function of the (Dirichlet) boundary condition:

Lemma 4.12. *Let $\Lambda\subset\mathbb{Z}^d$ be finite and fix an $\omega\in\Omega$. Then there is $K\colon\partial\Lambda\times\partial\Lambda\to[0,\infty)$, depending on Λ and ω, such that for any h that obeys $\mathcal{L}_\omega h(x)=0$ for $x\in\Lambda$,*

$$Q_\Lambda(h)=\frac12\sum_{x,y\in\partial\Lambda}K(x,y)\bigl[h(y)-h(x)\bigr]^2. \tag{4.58}$$

Moreover, $K(x,y)=K(y,x)$ for all $x,y\in\partial\Lambda$ and

$$\sum_{y\in\partial\Lambda}K(x,y)=\sum_{\substack{z\in\Lambda\\ \langle x,z\rangle\in\mathbb{B}(\Lambda)}}a_{xz} \tag{4.59}$$

for all $x\in\partial\Lambda$.

Proof. "Integrating" by parts we obtain

$$Q_\Lambda(h) = -\sum_{y\in\Lambda} h(y)(\mathcal{L}_\omega h)(y) + \sum_{\substack{y\in\partial\Lambda,\, x\in\Lambda \\ \langle x,y\rangle\in\mathbb{B}(\Lambda)}} a_{xy}\,\bigl[h(y)-h(x)\bigr]h(y). \tag{4.60}$$

Employing the fact that h is $\mathcal{L}_\omega$-harmonic, the first sum drops out. For the second sum we recall that $h(x) = \sum_{z\in\partial\Lambda} p_\Lambda(x,z)h(z)$, where $p_\Lambda(x,z)$ is the discrete Poisson kernel which can be defined by $p_\Lambda(x,z) := P_\omega^x(X_{\tau_{\partial\Lambda}} = z)$ for $\tau_{\partial\Lambda}$ denoting the first exit time from Λ of the random walk in conductances ω. Now set

$$K(y,z) := \sum_{\substack{x\in\Lambda \\ \langle x,y\rangle\in\mathbb{B}(\Lambda)}} a_{xy} p_\Lambda(x,z) \tag{4.61}$$

and note that $\sum_{z\in\partial\Lambda} K(y,z) = \sum_{x\in\Lambda,\,\langle x,y\rangle\in\mathbb{B}(\Lambda)} a_{xy}$. It follows that

$$\sum_{\substack{y\in\partial\Lambda,\, x\in\Lambda \\ \langle x,y\rangle\in\mathbb{B}(\Lambda)}} a_{xy}\,\bigl[h(y)-h(x)\bigr]h(y) = \sum_{y,z\in\partial\Lambda} K(y,z)\bigl[h(y)-h(z)\bigr]h(y). \tag{4.62}$$

The representation using the random walk and its reversiblity now imply that K is symmetric. Symmetrizing the last sum then yields the result. □

Remark 4.13. *We note that Lemma 4.12 holds even for vector valued functions; just replace $[h(y)-h(x)]^2$ by the norm squared of $h(y)-h(x)$. This applies to several derivations that are to follow; a point that we will leave without further comment.*

We can now prove Proposition 4.3 dealing with the convergence of $\nabla\Psi_\Lambda$ to $\nabla\psi$ in $\|\cdot\|_{\Lambda,2}$-norm, as $\Lambda := \Lambda_L$ fills up all of $\mathbb{Z}^d$.

Proof of Proposition 4.3. Abbreviate $h(x) := \psi(\omega,x) - \Psi_{\Lambda_L}(\omega,x)$. The bound (1.1) implies

$$\bigl\|\nabla(\Psi_{\Lambda_L}-\psi)\bigr\|_{\Lambda_L,2}^2 \le \frac{1}{\lambda}\,\frac{1}{|\Lambda_L|}\;\mathbb{E}\biggl(\sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} a_{xy}\bigl|h(y)-h(x)\bigr|^2\biggr). \tag{4.63}$$

Let $f\colon \Lambda\cup\partial\Lambda \to \mathbb{R}^d$ be the minimizer of

$$\inf\biggl\{\sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} \bigl|f(y)-f(x)\bigr|^2,\; f(z)=\chi(z) \text{ for all } z\in\partial\Lambda_L\biggr\}. \tag{4.64}$$

Since h is the minimizer of the corresponding Dirichlet energy with conductances $\{a_{xy}\}$ and boundary condition χ, we get using (1.1)

$$\begin{aligned}\sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} a_{xy}\big|h(y)-h(x)\big|^2 &\le \sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} a_{xy}\big|f(y)-f(x)\big|^2 \\ &\le \frac{1}{\lambda}\sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} \big|f(y)-f(x)\big|^2.\end{aligned}\tag{4.65}$$

Writing the last sum coordinate-wise and applying Lemma 4.12, we thus get

$$\sum_{\langle x,y\rangle\in\mathbb{B}(\Lambda_L)} a_{xy}\big|h(y)-h(x)\big|^2 \le \frac{1}{2\lambda}\sum_{x,y\in\partial\Lambda_L} K(x,y)\big|\chi(\omega,y)-\chi(\omega,x)\big|^2,\tag{4.66}$$

where the kernel $K(x,y)$ pertains to the homogeneous problem, i.e., the simple random walk. Note that these bounds hold for all configurations satisfying (1.1).

By shift covariance and sublinearity of the corrector (cf Proposition 4.2(2,4)), for each $\varepsilon>0$ there is $A=A(\varepsilon)$ such that

$$\mathbb{E}\big(\,\big|\chi(\cdot,x)-\chi(\cdot,y)\big|^2\big)\le A+\varepsilon|x-y|^2.\tag{4.67}$$

Using this and (4.66) in (4.63) yields

$$\big\|\nabla(\Psi_{\Lambda_L}-\psi)\big\|^2_{\Lambda_L,2}\le\frac{1}{2\lambda^2}\frac{1}{|\Lambda_L|}\sum_{x,y\in\partial\Lambda_L}K(x,y)\big(A+\varepsilon|x-y|^2\big).\tag{4.68}$$

But $\sum_{y\in\partial\Lambda_L}K(x,y)\le 1$ for each $x\in\partial\Lambda_L$ while $\sum_{x,y\in\partial\Lambda_L}K(x,y)|x-y|^2$ is, by Lemma 4.12, the Dirichlet energy of the function $x\mapsto x$ for conductances all equal to 1. Hence, the last sum in (4.68) is bounded by $A|\partial\Lambda_L|+\varepsilon|\mathbb{B}(\Lambda_L)|$. Taking $L\to\infty$ and $\varepsilon\downarrow 0$ finishes the proof. □

Remark 4.14. *As alluded to in the introduction, the L^2-convergence $\nabla\Psi_{\Lambda_L}\to\nabla\psi$ permits us to prove the formula (1.25) for $c_{\text{eff}}(t)$. The argument is similar to (albeit much easier than) what we used in the proof of Proposition 4.10. Indeed, we trivially decompose*

$$C_L^{\text{eff}}(t)=Q_{\Lambda_L}\big(t\cdot\Psi_{\Lambda_L}\big)=Q_{\Lambda_L}(t\cdot\psi)+\big(Q_{\Lambda_L}\big(t\cdot\Psi_{\Lambda_L}\big)-Q_{\Lambda_L}(t\cdot\psi)\big).\tag{4.69}$$

The stationarity of the gradients of ψ and the Spatial Ergodic Theorem imply that for any ergodic law $\mathbb{P}$ on conductances, $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$,

$$\frac{1}{|\Lambda_L|}Q_{\Lambda_L}(t\cdot\psi)\underset{L\to\infty}{\longrightarrow}\mathbb{E}\Big(\sum_{x=\hat{e}_1,\dots,\hat{e}_d}a_{0,x}(\omega)\big|t\cdot\psi(\omega,x)\big|^2\Big).\tag{4.70}$$

It follows from the construction of the harmonic coordinate that expression on the right coincides with the infimum in (1.25)*. (There is no gradient on the right-hand side of* (4.70) *because* $\psi(\omega,0):=0$*.) It remains to control the difference on the extreme right of* (4.69)*.*

Using the quadratic nature of Q_Λ*, the ellipticity assumption* (1.1) *and Cauchy-Schwarz,*

$$\frac{\mathbb{E}\bigl|Q_\Lambda\bigl(t\cdot\Psi_\Lambda\bigr)-Q_\Lambda(t\cdot\psi)\bigr|}{|\Lambda|} \le \frac{1}{\lambda}|t|^2\bigl\|\nabla(\Psi_\Lambda-\psi)\bigr\|_{\Lambda,2}^2+\frac{2}{\lambda}|t|^2\|\nabla\psi\|_2\bigl\|\nabla(\Psi_\Lambda-\psi)\bigr\|_{\Lambda,2}. \tag{4.71}$$

By Proposition 4.3 — which holds for any shift-ergodic (elliptic) law on conductances — the right-hand side tends to zero as $\Lambda:=\Lambda_L$ *increases to* $\mathbb{Z}^d$*. Since we know that* $|\Lambda_L|^{-1}C_L^{\text{eff}}(t)$ *is bounded and converges almost surely (e.g., by the Subadditive Ergodic Theorem), it converges also in* $L^1(\mathbb{P})$*. We conclude that the limit value* $c_{\text{eff}}(t)$ *is given by* (1.25)*.*

4.3.2 The Meyers estimate in finite volume

Key to the proof of Proposition 4.4 is the Meyers estimate. The term owes its name to Norman G. Meyers [Mey63] who discovered a bound on L^p-continuity (in the right-hand side) of the solutions of Poisson equation with second-order elliptic differential operators in divergence from, provided the associated coefficients are close to a constant. The technical ingredient underpinning this observation is the Calderón-Zygmund regularity theory for certain singular integral operators in $\mathbb{R}^d$. (Incidentally, as noted in [Mey63], Meyers' argument is a generalization of earlier work of Boyarskii, cf [Mey63, ref. 2 and 3] for systems of first-order PDEs and a version of his result was also derived, though not published, by Calderón himself; cf [Mey63, page 190]).

To ease the notation, in addition to (4.18), we will use the notation $\|f\|_p$ also for the canonical norm in $\ell^p(\Lambda)$,

$$\|f\|_p:=\Bigl(\sum_{x\in\Lambda}\bigl|f(x)\bigr|^p\Bigr)^{1/p}, \tag{4.72}$$

throughout the rest of this section.

Let us review the gist of Meyers' argument for functions on $\mathbb{Z}^d$. Our notation is inspired by that used in Naddaf and Spencer [NS98] and Gloria and Otto [GO11]. A general form of the second order difference operator $\mathcal{L}$ in divergence form is

$$\mathcal{L} := \nabla^\star \cdot A \cdot \nabla, \tag{4.73}$$

where $A = \{A_{ij}(x) \colon i,j = 1,\dots,d,\ x \in \mathbb{Z}^d\}$ are x-dependent matrix coefficients, $\nabla f(x)$ is a vector whose i-th component is $\nabla_i f(x) := f(x+\hat{e}_i) - f(x)$ and $\nabla^\star$ is its conjugate acting as $\nabla_i^\star f(x) := f(x) - f(x-\hat{e}_i)$. The above $\mathcal{L}$ is explicitly given by

$$(\mathcal{L}f)(x) = \sum_{i,j=1}^d \Bigl(A_{i,j}(x)\bigl[f(x+\hat{e}_i)-f(x)\bigr]-A_{i,j}(x-\hat{e}_j)\bigl[f(x+\hat{e}_i-\hat{e}_j)-f(x-\hat{e}_j)\bigr]\Bigr). \tag{4.74}$$

Now, if A is close to the identity matrix, it makes sense to write

$$\mathcal{L} = \Delta + \nabla^\star \cdot (A - \mathrm{id}) \cdot \nabla, \tag{4.75}$$

where we noted that the standard lattice Laplacian Δ corresponds to $\nabla^\star \cdot \mathrm{id} \cdot \nabla$. This formula can be used as a starting point of perturbative arguments.

Consider a finite set $\Lambda \subset \mathbb{Z}^d$ and let $g \colon \Lambda \cup \partial\Lambda \to \mathbb{R}^d$. Let f be a solution to the Poisson equation

$$-\mathcal{L}\, f = \nabla^\star \cdot g, \qquad \text{in } \Lambda, \tag{4.76}$$

with $f := 0$ on $\partial\Lambda$. Employing (4.75), we can rewrite this as

$$-\Delta f = \nabla^\star \cdot \bigl[g + (A - \mathrm{id}) \cdot \nabla f\bigr]. \tag{4.77}$$

The function on the right has vanishing total sum over Λ and hence it lies in the domain of the inverse $(\Delta)_\Lambda^{-1}$ of Δ with zero boundary conditions. Taking this inverse followed by one more gradient, and denoting

$$\mathcal{K}_\Lambda := \nabla(-\Delta)_\Lambda^{-1}\nabla^\star, \tag{4.78}$$

this equation translates to

$$\nabla f = \mathcal{K}_\Lambda \cdot \bigl[g + (A - \mathrm{id})\nabla f\bigr]. \tag{4.79}$$

A first noteworthy point is that this is now an autonomous equation for ∇f. A second point is that, if $\|\mathcal{K}_\Lambda\|_p$ is the norm of $\mathcal{K}_\Lambda$ as a map (on vector valued functions) $\ell^p(\Lambda) \to$

$\ell^p(\Lambda)$ and $\|A - \mathrm{id}\|_\infty$ is the least a.s. upper bound on the coefficients of $A(x) - \mathrm{id}$, uniform in x, we get

$$\|\nabla f\|_p \le \|\mathcal{K}_\Lambda\|_p \|A - \mathrm{id}\|_\infty \|\nabla f\|_p + \|\mathcal{K}_\Lambda\|_p \|g\|_p. \tag{4.80}$$

Assuming $\|\mathcal{K}_\Lambda\|_p \|A - \mathrm{id}\|_\infty < 1$ this yields

$$\|\nabla f\|_p \le \frac{\|\mathcal{K}_\Lambda\|_p \|g\|_p}{1 - \|\mathcal{K}_\Lambda\|_p \|A - \mathrm{id}\|_\infty}. \tag{4.81}$$

Furthermore, the condition $\|\mathcal{K}_\Lambda\|_p \|A - \mathrm{id}\|_\infty < 1$ ensures the very existence of a unique solution ∇f to (4.79) via a contraction argument; (4.81) then implies the continuity of $g \mapsto \nabla f$ in $\ell^p(\Lambda)$.

The aforementioned general facts are relevant for us because $\mathcal{L}_\omega$ is of the form (4.73). Indeed, set $A_{ij}(x) := \delta_{ij} a_{x,x+\hat{e}_i}$ and note that (4.74) reduces to (4.8). The finite-volume corrector

$$\chi_\Lambda(\omega, x) := \Psi_\Lambda(\omega, x) - x \tag{4.82}$$

then solves the Poisson equation

$$-\mathcal{L}_\omega \chi_\Lambda = \nabla^\star \cdot g, \quad \text{where} \quad g(x) := (a_{x,x+\hat{e}_1}, \dots, a_{x,x+\hat{e}_d}). \tag{4.83}$$

Thanks to (1.1), this g is bounded uniformly so, in order to have (4.81) for all finite boxes, our main concern is the following claim:

Theorem 4.15. *For each $p \in (1, \infty)$, the operator $\mathcal{K}_{\Lambda_L}$ is bounded in $\ell^p(\Lambda_L)$, uniformly in $L \ge 1$.*

Proof of Proposition 4.4 from Theorem 4.15. Let $p^* > 4$. Since (in our setting) $\|A - \mathrm{id}\|_\infty \le \lambda^{-1} - 1$, we may choose $\lambda \in (0, 1)$ close enough to one so that $\sup_{L\ge1} \|\mathcal{K}_{\Lambda_L}\|_{p^*} \|A - \mathrm{id}\|_\infty < 1$. From the above derivation it follows

$$\sup_{L \ge 1} \|\nabla \chi_{\Lambda_L}\|_{\Lambda_L, p^*} < \infty. \tag{4.84}$$

We claim that this implies

$$\|\nabla \chi\|_p < \infty, \qquad p < p^*. \tag{4.85}$$

Indeed, pick $\alpha > 0$ and note that, for any $\epsilon \in (0, \alpha)$,

$$\sum_{x\in\Lambda_L} 1\!\!1_{\{|\nabla\chi(\cdot,x)|>\alpha\}} \le \sum_{x\in\Lambda_L} 1\!\!1_{\{|\nabla\chi_{\Lambda_L}(\cdot,x)|>\alpha-\epsilon\}} + \sum_{x\in\Lambda_L} 1\!\!1_{\{|\nabla\chi_{\Lambda_L}(\cdot,x)-\nabla\chi(\cdot,x)|>\epsilon\}}. \tag{4.86}$$

Taking expectations and dividing by $|\Lambda_L|$, the left hand side becomes $\mathbb{P}(|\nabla\chi(\cdot,0)| > \alpha)$, while the second sum on the right can be bounded by $\epsilon^{-2}\|\nabla\chi_{\Lambda_L} - \nabla\chi\|^2_{\Lambda_L,2}$, which tends to zero as $L \to \infty$ by Proposition 4.3. Applying Chebyshev's inequality to the first sum on the right and taking $L \to \infty$ followed by $\epsilon \downarrow 0$ yields

$$\mathbb{P}\bigl(|\nabla\chi(\cdot,0)| > \alpha\bigr) \le \frac{1}{\alpha^{p^*}} \sup_{L\ge1} \|\nabla\chi_{\Lambda_L}\|^{p^*}_{\Lambda_L,p^*}. \tag{4.87}$$

Multiplying by α^{p-1} and integrating over $\alpha > 0$ then proves (4.85).

Returning to the claims in Proposition 4.4, inequality (4.85) is a restatement of (4.20). Since (4.84–4.85) imply the uniform boundedness of $\|\nabla(\chi_{\Lambda_L} - \chi)\|_{\Lambda_L,p}$, for each $p < p^*$, Lemma 4.9 then shows $\|\nabla(\chi_{\Lambda_L} - \chi)\|_{\Lambda_L,p} \to 0$, as $L \to \infty$ for all $p < p^*$. This proves (4.21) as well. □

4.3.3 Interpolation

In the proof of Theorem 4.15 we will follow the classical argument — by and large due to Marcinkiewicz — that is spelled out in Chapter 2 (specifically, proof of Theorem 1 in Section 2.2) of Stein's book [Ste70]. The reasoning requires only straightforward adaptations due to discrete setting and finite volume, but we still prefer to give a full argument to keep the present paper self-contained. A key idea is the use of interpolation between the strong ℓ^2-type estimate (Lemma 4.16) and the weak ℓ^1-type estimate for $\mathcal{K}_{\Lambda_L}$ (Lemma 4.17). Both of these of course need to hold uniformly in $L \ge 1$.

Lemma 4.16. *For any finite $\Lambda \subset \mathbb{Z}^d$, the $\ell^2(\Lambda)$-norm of $\mathcal{K}_\Lambda$ satisfies $\|\mathcal{K}_\Lambda\|_2 \le 1$.*

Proof. Let $\mathcal{H}$ be a Hilbert space and T a positive self-adjoint, bounded and invertible operator. Then for all $h \in \mathcal{H}$,

$$\bigl(h, T^{-1}h\bigr) = \sup_{g\in\mathcal{H}} \bigl\{2(g,h) - (g,Tg)\bigr\}. \tag{4.88}$$

We will apply this to $\mathcal{H}$ given by the space (of $\mathbb{R}$-valued functions) $\ell^2(\Lambda)$, $T := \epsilon - \Delta$ and $h := \nabla^\star \cdot f$ for some $f\colon \Lambda \to \mathbb{R}^d$ with zero boundary conditions outside Λ. Then

$$\begin{aligned}\big(\nabla^\star \cdot f, (\varepsilon - \Delta)^{-1}\nabla^\star \cdot f\big) &= \sup_{g\in\ell^2(\Lambda)} \big\{2(g, \nabla^\star \cdot f) - \varepsilon(g, g) + (g, \Delta g)\big\} \\ &= \sup_{g\in\ell^2(\Lambda)} \big\{2(\nabla g, f) - \varepsilon(g, g) - (\nabla g, \nabla g) - (f, f)\big\} + (f, f) \\ &= \sup_{g\in\ell^2(\Lambda)} \big\{-(\nabla g - f, \nabla g - f)\big\} + (f, f) \\ &\le (f, f),\end{aligned} \tag{4.89}$$

where we used that $\nabla^\star$ is the adjoint of ∇ in the space of $\mathbb{R}^d$-valued functions $\ell^2(\Lambda)$ and where the various inner products have to be interpreted either for $\mathbb{R}$-valued or $\mathbb{R}^d$-valued functions accordingly. Taking $\epsilon \downarrow 0$, the left-hand side becomes $(f, \mathcal{K}_\Lambda \cdot f)$. The claim follows. □

The second ingredient turns out to be technically more involved.

Lemma 4.17. *$\mathcal{K}_{\Lambda_L}$ is of weak-type (1-1), uniformly in $L > 1$. That is, there exists $\widehat{K}_1$ such that, for all $L > 1$, $f \in \ell^1(\Lambda_L)$ and $\alpha > 0$,*

$$\big|\{z \in \Lambda_L \colon |\mathcal{K}_{\Lambda_L} f(z)| > \alpha\}\big| \le \widehat{K}_1 \frac{\|f\|_1}{\alpha}. \tag{4.90}$$

Deferring the proof of this lemma to the next subsection, we now show how this enters into the proof of Theorem 4.15.

Proof of Theorem 4.15 from Lemma 4.17. We follow the proof in Stein [Ste70, Theorem 5, page 21]. We begin with the case $1 < p < 2$. Let $f \in \ell^p(\Lambda_L)$ and pick $\alpha > 0$. Let $f_1 := f 1\!\!1_{\{|f|>\alpha\}}$ and $f_2 := f 1\!\!1_{\{|f|\le\alpha\}}$. Then

$$\begin{aligned}\big|\{z \in \Lambda_L \colon |\mathcal{K}_{\Lambda_L} f(z)| > 2\alpha\}\big| \le \big|\{z \in \Lambda_L \colon |\mathcal{K}_{\Lambda_L} f_1| > \alpha\}\big| \\ + \big|\{z \in \Lambda_L \colon |\mathcal{K}_{\Lambda_L} f_2| > \alpha\}\big|.\end{aligned} \tag{4.91}$$

Lemmas 4.16 and 4.17 then yield

$$\big|\{z \in \Lambda_L \colon |\mathcal{K}_{\Lambda_L} f(z)| > \alpha\}\big| \le \widehat{K}_1 \frac{\|f_1\|_1}{\alpha} + \widehat{K}_2 \frac{\|f_2\|_2}{\alpha^2}, \tag{4.92}$$

with $\widehat{K}_1$ and $\widehat{K}_2$ independent of L. Multiplying by α^{p-1} and integrating, we infer

$$\begin{aligned}\|\mathcal{K}_{\Lambda_L} f\|_p^p &= p\int_0^\infty \alpha^{p-1}\big|\{z\in\Lambda_L\colon |\mathcal{K}_{\Lambda_L} f(z)|>\alpha\}\big|\,\mathrm{d}\alpha \\ &\le p\sum_z\int_0^\infty\Big(\widehat{K}_1\alpha^{p-2}|f(z)|1\!\!1_{\{|f|>\alpha\}}+\widehat{K}_2\alpha^{p-3}|f(z)|^2 1\!\!1_{\{|f|\le\alpha\}}\Big)\,\mathrm{d}\alpha \\ &= p\widehat{K}_1\sum_z|f(z)|\int_0^{|f(z)|}\alpha^{p-2}\,\mathrm{d}\alpha+p\widehat{K}_2\sum_z|f(z)|^2\int_{|f(z)|}^\infty\alpha^{p-3}\,\mathrm{d}\alpha \\ &= \frac{p\widehat{K}_1}{p-1}\sum_z|f(z)|^p+\frac{p\widehat{K}_2}{2-p}\sum_z|f(z)|^p,\end{aligned}\tag{4.93}$$

proving the assertion in the case $1<p<2$.

For $p\in(2,\infty)$, the fact that $\mathcal{K}_\Lambda$ is obviously symmetric implies that $\|\mathcal{K}_\Lambda\|_p=\|\mathcal{K}_\Lambda\|_q$, where q is the index dual to p. Hence $\sup_{L\ge1}\|\mathcal{K}_{\Lambda_L}\|_p<\infty$ for all $p\in(1,\infty)$. □

4.3.4 Weak type-(1,1) estimate

It remains to prove Lemma 4.17. The strategy is to represent the operator using a singular kernel that has a “nearly ℓ^1-integrable” decay. Let $G_\Lambda(x,y)$ be the Green function (i.e., inverse) of the Laplacian Δ on Λ with zero boundary condition on $\partial\Lambda$.

Lemma 4.18. *The operator $\mathcal{K}_\Lambda$ admits the representation*

$$\hat{e}_i\cdot\big[\mathcal{K}_\Lambda\cdot f(x)\big]=\sum_{y\in\Lambda}\sum_{j=1}^d\big[\nabla_i^{(1)}\nabla_j^{(2)}G_\Lambda(x,y)\big]f_j(y),\tag{4.94}$$

where the superscripts on the ∇'s indicate which of the two variables the operator is acting on.

Proof. Since both G_Λ and f vanish outside Λ, we have

$$\begin{aligned}\hat{e}_i\cdot\big[\mathcal{K}_\Lambda\cdot f(x)\big]=&\nabla_i\Big(\sum_{y\in\Lambda}G_\Lambda(\cdot,y)\big(\nabla^\star\cdot f\big)(y)\Big)(x)\\ =&\sum_{y\in\mathbb{Z}^d}\Big(\big(G_\Lambda(x+\hat{e}_i,y)-G_\Lambda(x,y)\big)\sum_{j=1}^d[f_j(y-\hat{e}_k)-f_j(y)]\Big)\\ =&\sum_{j=1}^d\sum_{y\in\mathbb{Z}^d}\big(G_\Lambda(x+\hat{e}_i,y+\hat{e}_j)-G_\Lambda(x,y+\hat{e}_j)\big)f_j(y)\\ &-\sum_{j=1}^d\sum_{y\in\mathbb{Z}^d}\big(G_\Lambda(x+\hat{e}_i,y)-G_\Lambda(x,y)\big)f_j(y).\end{aligned}\tag{4.95}$$

This is exactly the claimed expression. □

Crucial for the proof of the weak-type (1,1)-estimate in Lemma 4.17 is an integrable decay estimate on the gradient of the kernel of the operator $\mathcal{K}_\Lambda$:

Proposition 4.19. *There exists $C > 0$ independent of L such that*

$$\left|\nabla_i^{(2)}\nabla_j^{(1)}\nabla_k^{(2)} G_{\Lambda_L}(x,y)\right| \le \frac{C}{|x-y|^{d+1}} \tag{4.96}$$

for all $x, y \in \Lambda_L$ and $i, j, k \in \{1, \dots, d\}$.

Although (4.96) is certainly not unexpected, and perhaps even well-known, we could not find an exact reference and therefore provide an independent proof in Section 4.3.5. With this estimate at hand, we can now turn to the proof of Lemma 4.17.

Proof of Lemma 4.17 from Proposition 4.19. To ease the notation, we will write $\Lambda := \Lambda_L$ (note that all bounds will be uniform in L) and, resorting to components, write $\mathcal{K}_\Lambda$ for the scalar-to-scalar operator with kernel $\mathcal{K}_\Lambda^{(i,j)}(x,y) := \nabla_i^{(1)}\nabla_j^{(2)} G_\Lambda(x,y)$ for some fixed $i, j \in \{1, \dots, d\}$. For the most part, we adapt the arguments in Stein [Ste70, pages 30-33].

Take some function $f\colon \Lambda \to \mathbb{R}$, extended to vanish outside Λ, and pick $\alpha > 0$. Consider a partition of $\mathbb{Z}^d$ into cubes of side 3^r, where r is chosen so large that $3^{-rd}\|f\|_1 \le \alpha$. Naturally, each of the cubes in the partition further divides into 3^d equal-sized sub-cubes of side 3^{r-1}, which subdivide further into sub-cubes of side 3^{r-2}, etc. We will now designate these to be either *good cubes* or *bad cubes* according to the following recipe. All cubes of side 3^r are *ex definitio* good. With Q being one of these sub-cubes of side 3^{r-1}, we call Q good if

$$\frac{1}{|Q|}\sum_{z \in Q}\left|f(z)\right| \le \alpha, \tag{4.97}$$

and bad otherwise. For each good cube, we repeat the process of partitioning it into 3^d equal-size sub-cubes and designating each of them to be either good or bad depending on whether (4.97) holds or not, respectively. The bad cubes are not subdivided further.

Iterating this process, we obtain a finite set $\mathcal{B}$ of bad cubes which covers the (bounded) region $B := \bigcup_{Q \in \mathcal{B}} Q$. We define $G := \mathbb{Z}^d \setminus B$, the good region, and note that

$$\left|f(z)\right| \le \alpha, \qquad z \in G, \tag{4.98}$$

and

$$\alpha < \frac{1}{|Q|} \sum_{z \in Q} \bigl| f(z) \bigr| \leq 3^d \alpha, \qquad Q \in \mathcal{B}, \tag{4.99}$$

where the last inequality is due to the fact that the parent cube of a bad cube is good. Next we define the "good" function

$$g(z) := \begin{cases} f(z), & z \in G \\ \frac{1}{|Q|} \sum_{z \in Q} f(z), & z \in Q \in \mathcal{B}. \end{cases} \tag{4.100}$$

The "bad" function, defined by $b := f - g$, then satisfies

$$\begin{aligned} b(z) &= 0, & z &\in G, \\ \sum_{z \in Q} b(z) &= 0, & Q &\in \mathcal{B}. \end{aligned} \tag{4.101}$$

Since $\mathcal{K}_\Lambda f = \mathcal{K}_\Lambda g + \mathcal{K}_\Lambda b$, as soon as

$$\bigl|\{z : |\mathcal{K}_\Lambda g(z)| > \alpha/2\}\bigr| \leq \frac{\widehat{K}_1 \|f\|_1}{2\alpha} \quad \text{AND} \quad \bigl|\{z : |\mathcal{K}_\Lambda b(z)| > \alpha/2\}\bigr| \leq \frac{\widehat{K}_1 \|f\|_1}{2\alpha}, \tag{4.102}$$

the desired bound (4.90) will hold. We will now show these bounds in separate arguments.

Considering g first, we note that $\|g\|_2^2$ is bounded by a constant times $\alpha\|f\|_1$. Indeed, for $z \in B$ let Q_z denote the bad cube containing z. Then

$$\begin{aligned} \sum_{z \in \mathbb{Z}^d} g(z)^2 &= \sum_{z \in G} f(z)^2 + \sum_{z \in B} g(z)^2 \\ &\leq \alpha \sum_{z \in G} \bigl| f(z) \bigr| + \sum_{z \in B} \Bigl(\frac{1}{|Q_z|} \sum_{y \in Q_z} f(z) \Bigr)^2 \\ &\leq \alpha \|f\|_1 + 3^d \alpha \sum_{z \in B} \frac{1}{|Q_z|} \sum_{y \in Q_z} \bigl| f(z) \bigr| \\ &\leq (3^d + 1) \alpha \|f\|_1 \end{aligned} \tag{4.103}$$

by using (4.98) on G and (4.99) on B. By Chebychev's inequality and Lemma 4.16,

$$\bigl|\{z : |\mathcal{K}_\Lambda g(z)| > \alpha\}\bigr| \leq \frac{\|\mathcal{K}_\Lambda g\|_2^2}{\alpha^2} \leq \frac{(3^d + 1) \|\mathcal{K}_\Lambda\|_2^2 \, \|f\|_1}{\alpha}. \tag{4.104}$$

Note that this yields an estimate that is uniform in $\Lambda := \Lambda_L$ because $\|\mathcal{K}_\Lambda\|_2 \leq 1$ by Lemma 4.16.

Let us turn to the estimate in (4.102) concerning b. Let $\{Q_k\colon k = 1,\dots,|\mathcal{B}|\}$ be an enumeration of the bad cubes and let $b_k := b\mathbb{1}_{Q_k}$ be the restriction of b onto Q_k. Abusing the notation to the point where we write $\mathcal{K}_\Lambda(x,y)$ for the kernel governing $\mathcal{K}_\Lambda$, from (4.101) we then have

$$\mathcal{K}_\Lambda b_k(z) = \sum_{y\in Q_k} \bigl[\mathcal{K}_\Lambda(z,y) - \mathcal{K}_\Lambda(z,y_k)\bigr] b(y), \tag{4.105}$$

where y_k is the center of Q_k (remember that all cubes are odd-sized). Let $\tilde{Q}_k$ denote the cube centered at y_k but of three-times the size — i.e., $\tilde{Q}_k$ is the union of Q_k with the adjacent $3^d - 1$ cubes of the same side. The bound now proceeds depending on whether $z \in \tilde{Q}_k$ or not.

For $z \notin \tilde{Q}_k$, the distance between z and any $y \in Q_k$ is proportional to the distance between z and y_k. Proposition 4.19 thus implies

$$\bigl|\mathcal{K}_\Lambda(z,y) - \mathcal{K}_\Lambda(z,y_k)\bigr| \le C\,\frac{\operatorname{diam}(Q_k)}{|z-y_k|^{d+1}}, \qquad z \notin \tilde{Q}_k. \tag{4.106}$$

Moreover, thanks to (4.100),

$$\sum_{y\in Q_k} |b(y)| \le \sum_{y\in Q_k} \bigl(|f(y)| + |g(y)|\bigr) \le 2 \sum_{y\in Q_k} |f(y)|. \tag{4.107}$$

Using these in (4.105) yields

$$|\mathcal{K}_\Lambda b_k(z)| \le C\,\frac{\operatorname{diam}(Q_k)}{|z-y_k|^{d+1}} \sum_{y\in Q_k} |f(y)|. \tag{4.108}$$

Summing over all $z \notin \tilde{Q}_k$ and taking into account that $|z - y_k| \ge \operatorname{diam}(Q_k)$ for $z \in \tilde{Q}_k$, we conclude

$$\begin{aligned}\sum_{z\in\Lambda\setminus\tilde{Q}_k} |\mathcal{K}_\Lambda b_k(z)| &\le C\operatorname{diam}(Q_k) \sum_{y\in Q_k} |f(y)| \sum_{z\colon |z-y_k|\ge\operatorname{diam}(Q_k)} \frac{1}{|z-y_k|^{d+1}} \\ &\le \tilde{C} \sum_{y\in Q_k} |f(y)|\end{aligned} \tag{4.109}$$

for some constant $\tilde{C}$. Setting $\tilde{B} := \bigcup_k \tilde{Q}_k$ and summing over k, we obtain

$$\sum_{z\in\Lambda\setminus\tilde{B}} |\mathcal{K}_\Lambda b(z)| \le \tilde{C} \sum_{y\in B} |f(y)| \le \tilde{C}\|f\|_1, \tag{4.110}$$

which by an application of Chebychev's inequality yields

$$\bigl|\{z\in\Lambda\setminus\tilde B\colon |\mathcal{K}_\Lambda b(z)|\ge\alpha\}\bigr|\le\frac{\tilde C\|f\|_1}{\alpha}. \tag{4.111}$$

i.e., a bound of the desired form.

To finish the proof, we still need to take care of $z\in\tilde B$. Here we get (and this is the only step where we are forced to settle on *weak*-type estimates),

$$\begin{aligned}\bigl|\{z\in\tilde B\colon |\mathcal{K}_\Lambda b(z)|\ge\alpha\}\bigr|&\le|\tilde B|\le 3^d\sum_k|Q_k|\\&\le 3^d\sum_k\frac{1}{\alpha}\sum_{z\in Q_k}\bigl|f(z)\bigr|\le\frac{3^d\|f\|_1}{\alpha}.\end{aligned} \tag{4.112}$$

The bound (4.90) then follows by combining (4.104), (4.111) and (4.112). □

4.3.5 Triple gradient of finite-volume Green's function

In order to finish the proof of Theorem 4.15, we still need to establish the decay estimate in Proposition 4.19. This will be done by invoking a corresponding bound in the full lattice and reducing it onto a box by reflection arguments. (This is the sole reason why we restrict to rectangular boxes; more general domains require considerably more sophisticated methods.)

For $\varepsilon>0$, let G^ε denote the Green function associated with the discrete Laplacian Δ on $\mathbb{Z}^d$ with killing rate $\varepsilon>0$, i.e., $G^\varepsilon(\cdot,\cdot)$ is the kernel of the bounded operator $(\varepsilon-\Delta)^{-1}$ on $\ell^2(\mathbb{Z}^d)$. This function admits the probabilistic representation

$$G^\varepsilon(x,y)=\sum_{k=0}^\infty\frac{P^x\bigl(X_k=y\bigr)}{(1+\varepsilon)^{k+1}}, \tag{4.113}$$

where X is the simple random walk and P^x is the law of X started at x. This function depends only on the difference of its arguments, so we will interchangeably write $G^\varepsilon(x,y)=G^\varepsilon(x-y)$. We now claim:

Lemma 4.20. *There exists $\widehat C>0$ such that, for all $\varepsilon>0$, all $i,j,k\in\{1,\dots d\}$ and all $x\neq 0$,*

$$\bigl|\nabla_i\nabla_j\nabla_k G^\varepsilon(x)\bigr|\le\frac{\widehat C}{|x|^{d+1}}. \tag{4.114}$$

Sketch of proof. This is a mere extension (by adding one more gradient) of the estimates from in Lawler [Law91, Theorem 1.5.5]. (Strictly speaking, this theorem is only for the transient dimensions but, thanks to $\varepsilon > 0$, the same proofs would apply here.) The main idea is to use translation invariance of the simple random walk to write $G^{\epsilon}(x)$ as a Fourier integral and then control the gradients thereof under the integral sign. We leave the details as an exercise to the reader. □

We now state and prove a stronger form of Proposition 4.19.

Lemma 4.21. *There exists $C > 0$ such that, for all $L > 1$, $\varepsilon > 0$ and arbitrary $i, j, k \in \{1, \dots d\}$,*

$$|\nabla_i^{(2)} \nabla_j^{(1)} \nabla_k^{(2)} G_\Lambda^\varepsilon(x, y)| \leq \frac{C}{|x - y|^{d+1}} \tag{4.115}$$

for all $x, y \in \Lambda$ and all $i, j, k \in \{1, \dots, d\}$. Here, the superscripts on the operators indicate the variable the operator is acting on.

Proof. Throughout, we fix $L \in \mathbb{N}$ and denote $\Lambda := \Lambda_L$. The proof is based on the Reflection Principle for the simple random walk on $\mathbb{Z}^d$. To start, denote

$$\begin{aligned} \Lambda_0 &:= \Lambda_L = \{0, \dots, L\}^d, \\ \Lambda_i &:= \mathbb{Z}^i \times \{0, \dots, L\}^{d-i}, \qquad i = 1, \dots, d-1, \\ \Lambda_d &:= \mathbb{Z}^d, \end{aligned} \tag{4.116}$$

(abusing our earlier notation), write $X^{(i)}$ for the i-th component of X and let

$$\tau_0^i := \inf\{k \geq 0 \colon X_k^{(i)} = 0\}, \qquad \tau_L^i := \inf\{k \geq 0 \colon X_k^{(i)} = L\}.$$

For $y \in \Lambda_i$ with components $y = (y_1, \dots, y_d)$, and integer-valued indices $n \in \mathbb{Z}$, put

$$\begin{aligned} r_{2n}^i(y) &:= (y_1, \dots, 2nL + y_i, \dots, y_d) \\ r_{2n+1}^i(y) &:= (y_1, \dots, 2(n+1)L - y_i, \dots, y_d). \end{aligned}$$

Our first claim is that, for $i \in \{1, \dots, d\}$,

$$P^x\big(X_k = y, \tau_0^i > k, \tau_L^i > k\big) = \sum_{n \in \mathbb{Z}} (-1)^n P^x\big(X_k = r_n^i(y)\big). \tag{4.117}$$

In order to show (4.117), fix $i \in \{1, \dots, d\}$ and $x, y \in \Lambda_i$ and let A_m^k for $k, m \in \mathbb{N}$ denote the set of paths of length k starting in x and ending in $r_n^i(y)$ (for some $n \in \mathbb{Z}$) that visit the set $\{x_i = L\mathbb{Z}\}$ exactly m times. Moreover, for a path p, let $s(p) := 0$ if the path p ends in an even vertex (that is, $r^i_{2n}(y)$ for some n) and $s(p) := 1$ if it ends in an odd vertex. We note that, for $m > 0$,

$$\sum_{p \in A^k_m} (-1)^{s(p)} = 0. \tag{4.118}$$

To see this, we consider the mapping from A^k_m onto itself defined by taking a path and reflecting the segment between the last visit to $L\mathbb{Z}$ and the endpoint around the point where it last visited $L\mathbb{Z}$. This is obviously a bijection from A^k_m onto itself which changes the sign of $(-1)^{s(p)}$. It follows that the sum must vanish. As all paths in A^k_m have the same probability, we may in each summand in (4.118) multiply the probability of each respective path and obtain

$$\begin{aligned} 0 = \sum_{p \in A^k_m} (-1)^{s(p)} P^x(X_{0,\dots,k} = p) &= \sum_{p \in A^k_m,\, n \in \mathbb{Z}} (-1)^{s(p)} P^x(X_{0,\dots,k} = p, X_k = r^i_n(y)) \\ &= \sum_{p \in A^k_m,\, n \in \mathbb{Z}} (-1)^n P^x(X_{0,\dots,k} = p, X_k = r^i_n(y)) \\ &= \sum_{n \in \mathbb{Z}} (-1)^n P^x(X_{0,\dots,k} \in A^k_m, X_k = r^i_n(y)) \qquad \text{for all } m \geq 0 \end{aligned} \tag{4.119}$$

with $X_{0,\dots,k}$ denoting the path of the random walk up to time k. We now verify (4.117) by

$$\begin{aligned} P^x\big(X_k = y, \tau^i_0 > k, \tau^i_L > k\big) &= P^x\big(X_{0,\dots,k} \in A^k_0\big) \\ &= \sum_{n \in \mathbb{Z}} (-1)^n P^x\big(X_{0,\dots,k} \in A^k_0, X_k = r^i_n(y)\big) \\ &\overset{(4.119)}{=} \sum_{m \geq 0,\, n \in \mathbb{Z}} (-1)^n P^x\big(X_{0,\dots,k} \in A^k_m, X_k = r^i_n(y)\big) \\ &= \sum_{n \in \mathbb{Z}} (-1)^n P^x\big(X_k = r^i_n(y)\big). \end{aligned}$$

This obviously holds regardless of any restriction of the other components of the walk, which means that we have in particular

$$P^x\big(X_k = y,\, \tau^j_0 > k, \tau^j_L > k \quad \forall j > i\big) = \sum_{n \in \mathbb{Z}} (-1)^n P^x\big(X_k = r^{i+1}_n(y),\, \tau^j_0 > k, \tau^j_L > k \quad \forall j > i{+}1\big) \tag{4.120}$$

for each $i \in \{0, \ldots, d-1\}$. Let us now establish the desired representation for the Green function. For any $i \in \{0, \ldots, d\}$, the Green function $G^{\varepsilon}_{\Lambda_i}$ on Λ_i with zero boundary condition is given by

$$G^{\varepsilon}_{\Lambda_i}(x,y) = \sum_{k=0}^{\infty} (1+\epsilon)^{-k-1} P^x\big(X_k = y,\, \tau_0^j > k, \tau_L^j > k \quad \forall j > i\big). \tag{4.121}$$

Applying (4.120) to every probability term, we obtain for each $i \in \{0, \ldots, d-1\}$

$$G^{\varepsilon}_{\Lambda_i}(x,y) = \sum_{n \in \mathbb{Z}} (-1)^n G^{\varepsilon}_{\Lambda_{i+1}}(x, r_n^{i+1}(y)). \tag{4.122}$$

Consecutive application of this equality gives

$$G^{\varepsilon}_{\Lambda}(x,y) = \sum_{z \in \mathbb{Z}^d} (-1)^{z_1 + \ldots + z_d} G^{\varepsilon}_{\mathbb{Z}^d}(x, r_z(y)) \tag{4.123}$$

for all $x, y \in \Lambda$, where we abbreviate $r_z = r^1_{z_1} \circ \cdots \circ r^d_{z_d}$. From Lemma 4.20, we thus obtain

$$\big|\nabla_i^{(2)} \nabla_j^{(1)} \nabla_k^{(2)} G^{\varepsilon}_{\Lambda}(x,y)\big| \le \sum_{z \in \mathbb{Z}^d} \big|\nabla_i^{\star} \nabla_j \nabla_k^{\star} G^{\varepsilon}(x - r_z(y))\big| \le \sum_{z \in \mathbb{Z}^d} \frac{\widehat{C}}{|\, x - r_z(y)|^{d+1}} \tag{4.124}$$

for all $x, y \in \Lambda$. Now we are ready to conclude the argument. Let $x, y \in \Lambda$ and abbreviate

$$z_{\max} = \max_{i=1}^{d} |z_i|.$$

Whenever $z_{\max} \le 1$, we have $|\, x - r_z(y)| \ge |\, x - y|$ as reflection always increases the distance between points in Λ. If $z_{\max} > 1$, we may even estimate $|\, x - r_z(y)| \ge d^{-1/2} L |\, z| \ge d^{-1} |\, x - y||\, z|$. The latter is verified quickly using $d^{1/2} z_{\max} \ge |z| \ge z_{\max}$ and the fact that $z_{\max}$ is at least 2 in this case. Therefore, we obtain

$$\begin{aligned} \big|\nabla_i^{(2)} \nabla_j^{(1)} \nabla_k^{(2)} G^{\varepsilon}_{\Lambda}(x,y)\big| &\le \sum_{z\,:\, z_{\max} \le 1} \frac{\widehat{C}}{|\, x-y|^{d+1}} + \sum_{z\,:\, z_{\max} > 1} \frac{d^{d+1} \widehat{C}}{|\, x-y|^{d+1} |\, z|^{d+1}} \\ &\le \frac{\widehat{C}}{|\, x-y|^{d+1}} \Big(3^d + d^{d+1} \sum_{z \ne 0} \frac{1}{|\, z|^{d+1}} \Big), \end{aligned} \tag{4.125}$$

which is the desired estimate. □

We are now ready to complete the proof of Theorem 4.15:

Proof of Proposition 4.19. Although the $\varepsilon \downarrow 0$ limit of G^ϵ exists only in $d \geq 3$, for gradients we have $\nabla G(x,y) = \lim_{\varepsilon\downarrow 0} \nabla G^\epsilon(x,y)$ in all $d \geq 1$. Since the bound in Lemma 4.21 holds uniformly in $\varepsilon > 0$, we get the claim in all $d \geq 1$. □

4.4 Perturbed harmonic coordinate

In this section we will prove Propositions 4.5 and 4.6. Abandoning our earlier notation, let

$$G_\Lambda(x,y;\omega) = (-\mathcal{L}_\omega)^{-1}(x,y) \tag{4.126}$$

denote the Green function in Λ with Dirichlet boundary condition for conductance configuration ω. (Thus, the simple-random walk Green function from Section 4.3 corresponds to $\omega := 1$.) The Green function is the fundamental solution to the Poisson equation, i.e.,

$$\begin{cases} -\mathcal{L}_\omega G_\Lambda(x,z,\omega) = \delta_x(z) & \text{if } z \in \Lambda, \\ G_\Lambda(x,z,\omega) = 0, & \text{if } z \in \partial\Lambda, \end{cases} \tag{4.127}$$

where $\delta_x(z)$ is the Kronecker delta. Note that G_Λ is defined for all $\omega \in \Omega$. The solution to (4.127) is naturally symmetric,

$$G_\Lambda(x,y;\omega) = G_\Lambda(y,x;\omega), \qquad x,y \in \Lambda, \tag{4.128}$$

and so we can extend it to a function on $\Lambda \cup \partial\Lambda$ by setting $G_\Lambda(x,\cdot;\omega) = 0$ whenever $x \in \partial\Lambda$. Here is a generalized form of the representation (4.23):

Lemma 4.22 (Rank-one perturbation)**.** *For a finite $\Lambda \subset \mathbb{Z}^d$ let $x, y \in \Lambda$ be nearest neighbors. For any ω, ω' such that $\omega'_b = \omega_b$ except at $b := \langle x,y\rangle$, and any $z \in \Lambda \cup \partial\Lambda$,*

$$\begin{aligned} &\Psi_\Lambda(\omega',z) - \Psi_\Lambda(\omega,z) \\ &\quad = -(\omega'_{xy} - \omega_{xy})\big[G_\Lambda(z,y;\omega') - G_\Lambda(z,x;\omega')\big]\big[\Psi_\Lambda(\omega,y) - \Psi_\Lambda(\omega,x)\big]. \end{aligned} \tag{4.129}$$

Proof. Suppose $\omega, \omega' \in \Omega$ are such that ω' equals ω except at the edge $b := \langle x,y\rangle$, where $\omega'_b := \omega_b + \epsilon$. Define the function $\Phi_\Lambda \colon \Lambda \cup \partial\Lambda \to \mathbb{R}^d$ by

$$\Phi_\Lambda(z) := \Psi_\Lambda(\omega,z) - \epsilon\big[G_\Lambda(z,y;\omega') - G_\Lambda(z,x;\omega')\big]\big[\Psi_\Lambda(\omega,y) - \Psi_\Lambda(\omega,x)\big]. \tag{4.130}$$

We claim that

$$\mathcal{L}_{\omega'}\Phi_\Lambda = 0 \qquad \text{in } \Lambda. \tag{4.131}$$

Since for $z \in \partial\Lambda$ we have $\Phi_\Lambda(z) = \Psi_\Lambda(\omega, z) = z$, this will imply $\Phi_\Lambda(\cdot) = \Psi_\Lambda(\omega', \cdot)$ thanks to the uniqueness of the solution of the Dirichlet problem.

In order to show (4.131), we first use (4.127–4.128) to get

$$\mathcal{L}_{\omega'}\Phi_\Lambda(z) = \mathcal{L}_{\omega'}\Psi_\Lambda(\omega, z) - \epsilon\big[\delta_y(z) - \delta_x(z)\big]\big[\Psi_\Lambda(\omega, y) - \Psi_\Lambda(\omega, x)\big]. \tag{4.132}$$

To deal with the term $\mathcal{L}_{\omega'}\Psi_\Lambda(\omega, z)$, we think of of $\mathcal{L}_{\omega'}$ as a matrix of dimension $|\Lambda|$. For its coefficients $\mathcal{L}_\omega(z, z') := \langle\delta_z, \mathcal{L}_\omega\delta_{z'}\rangle_{\ell^2(\Lambda)}$ we obtain

$$\mathcal{L}_{\omega'}(z, z') = \mathcal{L}_\omega(z, z') + \epsilon\big[\delta_y(z) - \delta_x(z)\big]\big[\delta_y(z') - \delta_x(z')\big]. \tag{4.133}$$

Using that $\mathcal{L}_\omega\Psi_\Lambda(\omega, z) = 0$ for $z \in \Lambda$, we now readily confirm (4.131). □

Proof of Proposition 4.5. Set $y := x + \hat{e}_i$ and denote $\nabla_i f(z) := f(z + \hat{e}_i) - f(z)$. Lemma 4.22 shows

$$\nabla_i\Psi_\Lambda(\omega', x) = \Big[1 - (\omega_b' - \omega_b)\nabla_i^{(1)}\nabla_i^{(2)}G_\Lambda(x, x, \omega')\Big]\nabla_i\Psi_\Lambda(\omega, x), \tag{4.134}$$

where the superindices on ∇ indicate which variable is the operator acting on. To prove the claim we need to show

$$\big[\nabla_i^{(1)}\nabla_i^{(2)}G_\Lambda(x, x, \omega)\big]^{-1} = \inf\big\{Q_\Lambda(f)\colon f(y) - f(x) = 1,\ f_{\partial\Lambda} = 0\big\}, \tag{4.135}$$

where the conductances in Q_Λ correspond to ω. For this, let f be the minimizer of the right-hand side. The method of Largrange multipliers shows

$$-\mathcal{L}_\omega f(z) = \alpha\big[\delta_y(z) - \delta_x(z)\big]. \tag{4.136}$$

Thanks to (4.127), this is solved by

$$f(z) = \alpha\big[G_\Lambda(y, z; \omega) - G_\Lambda(x, z; \omega)\big] = \alpha\nabla_i^{(1)}G_\Lambda(x, z; \omega) \tag{4.137}$$

which in light of the constraint $f(y) - f(x) = 1$ gives $\alpha = [\nabla_i^{(1)}\nabla_i^{(2)}G_\Lambda(x, x, \omega)]^{-1}$. Since also $Q_\Lambda(f) = \langle f, -\mathcal{L}_\omega f\rangle_{\ell^2(\Lambda)}$, (4.136) gives $Q_\Lambda(f) = \alpha$ and so (4.135) holds. The correspondence (4.23) then follows from (4.134–4.135); the identity (4.24) results by differentiation of the left-hand side with respect to ω_b'. □

Finally, it remains to establish the limit (4.25), including all of its stated properties:

Proof of Proposition 4.6. Thanks to ellipticity restriction (1.1), we have a bound on this quantity in terms of the lattice Laplacian. This shows that, for some $c = c(\lambda) \in (0, 1)$,

$$c < \nabla_i^{(1)} \nabla_i^{(2)} G_\Lambda(x, x, \omega') < 1/c \tag{4.138}$$

uniformly in Λ. Moreover, $\Lambda \mapsto \nabla_i^{(1)} \nabla_i^{(2)} G_\Lambda(x, x, \omega')$ is obviously non-decreasing in Λ and so the limit exists. The formula (4.26) and the claimed stationarity then follow as well. □

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