On the speed of Random Walks among Random Conductances (in 5 minutes!)

Interaction between Analysis and Probability in Physics, Oberwolfach

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1. First minute
   - The model
2. Second minute
   - The speed problem
3. Third & fourth minute
   - A log-moments issue
4. Fifth minute
   - Can you do better?
5. Extra 10 seconds
   - A nice picture

The model

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Lattice $\mathbb{Z}^d$, assign to any bond (x, y) a random weight ω_{xy} (conductance) s.t.

- $\omega_{xy} = \omega_{yx}$ (symmetry),
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The discrete-time *Random Walk among Random Conductances* (RWRC) $(X_n)_{n\in\mathbb{N}}$ has probability transitions

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It is is reversible (!) w.r.t.

$$\pi(x) = \sum_{z \sim x} \omega_{xz}.$$

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Two well known cases:

- i.i.d. conductances $\implies \mathbb{P}\big(\lim_{n\to\infty} \frac{X_n}{n} = 0\big) = 1$ (*point of view of the particle*);
- bounded conductances $\implies \mathbb{P}\big(\lim_{n\to\infty} \frac{X_n}{n} = 0\big) = 1$ (e.g. *Rayleigh principle*).

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Proposition

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What if the log-moments are finite only for $\alpha < 1$?

(Beautiful picture at the blackboard)

Note that

$$\mathbb{E}[\log^{\alpha}\omega_{xy}] \simeq \sum_{k=1}^{\infty} Pr(h(y) > k)\, k^{A\alpha-1}, \quad 1 < A < \frac{1}{\alpha}$$

where $h(y)$ is the distance from the farthest leaf in the branch of y.
In the example

$$\mathbb{P}(h(y) \geq k) \geq \frac{C}{k^{1/2}},$$

so we don't have finite moments for $\alpha > 1/2$.

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Can we push α as close as we want to 1?

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What is the speed of the RWRC on such a tree?

Exercise

Find such a tree! (hint: umbrellas)

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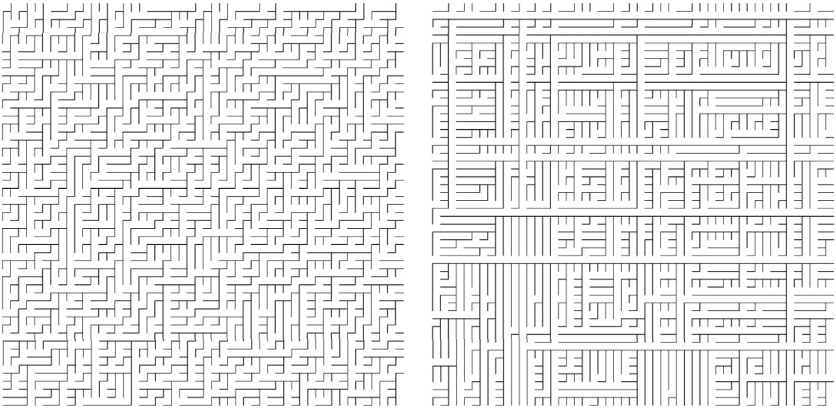

(picture stolen from *Shortest spanning trees and a counterexample for Random Walks in Random Environments*, by M. Bramson, O. Zeitouni and M. Zerner (2006))