Epidemics on inhomogeneous spatial random graphs

Joint work with Vincent Bansaye (École Polytechnique)

Spatial Epidemic Models (Including Graphs and Graphons), Paris, France

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Complex networks

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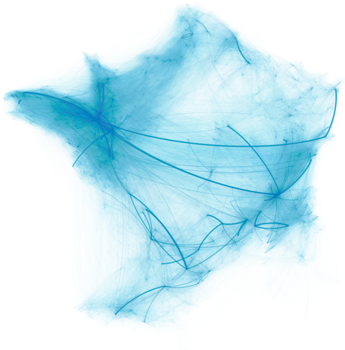

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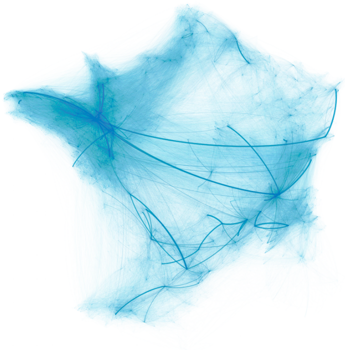

Scale-free

Law of degrees decays polynomially:

$$\mathbb{P}(D_x \geq t) \simeq t^{-\gamma}$$

Small world

Graph dist $\simeq$ log(Euclidean dist)

Positive clustering coefficient

Probability that two of my friends are friends is high.

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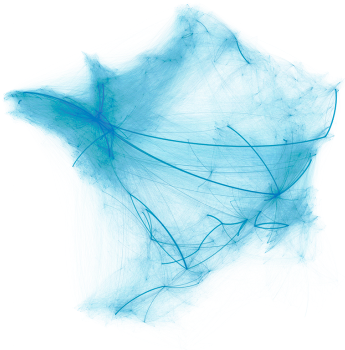

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Norros-Reittu, Chung-Lu	✓	✓	✗
Watts-Strogatz	✗	✓	✓

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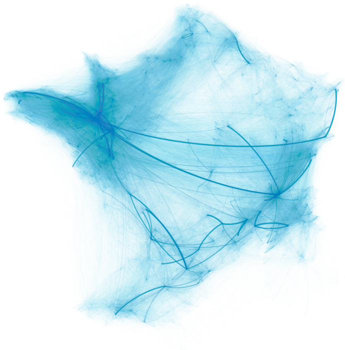

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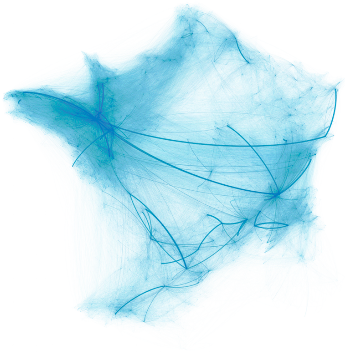

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- Go beyond mean field, regular lattices, explicit large structures...
- Combine motion + demography (metapopulation)

The model

Random graph $G = (V, E)$
$G = G(\omega)$ with law $\mathbb{P}$:

- $V = PPP(\mathbb{R}^n)$
- $E \ni \{x, y\}$ with $\mathbb{P}(x \leftrightarrow y)$

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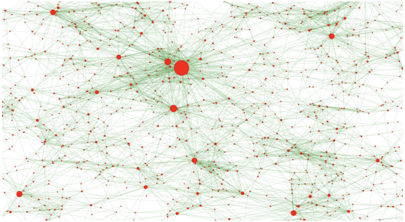

Examples:
- $\mathbb{P}(x \leftrightarrow y) = 1$ (complete graph)
- $\mathbb{P}(x \leftrightarrow y) = \|x - y\|^{-\alpha}$ (long-range percolation)
- $\mathbb{P}(x \leftrightarrow y) \simeq W_x W_y \|x - y\|^{-\alpha}$ (scale-free percolation)

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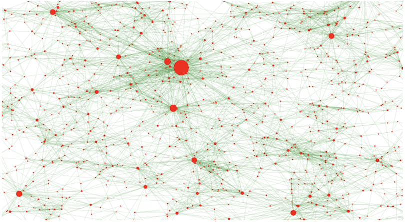

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Dynamics $(\eta_t(\cdot))_{t \geq 0}$.
Fix $G(\omega)$. Under P^ω independent particles that

- Jump from x to y at rate $r(x, y) = \mathrm{e}^{-\|x-y\|} \mathbb{1}_{\{x \leftrightarrow y\}}$
- Give birth to new particle at rate b on site.
- Die at rate d.

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- OPTION 2

  View model as first stages of SIR epidemics in a large population: b is then infection rate, d is the recovery rate.

Note: Approximation valid on a time window where infected population remains locally small compared to population size.

Existence

Theorem (with V. Bansaye)

For $\mathbb{P}$-a.a. realizations $G(\omega)$, suppose $E^\omega[\eta_0(x)] \leq M$, $\forall x \in V(\omega)$. Then

$$(\eta_t)_{t\in[0,T]} \text{ is well defined.}$$

For functions $f_H(\eta) = \sum_{x\in V} H(x)\eta(x)$ the "generator" of the process "is"

$$\mathcal{L} f_H(\eta) = \sum_{x,y\in V} \eta(x) r(x,y) \big(H(y) - H(x)\big) + \sum_{x\in V} \eta(x) \big(b - d\big) H(x) \, .$$

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- Holds for any choice of $\mathbb{P}(x \leftrightarrow y)$!
- $b = d = 0$ independent random walks, easy.
- Available techniques (Liggett, Andjel, Ganguly-Ramanan) don't apply:
  - Jump rates are not bounded
  - Restriction on initial conditions

Sketch of proof

Fix a compact $K \subset \mathbb{R}^d$.

Aim: only finite number of events in K up to T.

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STEP 1: finite subgraph

Existence when suppressing jumps out of a finite subgraph B + show

$$E^\omega[\mathit{ghosts}(K) + \mathit{particles}(K) \text{ at } T] \le C_K(\omega) M \mathrm{e}^{bT} \quad \text{(uniform in } B\text{!)}$$

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Extend to whole $G(\omega)$ but finite initial particles: take increasing sequence of subgraphs B_N + range of particles stays finite a.s.

STEP 3: full graph, infinite initial conditions

Extend to infinite initial conditions by monotonicity and previous bound:

$$E^\omega[\mathit{ghosts}(K) + \mathit{particles}(K) \text{ at } T]$$

$$\overset{\text{(MON)}}{=} \lim_{N\to\infty} E^\omega[\mathit{ghosts}(K) + \mathit{particles}(K) \text{ at } T \text{ with } \eta_0^N] \leq C_K(\omega) M \mathrm{e}^{bT}.$$

Main result: hydrodynamic limits

- Sped-up process $(\eta_t^N)_{t\in[0,T]}$:

$$\mathcal{L}^N f_H(\eta) = \sum_{x\in V} \eta(x) L^N H(x/N) + \sum_{x\in V} \eta(x)\big(b-d\big) H(x/N)\,.$$

L^N generator of random walk on V/N with rates $N^2 r(\cdot,\cdot)$:

$$L^N H(x/N) = \sum_{y\in V} N^2 r(x,y)\big(H(y/N) - H(x/N)\big)\,.$$

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- Measure-valued process $(\pi_t^N)_{t\in[0,T]}$:

$$\pi_t^N = \pi^N(\eta_t^N) = N^{-n} \sum_{x\in V} \eta_t^N(x)\delta_{x/N} \in \mathcal{M}(\mathbb{R}^n)\,.$$

Theorem (with V. Bansaye)

ASSUMPTION 1: $\eta_0^N(x) \leq M + \textit{Poisson}$, *for each* $x \in V$.

ASSUMPTION 2: $\exists \rho_0 : \mathbb{R}^n \to [0,\infty)$ *s.t. for any* $H \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$N^{-n} \sum_{x \in V} \eta_0^N(x) H(x/N) \xrightarrow[P^\omega]{N\to\infty} \int_{\mathbb{R}^n} H(x)\rho_0(x)\,\mathrm{d}x\,.$$

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The sequence of processes $\{(\pi_t^N)_{t \in [0,T]}\}_{N \in \mathbb{N}}$ *converges in law in* $\mathcal{D}([0,T], \mathcal{M}(\mathbb{R}^n))$ *to the deterministic trajectory* $(\rho(t,u)\,\mathrm{d}u)_{t \in [0,T]}$, *where* $\rho(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is the unique weak solution of*

$$\begin{cases} \partial_t \rho = \sigma \Delta \rho + (b-d)\rho \\ \rho(0,\cdot) = \rho_0 \end{cases} .$$

Here

$$\sigma^2 := \frac{1}{2} \inf_{\psi \in B(\Omega)} \mathbb{E}_0 \Big[\sum_{y \in V} r(0,y) \big(y_1 + \psi(\theta_y \omega) - \psi(\omega) \big)^2 \Big] .$$

Strategy of the proof

STEP 1: Corrected empirical measure

$L^N H$ might be irregular (not in $L^2(\mu_N)$).

Idea: L^N is a discretization of $\sigma\Delta \Longrightarrow$ substitute H with H_N^λ:

$$(\lambda - L^N) H_N^\lambda = (\lambda - \sigma\Delta) H \,.$$

PIN 1: need to show that H_N^λ converges to H in L^1 and L^2.

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STEP 2: Tightness of $\{(\pi_t^N)_{t\in[0,T]}\}_{N\in\mathbb{N}}$

$$M_t^N = \langle \pi_t^N, H_N^\lambda \rangle - \langle \pi_0^N, H_N^\lambda \rangle - \int_0^t \langle \pi_s^N, L^N H_N^\lambda + (b-d) H_N^\lambda \rangle \mathrm{d}s$$

shown to be an L^2 martingale converging to 0. Tightness from Aldous' crit.
PIN 2: need control of $\sup_{t\in[0,T]} \langle \pi_t^N, f \rangle = \sup_{t\in[0,T]} N^{-n} \sum_{x\in V} \eta_t^N(x) f(x/N)$.

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STEP 3: Identification of the limit

Limit of (π_t^N) is unique and has density $\rho(t,u)$: $\quad \partial_t \rho = \sigma\Delta\rho + (b-d)\rho$.

Stochastic homogenization

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[Faggionato, 2023 & 2023+]: Take a point process and rates s.t. $\mathbb{P}$-a.s.

- A1 $\mathbb{P}$ is stationary and ergodic
- A2 Finite and positive intensity $0 < \mathbb{E}[\mu_\omega([0,1]^n)] < \infty$
- A3 $\theta_g\omega \neq \theta_{g'}\omega$
- A4 μ_ω is stationary and $r_{\theta_g\omega}(x,y) = r_\omega(\tau_g x, \tau_g y)$
- A5 $r(x,y) = r(y,x)$
- A6 The graph is connected
- A7 $\mathbb{E}_0[\sum_{x\in V} r(0,x)\|x\|^k] < \infty$ for $k = 0, 2$
- A8 $L^2(\mathbb{P}_0)$ is separable

Non-reversible Kipnis-Varadhan

PIN 2: Control $\sup_{t\in[0,T]} N^{-n}\sum_{x\in V}\eta_t^N(x)f(x/N)\,.$

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STEP 1: Case $b=d=0$. Start with reversible measure ($\otimes$Poisson)

$$P^\omega\Big(\sup_{t\in[0,T]} N^{-n}\sum_{x\in V}\tilde\eta_t^N(x)f(x/N)>A\Big)\le c\,\frac{|||f|||_N}{A}$$

with $|||f|||_N^2 := \|f\|^2_{L^1(\mu_N)} + \|f\|^2_{L^2(\mu_N)}N^{-n}\|L^N f\|_{L^2(\mu_n)}$.

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- STEP 2: Extension to non-reversible case.

$$P^\omega\Big(\sup_{t\in[0,T]} N^{-n} \sum_{x\in V} \eta_t^N(x) f(x/N) > A \Big) \le c_1 \mathrm{e}^{c_2 bT} \frac{|||f|||_N}{A}$$

Proof: For each initial particle look at single branch of the ancestral line, say of length ℓ. Dominate with indep. r.w.'s with percolated initial distribution: keep each particle with $\mathbb{P}$(particle has ℓ births up to time T). Use reversible K-V (1). Union bound.

☺ Results are true if A1,...,A8 are satisfied!

Can consider more general point processes, graphs, jumping rates...

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Future developments

- What happens when jumping rates decay slower?

- We only accelerated by N^2 the jumps.
What happens if consider faster births/deaths? (rapid stirring)

- Consider more realistic birth/death mechanism.
Example: For $d_x(\eta) = d + c\eta(x)$ should obtain in the limit

$$\partial_t \rho = \sigma \Delta \rho + (b - d - c\rho)\rho \,.$$

Or births/deaths that depend via a non-local kernel on the population size in a surrounding region.

Thank you!