FORMALITY OF THE FRAMED LITTLE 2-DISCS OPERAD AND SEMIDIRECT PRODUCTS

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Abstract. We prove that the operad of framed little 2-discs is formal. Tamarkin and Kontsevich each proved that the unframed 2-discs operad is formal. The unframed 2-discs is an operad in the category of S^1-spaces, and the framed 2-discs operad can be constructed from the unframed 2-discs by forming the operadic semidirect product with the circle group. The idea of our proof is to show that Kontsevich's chain of quasi-isomorphisms is compatible with the circle actions and so one can essentially take the operadic semidirect product with the homology of S^1 everywhere to obtain a chain of quasi-isomorphisms between the homology and the chains of the framed 2-discs.

1. Introduction

We begin by recalling two closely related operads. First, let D_2 denote the *little 2-discs operad* of Boardman and Vogt. In arity n it is the space of embeddings of the union of n discs into a standard disc, where each disc is embedded by a map which is a translation composed with a dilation. At the level of spaces, group complete algebras over this operad are 2-fold loop spaces, and at the level of homology an algebra over $H_*(D_2)$ is precisely a Gerstenhaber algebra.

A variant of the D_2 operad is the *framed little 2-discs* operad, denoted fD_2, introduced by Getzler [2]. Here the little discs are allowed to be embedded by a composition of a dilation, rotation, and translation. The (unframed) little 2-discs operad D_2 is an operad in the category of S^1-spaces, where the circle acts by conjugation. Markl and Salvatore-Wahl [9] presented the framed little 2-discs operad as the semidirect product of the circle group S^1 with D_2. In particular, $fD_2(n) = D_2(n) \times (S^1)^n$. Getzler observed that algebras over the homology operad $H_*(fD_2)$ are precisely Batalin-Vilkovisky algebras, and at the space level Salvatore-Wahl proved that a group complete algebra over fD_2 is a 2-fold loop space on a based space with a circle action.

The operad D_2 is homotopy equivalent to the Fulton-MacPherson operad $FM = FM_2$ [8] (we drop the subscript since we will only be discussing 2-discs in this note); the space $FM(n)$ is a compactification of the configuration space of n ordered distinct points in the plane modulo translations and positive dilations. As with D_2, the circle acts on FM by rotations. The semidirect product construction for this action gives the *framed Fulton-MacPherson operad* $fFM = fFM_2$, which is homotopy equivalent to fD_2, and such that $fFM(n) = FM(n) \times (S^1)^n$. Both FM and fFM are operads of semi-algebraic sets.

Tamarkin [11] and Kontsevich [6, 5] proved the following formality theorem.

Theorem 1.1. *The operad $C_*(FM)$ of chains on FM with real coefficients is quasi-isomorphic to its homology operad $H_*(FM)$, the Gerstenhaber operad.*

(One can also use singular or semi-algebraic chains in the statement; we will return to this point later.) Kontsevich's proof seems more geometric and has the advantage of extending to a proof of

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formality for the little k-discs for all $k \geq 2$; this proof has been explained in greater detail by Lambrechts and Volic [7].

In general, formality of an operad is a powerful property with many theoretical and computational applications. The above operad formality theorem plays an important role in Tamarkin's proof [12] of Kontsevich's deformation quantization theorem. Our purpose in writing this note is to show that Kontsevich's proof of formality of the operad FM can be adapted to show the formality of the operad fFM. Our main result is:

Theorem 1.2. *The operad $C_*(fFM)$ of chains on fFM with real coefficents is quasi-isomorphic to its homology operad $H_*(fFM) = BV$, the Batalin-Vilkovisky operad.*

An independent proof of this formality, built from Tamarkin's method rather than Kontsevich's, is due to Severa [10]. One interesting application of this formality result is given in [1], where it is used to construct homotopy BV algebra structures on objects such as the chains on double loop spaces.

Unline fD_2, the operad fFM in fact has the structure of a *cyclic operad*, and so it is natural to ask if the formality can be made compatible with the cyclic structure. After the present work was completed we found a proof [3] of the stronger result that fFM is formal as a cyclic operad, although that proof is significantly more involved and required the introduction of a new type of graph complex.

Kontsevich's proof showed formality of the little k-discs operad for *all* k, and so it is reasonable to ask if the framed k-discs operads are all formal as well (as operads, or better yet as cyclic operads). We plan to address this question in future work. The proofs given in this paper, [10], and [3] address only the case $k = 2$. These arguments do not work for $k > 2$ for various reasons. In the Tamarkin formality argument it is essential that the operad spaces are $K(\pi, 1)$s, and this is no longer true for $k > 2$. The argument in this paper does not immediately extend to higher k becasue one would have to replace the group $S^1 = SO(2)$ with $SO(k)$ and find a quasi-isomorphism $H^*SO(k) \to \Omega^*SO(k)$ that is compatible with Kontsevich's integration map; we do not know if this is possible. There are similar obstacles to adapting the argument in [3] to higher k.

1.1. **Outline of the proof.** First recall the outline of Kontsevich's proof of Theorem 1.1. It goes by constructing a certain DG-algebra $G(n)$ of graphs together with a quasi-isomorphism $I : G(n) \to \Omega^*(FM(n))$ to the DG-algebra of semi-algebraic forms, and a projection $G(n) \to H^*(G(n)) = H^*(FM(n))$ that is also a quasi-isomorphism. Both of these quasi-isomorphisms are essentially morphisms of DGA cooperads (this is not quite true — the subtleties here are discussed nicely in [7]). By dualizing one can obtain from this a chain of quasi-isomorphisms giving formality of FM.

What we show in this note is that Kontsevich's formality proof is in compatible in a precise sense with the circle action. The circle action on FM makes $H^*(FM)$ into a cooperad in $H := H^*(S^1)$-comodules. Kontsevich's DGA cooperad of admissible graphs G has a differential given by contracting edges; we define a degree -1 derivation Δ on $G(n)$ given by deleting edges. This derivation defines a H-comodule structure on $G(n)$; we check that that this comodule structure is compatible with the cooperad structure and that the projection $G(n) \to H^*(FM(n))$ is a morphism of H-comodules. Using the quasi-isomorphism $H^*(S^1) \to \Omega^*(S^1)$, this morally allows us to form a diagram of quasi-isomorphisms of semidirect product cooperads

$$\Omega^*(FM \rtimes S^1) \leftarrow G \rtimes H \to H^*(FM) \rtimes H \cong H^*(FM \rtimes S^1).$$

However, the proof is not quite so simple because the functor of semi-algebraic forms is contravariant monoidal and so $\Omega^*(FM \rtimes S^1)$ is not a cooperad on the nose. Nevertheless, this issue can be overcome easily, exactly as discussed in [7] in the unframed case.

2. A degree -1 derivation on admissible graphs

Consider the circle action $\rho_n : S^1 \times FM(n) \to FM(n)$, and let H denote the coalgebra $H^*(S^1) = \mathbb{R}[d\theta]$. The circle action induces an H-comodule structure on $H^*(FM(n))$; the coaction is given by the formula

$$\rho_n^*(x) = [d\theta] \otimes \Delta(x) + 1 \otimes x$$

where

$$\Delta : H^*(FM(n)) \to H^{*-1}(FM(n))$$

is a degree -1 derivation. Clearly the H-comodule structure and the derivation Δ determine each other.

We shall now lift the H-comodule structure to Kontsevich's admissible graph complex $G(n)$ by lifting the derivation Δ. Recall that $G(n)$ is the complex of real vector spaces spanned by admissible graphs on n external vertices [5]. The grading is defined by

$$(\#\text{ edges}) - 2(\#\text{ internal vertices}).$$

Graphs are *admissible* if they are at least trivalent at each internal vertex and satisfy a few additional conditions. Each graph is equipped with a total ordering of its edges, and a permutation of the edges acts on the corresponding generator of $G(n)$ by its sign. Given a graph g with edges $e_1, \ldots, e_k$, the differential is defined by

$$dg = \sum_i (-1)^i g/e_i$$

where g/e_i is the graph obtained from g by collapsing the edge e_i. Any non-admissible terms occuring in the sum are set to zero. Recall that the complex $G(n)$ has a graded commutative algebra structure given by disjoint union of internal vertices and the union of edges. The Kontsevich integral defines a morphism of differential graded algebras $I : G(n) \to \Omega^*(FM(n))$ (the target is the algebra of semi-algebraic forms defined in [4]) and this is a quasi-isomorphism.

Definition 2.1. *A linear operator $\Delta : G(n) \to G(n)$ of degree -1 is defined as follows. Given a graph $g \in G(n)$ with ordered set of edges $e_1, \ldots, e_k$,*

$$\Delta(g) = \sum_i (-1)^{i+1}(g - e_i)$$

where $g - e_i$ is the graph obtained by deleting the edge e_i from g (without identifying the endpoints together). If a summand is a non-admissible graph then we set it to zero.

Proposition 2.2. *The operator Δ satisfies:*

(1) $\Delta^2 = 0$;
(2) *it is a derivation of the algebra $G(n)$;*
(3) *it graded commutes with the differential d of $G(n)$, i.e. $d\Delta = -\Delta d$.*

Hence the rule

$$g \mapsto 1 \otimes g + [d\theta] \otimes \Delta(g)$$

gives $G(n)$ the structure of a DG-comodule over the coalgebra $H := H^*(S^1)$.

Let $\theta_{ij} : FM(n) \to S^1$ be the map measuring the angle of the line from the i-th to the j-th point with the first coordinate axis. The algebra $G(n)$ is freely generated by *indecomposable* graphs, those that do not get disconnected by removing a small neighbourhood of the set of external vertices. If g is indecomposable with no internal vertices then it has only an edge between some vertices i and j, and we denote it $g = \alpha_{ij}$. Kontsevich considers the algebra map

$$q_n : G(n) \to H^*(G(n)) = H^*(FM(n))$$

sending all graphs with internal vertices to 0, and such that

$$q_n(\alpha_{ij}) = \theta_{ij}^*(d\theta) := d\theta_{ij}.$$

Proposition 2.3 (Kontsevich, Lambrechts-Volic)**.** *The collection of maps $\{q_n\}$ assemble to a quasi-isomorphism of DG-cooperads $q : G \to H^*(FM)$.*

Proposition 2.4. *The projection $q_n : G(n) \to H^*(FM(n))$ is a map of H-comodules, i.e. $q \circ \Delta = \Delta \circ q$.*

Proof. For any graph g, the summands of $\Delta(g)$ have the same number of internal vertices as g. Therefore if g is indecomposable with some internal vertices then $q(\Delta(g)) = \Delta(q(g)) = 0$. If g is indecomposable with no internal vertices, then $g = \alpha_{ij}$ for some i, j, and $\Delta(g) = 1$ (the unit of the algebra $G(n)$ is the graph on n external vertices with no edges). Then $q(g) = [d\theta_{ij}]$. Since the map θ_{ij} is S^1-equivariant, we have that

$$\Delta([d\theta_{ij}]) = \theta_{ij}^*(\Delta([d\theta])) = \theta_{ij}^*(1) = 1$$

and so $\Delta(q(g)) = 1 = q(\Delta(g))$. □

3. Compatibility of Δ with the Kontsevich integral and the cooperad structures

We show in the next lemma that the operator Δ is compatible with the integration map $I : G(n) \to \Omega^*(FM(n))$.

Lemma 3.1. *For $g \in G(n)$,*

$$\rho_n^*(I(g)) = d\theta \times I(\Delta(g)) + 1 \times I(g) \in \Omega^*(S^1 \times FM(n)).$$

Proof. If $g = \alpha_{ij}$, then $I(g) = d\theta_{ij}$, $\Delta(g) = 1$, and $\rho_n^*(d\theta_{ij}) = d\theta \times 1 + 1 \times d\theta_{ij}$. If g is a graph with no internal vertices, then it is a product of forms $d\theta_{ij}$ and the result follows by multiplying the corresponding expressions, since ρ_n^*, I are algebra maps and Δ is a derivation. If g has k internal vertices then $I(g) = p_*(I(h))$, for some $h \in G(n + k)$, where p_* denotes the push-forward along the semi-algebraic bundle projection $p : F(n + k) \to F(n)$. It follows from the definition of I that $p_*(I(\Delta(h))) = I(\Delta(g))$. The diagram

$$\begin{tikzcd} S^1 \times F(n+k) \arrow[r, "\rho_{n+k}"] \arrow[d, "S^1 \times p"'] & F(n+k) \arrow[d, "p"] \\ S^1 \times F(n) \arrow[r, "\rho_n"] & F(n) \end{tikzcd}$$

is a pullback of semi-algebraic sets. By Proposition 8.13 in [4]

$$(\rho_n)^* \circ p_* = (S^1 \times p)_* \circ \rho_{n+k}^*.$$

Since h has no internal vertices

$$\rho_{n+k}^*(I(h)) = d\theta \times I(\Delta(h)) + 1 \times I(h)$$

and so

$$\begin{aligned} \rho_n^*(I(g)) &= \rho_n^*(p_*(I(h))) = (S^1 \times p)_*(\rho_{n+k}^*(I(h))) \\ &= d\theta \times p_*(I(\Delta(h))) + 1 \times p_*(I(h)) \\ &= d\theta \times I(\Delta(g)) + 1 \times I(g). \end{aligned}$$ □

We show next that the operator Δ is compatible with the cooperad structure of G constructed by Kontsevich. The tensor product of H-comodules is a H-comodule, such that Δ on the tensor product is defined by the Leibniz rule.

Proposition 3.2. *The cooperad structure map*

$$\circ_i : G(m + n - 1) \to G(m) \otimes G(n)$$

(with $1 \le i \le m$) commutes with Δ; i.e. it is a map of H-comodules.

Proof. Given a graph $g \in G(m+n-1)$,

$$\circ_i(g) = \sum_j (-1)^{s(j)} g'_j \otimes g''_j,$$

where j ranges over partitions of the set V of internal vertices into two sets V'_j and V''_j. Then $g''_j \in G(n)$ is the full subgraph of g containing the external vertices $\{i, \dots, i+n-1\}$ (relabelled) and the internal vertices in V''_j . The graph g'_j is obtained from g by collapsing g''_j to a single external vertex, and relabelling external vertices. The sign $s(j)$ is the sign of the permutation moving the edges from the ordering of g to the ordering of g'_j followed by the ordering of g''_j. If such graphs have repeated edges or are not admissible then they are identified to zero. From the definition one sees that

$$\circ_i(\Delta(g)) = \sum_j (-1)^{s(j)} (\Delta(g'_j) \otimes g''_j + (-1)^{|g'_j|} g'_j \otimes \Delta(g''_j)) = \Delta(\circ_i(g)).$$ □

4. From H-comodules to semidirect product cooperads

By Proposition 3.2 above one can form the semidirect product cooperad $G \rtimes H$ with $(G \rtimes H)(n) = G(n) \otimes H^{\otimes n}$, by extending to the differential graded setting the construction in section 4 of [9], and dualizing it. Explicitly the cooperad structure maps

$$\circ_i : (G \rtimes H)(m+n-1) \to (G \rtimes H)(m) \otimes (G \rtimes H)(n)$$

are the algebra maps defined by sending

$$\begin{aligned} d\theta_k &\mapsto d\theta'_i + d\theta''_{k-i+1} \text{ for } i \le k \le n+i-1, \\ d\theta_k &\mapsto d\theta'_k \text{ for } k < i, \\ d\theta_k &\mapsto d\theta'_{k-i+1} \text{ for } k \ge n+i, \end{aligned}$$

and for $g \in G(m+n-1)$,

$$g \mapsto \sum_j (-1)^{s(j)} (g'_j \otimes d\theta'_i \otimes \Delta(g''_j) + g'_j \otimes g''_j).$$

By Proposition 2.4 and Proposition 2.3 the collection q induces a quasi-isomorphism of cooperads

$$G \rtimes H \to H^*(FM) \rtimes H = H^*(fFM).$$

The semi-algebraic differential forms on FM do not exactly constitute a cooperad because semi-algebraic forms is a contravariant monoidal functor. The cross product of forms $\Omega^*(FM(m)) \otimes \Omega^*(FM(n)) \to \Omega^*(FM(m) \times FM(n))$ (which is a quasi-isomorphism) and the operad composition $\circ_i : FM(m) \times FM(n) \to FM(m+n-1)$ induce a zigzag

$$\Omega^*(FM(m+n-1)) \to \Omega^*(FM(m) \times FM(n)) \leftarrow \Omega^*(FM(m)) \otimes \Omega^*(FM(n)).$$

Nevertheless, there is a compatibility rule between operadic composition in G and FM.

Lemma 4.1 (Lemma 8.19 of [7])**.** *The pullback along the operad composition map $\circ_i^{FM} : FM(m) \times FM(n) \to FM(m+n-1)$, for $g \in G(m+n-1)$, gives*

$$(\circ_i^{FM})^*(I(g)) = \sum_j (-1)^{s(j)} I(g'_j) \times I(g''_j)$$

where $\circ_i^G(g) = \sum_j (-1)^{s(j)} g'_j \otimes g''_j$.

We state next an analogous compatibility condition for the *framed* case. There are quasi-isomorphisms

$$\Omega^*(FM(n)) \otimes H^{\otimes n} \to \Omega^*(FM(n)) \otimes \Omega^*(S^1)^{\otimes n} \to \Omega^*(FM(n) \times (S^1)^n) = \Omega^*(fFM(n)).$$

The first map sends fundamental classes of circles to volume forms, and the second map is the cross product of forms. The composition with the Kontsevich integral gives a quasi-isomorphism of algebras

$$I' : G(n) \otimes H^{\otimes n} = (G \rtimes H)(n) \to \Omega^*(fFM(n)).$$

Lemma 4.2. *The diagram*

$$\begin{tikzcd}
(G \rtimes H)(m+n-1) \arrow[rr, "\circ_i^{G \rtimes H}"] \arrow[d, "I'"] & & (G \rtimes H)(m) \otimes (G \rtimes H)(n) \arrow[d, "I' \otimes I'"] \\
\Omega^*(fFM(m+n-1)) \arrow[r, "(\circ_i^{fFM})^*"] & \Omega^*(fFM(m) \times fFM(n)) & \Omega^*(fFM(m)) \otimes \Omega^*(fFM(n)) \arrow[l]
\end{tikzcd}$$

commutes.

Proof. By definition of semidirect product the composition in fFM is

$$(x, z_1, \dots, z_m) \circ_i^{fFM} (y, w_1, \dots, w_n) = (x \circ_i^{FM} \rho_m(z_i, y), z_1, \dots, z_{i-1}, z_i w_1, \dots, z_i w_n, z_{i+1}, \dots, z_m).$$

The lemma follows from this, Lemma 3.1 and Lemma 4.1. □

We proceed similarly as in section 10 of [7] observing that integration on semi-algebraic chains of forms associated to graphs defines pairings

$$C_*(fFM(n)) \otimes (G \rtimes H)(n) \to \mathbb{R}$$

sending $c \otimes g \mapsto \int_c I(g)$, and their adjoints give a quasi-isomorphism of operads

$$C_*(fFM) \to (G \rtimes H)^*.$$

This together with the fact that $q^* : H_*(fFM) \to (G \rtimes H)^*$ is a quasi-isomorphism of operads establishes theorem 1.2.

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