The spectrum of dense kernel-based random graphs

joint work with
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July 15th, 2025

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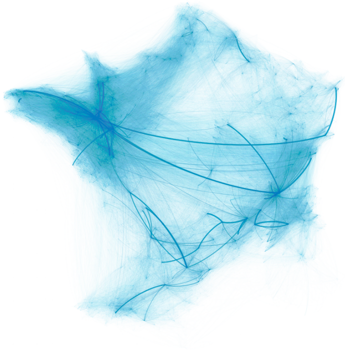
RoMaDS

Complex networks

Complex networks



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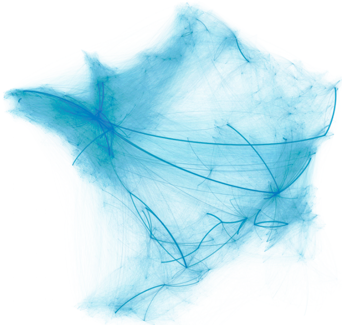

Scale-free

Law of degrees decays polynomially:

$$\mathbb{P}(D_x \geq t) \simeq t^{-\gamma}$$

Small world

Graph dist $\simeq$ log(Euclidean dist)

Positive clustering coefficient

Probability that two of my friends are friends is high.

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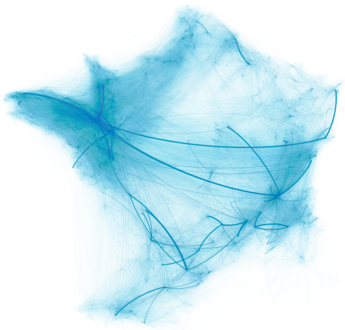

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Inhomogeneous spatial random graphs!

- Inhomogeneous –> Nodes have importances
- Spatial –> Distances influence connection probability

Kernel-based random graphs (KBRG)

[Gracar & al.'22, Jorritsma & al.'23]

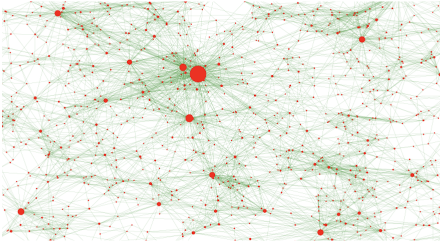
Random graph $G_N = G_N(\tau, \alpha, \sigma)$ with vertex set $V_N = \{1, 2, \ldots, N\}^d$

- Sample i.i.d. weights $(W_x)_{x \in V_N}$:

$$\mathbb{P}(W_x > t) = t^{-(\tau-1)}$$

- $$\mathbb{P}(x \leftrightarrow y \mid W_x, W_y) = \frac{\kappa_\sigma(W_x, W_y)}{\|x-y\|^\alpha}$$

  with $\kappa_\sigma(w, v) = (w \vee v)(w \wedge v)^\sigma$



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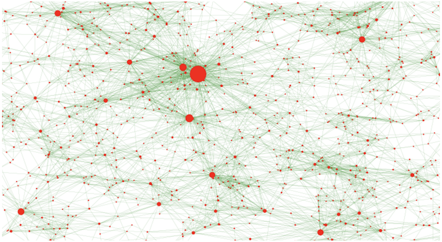

Examples:

- $\kappa_{\mathrm{SFP}} = wv$ ($\sigma = 1$, Scale-free percolation)
- $\kappa_{\mathrm{PA}} = (w \vee v)(w \wedge v)^{\frac{\alpha(\tau-1)}{d}}$ (Pref. Attachment type)
- $\kappa_{\mathrm{strong}} = w \vee v$ ($\sigma = 0$)
- $\kappa_{\mathrm{trivial}} \equiv 1$ (Long-range percolation)

Questions

Degrees, distances, percolation...

[Deijfen+'13,Jacob+'15,Heydenreich+'17,Bringmann+'19, Deprez+'19,Gracar+'19,Dalmau,S.'21,Hao+'23,Jorritsma+'24...]

Dynamics

RW[Heydenreich+'17,Gracar+'22],**ContactProcess**[Linker+'21, Gracar+'24,Barnier,S.+'25],**FPP**[Komjáthy+'20,'23], **ParticleSystems**[Bansaye,S.'24], ...

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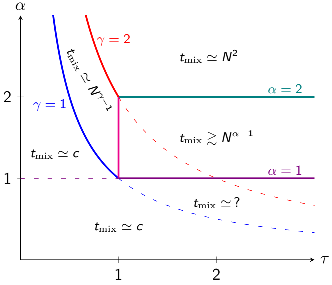
Mixing time [Kiwi+'18,Cipriani,S.'24]



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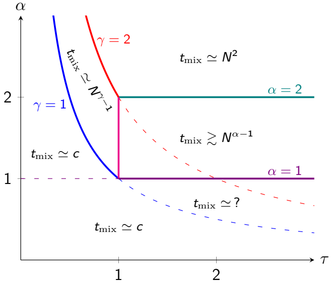

What about the whole spectrum?

Inhomogeneous ER [Avena+'23,Bose+'22], **Eigenvalue statistics** [Ajanki+'19, Ducatez+'24,Zhu+'24], **Quantum percolation** [Anantharaman+'21,Bordenave+'11]...

Empirical spectral distribution (ESD)

Let $d = 1$. For $x, y \in \{1, \dots N\}$ recall $\mathbb{P}(x \leftrightarrow y \mid W_x, W_y) = \frac{\kappa_\sigma(W_x, W_y)}{\|x-y\|^\alpha}$.

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Theorem (Wigner for GOE)

If $\mathcal{Z}_{ij} \sim \mathcal{N}(0,1)$, $\mathcal{Z}_{ii} \sim \mathcal{N}(0,\,2)$ are indep. (standard) Gaussians and

$$Z_N = (\mathcal{Z}_{i \wedge j,\, i \vee j})_{1 \le i \le j \le N}\,.$$

Then,weakly in probability,

$$\lim_{N \to \infty} ESD\left(\frac{Z_N}{\sqrt{N}}\right) = \mu_{sc}\,.$$

Scaling

What is the right scaling?

- Rule of thumb: rescale by $\sqrt{c_N}$ such that

$$\mathbb{E}\left[\mathrm{tr}\left(\left(\frac{A_N}{\sqrt{c_N}}\right)^2\right)\right] \overset{!}{=} 1$$

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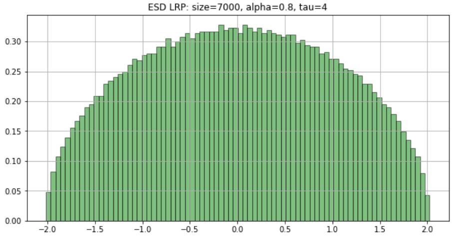
- $\alpha < 1\,(=d)$: dense graph

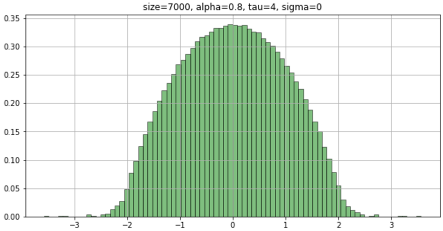
Goal: study (weakly in probability)

$$\lim_{N\to\infty} ESD\left(\frac{A_N}{\sqrt{c_N}}\right) = ???$$

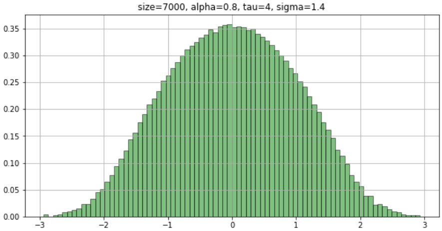
Simulations



Trivial kernel $\kappa \equiv 1$
(convergence to $\mu_{\rm sc}$ in [Ayadi, Slim '09])



Strong kernel $\kappa(u, v) = u \vee v$



PA $\kappa(u, v) = (w \vee v)(w \wedge v)^{\alpha(\tau-1)}$

Main result

Theorem (Cipriani, Hazra, Malhotra, S. '25)

Let $\alpha < 1 (= d)$, $\tau > 2$ *and* $\sigma < \tau - 1$. *Then...*

(i) ***Existence***

There exists a symmetric deterministic measure $\mu_{\sigma,\tau}$ *on* $\mathbb{R}$ *such that*

$$\lim_{N\to\infty} ESD\left(\frac{A_N}{\sqrt{c_N}}\right) = \mu_{\sigma,\tau}$$

$\mu_{\sigma,\tau}$ *does not depend on* α *!!*

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$\mu_{\sigma,\tau}$ does not depend on α !!

(ii) ***Moments***

$\mu_{\sigma,\tau}$ is heavy tailed with 2p-finite moments for all $p < \frac{\tau-1}{\sigma\vee 1}$. In particular

$$\int_{\mathbb{R}} x^2 \mu_{\sigma,\tau}(\mathrm{d}x) = (\tau-1)^2 \int_1^\infty \int_1^\infty \frac{1}{(x\wedge y)^{\tau-\sigma}(x\vee y)^{\tau-1}} \mathrm{d}x\mathrm{d}y \in (0,\infty).$$

...even if weights have infinite variance!

Main result

Theorem (Cipriani, Hazra, Malhotra, S. '25)

(iii) ***Absolute continuity***

$\mu_{\sigma,\tau}$ *is absolutely continuous with respect to Lebesgue.*

See continuum quantum percolation...

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(iv) ***Stieltjes transform***

For $\tau > 3$ and $\sigma < \tau - 2$, the limiting measure $\mu_{\sigma,\tau}$ can be characterized via its Stieltjes transform: $\exists!\, a : \mathbb{C}^+ \times [1,\infty)$ analytic s.t.

$$S_{\mu_{\sigma,\tau}}(z) = \int_{\mathbb{R}} \frac{\mu_{\sigma,\tau}(\mathrm{d}x)}{x - z} = \int_1^\infty a(x,z)\mu_W(\mathrm{d}x)\,.$$

$a(\cdot,\cdot)$ is the unique solution of a fixed point equation in a suitable Banach space.

Scale-Free Percolation

Theorem (Cipriani, Hazra, Malhotra, S. '25)

For $\sigma = 1$ *(Scale-Free Percolation model) we have*

$$\lim_{N\to\infty} ESD\left(\frac{A_N}{\sqrt{c_N}}\right) = \mu_{1,\tau} = \mu_{\rm sc} \boxtimes \mu_W$$

where $\boxtimes$ *is the free multiplicative convolution. In particular*

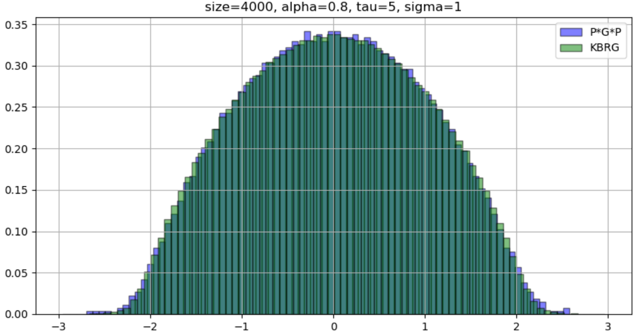
$$\mu_{1,\tau}(x,\infty) \sim \tfrac{1}{2}\left(E[W]\right)^{\tau-1} x^{-2(\tau-1)} \text{ as } x \to \infty.$$

Free multiplicative convolution [edit]

Let μ and ν be two probability measures on the interval $[0,+\infty)$, and assume that X is a random variable in a non commutative probability space with law μ and Y is a random variable in the same non commutative probability space with law ν. Assume finally that X and Y are freely independent. Then the **free multiplicative convolution** $\mu \boxtimes \nu$ is the law of $X^{1/2}YX^{1/2}$ (or, equivalently, the law of $Y^{1/2}XY^{1/2}$. Random matrices interpretation: if A and B are some independent n by n non negative Hermitian (resp. real symmetric) random matrices such that at least one of them is invariant, in law, under conjugation by any unitary (resp. orthogonal) matrix and such that the empirical spectral measures of A and B tend respectively to μ and ν as n tends to infinity, then the empirical spectral measure of AB tends to $\mu \boxtimes \nu$.[7]

To keep in mind:

$$Law(\sqrt{W_N} Z_N \sqrt{W_N}) \approx \mu_{sc} \boxtimes \mu_W$$



Scale-free percolation $\kappa(u,v) = uv$ VS "$\mu_{sc} \boxtimes \mu_W$"

Sketch of the proof

Limiting measure is heavy-tailed so method of moments does not work!

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1. **Truncation**
   Create new adjacency matrix $A_{N,m}$ by truncating the weights:
   $$W_x^m := W_x \mathbb{1}_{\{W_x \le m\}} \,.$$
2. **Gaussianization**
   Further replace entries $A_{N,m}$ with centered Gaussians [Chatterjee '05]
   $$A_{N,m,gauss}(i,j) \sim \mathcal{N}\left(0, \frac{\kappa_\sigma(W_i^m, W_j^m)}{\|i-j\|^\alpha}\right)$$
3. **Identification of the limit**
   Moments (Wick's formula) + concentration $\Longrightarrow$ exists limiting $\mu_{\sigma,\tau,m}$
4. **Send $m \to \infty$**

$$ESD\left(\frac{A_N}{\sqrt{c_N}}\right) \approx ESD\left(\frac{A_{N,m}}{\sqrt{c_N}}\right) \approx ESD\left(\frac{A_{N,m,gauss}}{\sqrt{c_N}}\right) \xrightarrow{N\to\infty} \mu_{\sigma,\tau,m} \approx \mu_{\sigma,\tau}$$

Open questions

- Tails for $\sigma \neq 1$?
- Other random matrices or observables associated to KBRGs? Laplacian see [Hazra, Malhotra '25]
- Other regimes in the dense case?
- Sparse case?

Thank you!