Homotopy type of Euclidean configuration spaces

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Abstract

We describe the homotopy type of Euclidean configuration spaces. They admit a minimal cellular model, whose cells are attached via higher order Whitehead products.

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The Euclidean ordered configuration space $F(\mathbb{R}^n, k)$ is the space of pairwise disjointed k-tuples of elements in $\mathbb{R}^n$. This space appears for example in the problem of N-bodies; in the homotopy theory of iterated loop spaces [4]; in knot theory and deformation theory [2].

It is easy to see that $F(\mathbb{R}; k)$ has $k!$ contractible components, by linear contraction. It is well known [1] that $F(\mathbb{R}^2; k)$ is the classifying space of the pure braid group on k strings. We will describe the homotopy type of $F(\mathbb{R}^n, k)$ for $n \geq 3$. It is easy to see that $F(\mathbb{R}^n; 2) \simeq S^{n-1}$. The case $k = 3$ was studied by Massey [3]. We will show that $F(\mathbb{R}^n, k)$ has the homotopy type of a CW-complex with a minimal number of cells, attached via higher order Whitehead products. This result was conjectured by H.-J. Baues. I am grateful to him for various conversations, and to M. Xicotencatl for an essential comment. I thank C.-F. Boedigheimer and his students for several discussions and for the nice atmosphere at the Mathematical Institute of the University of Bonn.

The integral cohomology ring of $F(\mathbb{R}^n, k)$ has been computed by Cohen [1]. We recall his result. For $i, j \in \{1, \ldots, n+1\}$ and $i \neq j$, let $\theta_{i,j} : F(\mathbb{R}^{n+1}, k) \to S^n$ be the map picking up the unit vector pointing from the j-th to the i-th particle. Explicitly $\theta_{i,j}(x_1, \ldots, x_{n+1}) = (x_i - x_j)/|x_i - x_j|$. Let $e_n^* \in H^n(S^n)$ be the generator yielding the standard orientation. We define $\alpha_{i,j}^* := \theta_{i,j}^*(e_n^*)$.

Theorem 1. *[1] The cohomology ring $H^*(F(\mathbb{R}^{n+1}, k); \mathbb{Z})$, for $n \geq 1$, is the graded commutative ring generated by the classes $\{\alpha_{i,j}^*\}_{i \neq j}$ under the following relations:*

1. $\alpha_{i,j}^* = (-1)^{n+1} \alpha_{j,i}^*$;
2. $(\alpha_{i,j}^*)^2 = 0$;
3. $\alpha_{i,j}^* \alpha_{j,l}^* + \alpha_{j,l}^* \alpha_{l,i}^* + \alpha_{l,i}^* \alpha_{i,j}^* = 0$.

We observe that relation 2 is automatic if n is odd. The theorem implies that the cohomology is torsion free. In particular $H^n(F(\mathbb{R}^{n+1}, k))$ is the free abelian group generated by $\{\alpha_{i,j}\}_{i>j}$. We exhibit now the dual basis of the homology group $H_n(F(\mathbb{R}^{n+1}; k))$. Let $p_l : F(\mathbb{R}^{n+1}, k) \to \mathbb{R}^{n+1}$ denote the projection onto the l-th factor. Let $a_{i,j} : S^n \to F(\mathbb{R}^{n+1}, k)$ be an embedding determined on the factors by $p_j a_{i,j}(\xi) = 0$, $p_i a_{i,j}(\xi) = \xi$ and $p_l a_{i,j} = a_l$ for $l \notin \{i, j\}$, where a_l is a constant vector of norm $|a_l| > 1$ and $a_l \neq a_{l'}$ if $l \neq l'$. This means that all configuration points in the image are fixed except the i-th point, that moves around the j-th point along a sphere that contains no further points. The homotopy class of $a_{i,j}$ is uniquely defined. Let $e_n \in H_n(S^n)$ be the generator associated to the standard orientation. We define $\alpha_{i,j} = (a_{i,j})_*(e_n) \in H_n(F(\mathbb{R}^{n+1}, k))$.

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Lemma 2. *The homology basis* $\{\alpha_{i,j}\}_{i>j}$ *is dual to the cohomology basis* $\{\alpha^*_{i,j}\}_{i>j}$.

Proof. For $i > j$ and $i' > j'$ it is easy to see that the composite $\theta_{i',j'} a_{i,j} : S^n \to S^n$ has degree 1 if $i = i'$ and $j = j'$, and degree 0 otherwise. □

As consequence of Theorem 1 Cohen describes the additive structure of the cohomology.

Lemma 3. *[1] An additive basis of* $H^{tn}(F(\mathbb{R}^{n+1}, k))$, *for* $n \geq 1$ *and* $t \in \{1, \ldots, k-1\}$ *consists of the products* $\{\alpha^*_{i_1,j_1} \ldots \alpha^*_{i_t,j_t}\}$, *with* $i_u > j_u$ *and* $i_1 < \cdots < i_t$. *The remaining cohomology groups in positive degree are trivial.*

We shall call the basis in the lemma the *canonical* basis. Its elements correspond to certain oriented graphs. An *oriented forest* is an oriented graph without cycles such that each vertex has at most one incoming edge.

Lemma 4. *The elements of the canonical basis of degree* tn *correspond bijectively to those oriented forests on the set of vertices* $\{1, \ldots, k\}$ *with* t *edges, such that the induced partial ordering on the vertices coincides with the standard ordering of the natural numbers.*

Proof. An element of the canonical basis $\alpha^*_{i_1,j_1} \ldots \alpha^*_{i_t,j_t}$ correponds to the forest G having an edge from j_u to i_u for each $u \in \{1, \ldots, t\}$. □

We will associate an embedding $a_G : (S^n)^t \to F(\mathbb{R}^{n+1}, k)$ to each such *connected* forest G. For each edge $e : i \to j$ of G we write $i \prec j$ and we assign a positive real number $d(i,j)$, the length, under the following conditions.

1. If $l_0 \prec l_1 \prec \cdots \prec l_v$, then
$$d(l_0, l_1) > d(l_1, l_2) + \cdots + d(l_{v-1}, l_v);$$
2. If in addition $l_0 = m_0 \prec m_1 \prec \cdots \prec m_w = j$, and $l_1 < m_1$, then
$$d(l_0, l_1) - d(l_1, l_2) - \cdots - d(l_{v-1}, l_v) > d(m_0, m_1) + d(m_1, m_2) + \cdots + d(m_{w-1}, m_w)\ .$$

This can be achieved by steps. Recall that root of a tree is the unique vertex that is not a target. We assign the lengths of the edges originating from the root so that smaller lengths correspond to larger targets. Then we move up the tree and repeat the procedure for each vertex with sufficiently small lengths.

Definition 5. *Let* $\{e_u : j_u \to i_u\}$, $u \in \{1, \ldots, t\}$, *be the edges of* G, *and* i *the root. The embedding* $a_G : (S^n)^t \to F(\mathbb{R}^{n+1}, k)$ *sends* $(x_1, \ldots, x_t)$ *to the configuration of points such that the* i*-th point is the origin and the vector from the* j_u*-th to the* i_u*-th point has direction* x_u *and length* $d(i_u, j_u)$.

The image of the embedding can be thought of as the space of configurations of an iterated planetary system, where the root is the sun, its first successors the planets, the next successors the satellites and so on, and $d(i,j)$ is the radius of the orbit of i around j.

We construct the embedding for a disconnected forest G by requiring that, for any root i, $a_G(x_1, \ldots, x_t)$ has a constant vector v_i as i-th component. If $j \neq i$ is also a root, then $|v_i - v_j| > \sum_{u=1}^t d(i_u, j_u)$. The image is the configuration space of many planetary systems far apart from each other.

Let $e_{n,t}$ be the standard generator of $H_{tn}((S^n)^t) \cong \mathbb{Z}$. We denote $\alpha_G := (a_G)_*(e_{n,t}) \in H_{tn}(F(\mathbb{R}^{n+1}, k))$.

Lemma 6. *The elements $\{\alpha_G\}_G$ form a homology basis dual to the canonical cohomology basis.*

Proof. The composite $\theta_{i,j}a_G : (S^n)^t \to S^n$ is nullhomotopic if i and j do not lie in the same connected component of G. Otherwise $\theta_{i,j}a_G$, with $i > j$, is homotopic to the projection onto the factor associated to the following edge $e : u \to v$ of G. If $j \prec x_1 \prec \dots i$, then $u = j$ and $v = x_1$; if $z \prec x_1 \prec \dots i$ and $z \prec y_1 \prec \dots j$, then $u = z$ and $v = \max(x_1, y_1)$. All this can be seen by shrinking linearly all edges but possibly e. Hence $a_G^*(\alpha_{i,j}) \in H^n((S^n)^t)$ is zero, or a multiplicative generator associated to an edge with target $v \leq i$. The equality holds if and only if G contains the edge from j to i. If a forest H with t edges labels an element of the canonical basis $\alpha_H^* := \alpha_{b_1,c_1}^* \dots \alpha_{b_t,c_t}^*$, and $G \neq H$, then $< \alpha_G, \alpha_H^* >=< e_{n,t}, a_G^*(\alpha_H^*) >= 0$ either because a factor in the expansion $a_G^*(\alpha_{b_1,c_1}^*) \dots a_G^*(\alpha_{b_t,c_t}^*)$ is zero, or because it appears twice. On the other hand $< \alpha_G, \alpha_G^* >= \pm 1$. □

Now we can state the main theorem.

Theorem 7. *The configuration space $F(\mathbb{R}^{n+1}, k)$, for $n \geq 2$, has the homotopy type of a CW-complex with a* nt*-cell for each element of the canonical basis of length* t*. Each cell is attached via a generalized Whitehead product.*

Proof. We contruct the skeletons of a cellular approximation by induction. The 0-skeleton is a point. Suppose that we have constructed a $(m-1)n$-dimensional CW-complex $X_{(m-1)n}$ together with a $(m-1)n$-equivalence $\beta_{m-1} : X_{(m-1)n} \to F(\mathbb{R}^{n+1}, k)$. Actually β_{m-1} is a $(mn-1)$-equivalence because the next non-trivial homology group of $F(\mathbb{R}^{n+1}, k)$ occurs in dimension mn. Let G be a forest as before, with m edges. We may deform a_G to a'_G so that the following diagram commutes for some map γ.

$$\begin{tikzcd} (S^n)^m_{(m-1)n} \arrow[r, hook, "j"] \arrow[d, "\gamma"'] & (S^n)^m \arrow[d, "a'_G"] \\ X_{(m-1)n} \arrow[r, "\beta_{m-1}"] & F(\mathbb{R}^{n+1}, k)\,. \end{tikzcd}$$

The subscript denotes the dimension of the skeleton. The pushout of j and γ is obtained by attaching a mn-cell to $X_{(m-1)n}$ and is equipped with a natural map to $F(\mathbb{R}^{n+1}, k)$. If we repeat this construction for each element of the canonical basis of length m, we obtain a CW-complex X_{mn} together with a map $\beta_m : X_{mn} \to F(\mathbb{R}^{n+1}, k)$.

By lemma 6 β_m^* is a bijection in degree mn. Thus β_m is a mn-equivalence, as $F(\mathbb{R}^{n+1}, k)$ is simply connected. The process terminates for $m = k-1$, so β_{k-1} is a weak homotopy equivalence. Actually β_{k-1} is a homotopy equivalence because $F(\mathbb{R}^{n+1}, k)$ is a smooth manifold. The number of cells is minimal since they correspond to homology generators. The second statement of the theorem holds because the attaching map of the top cell of a product of spheres represents the higher order Whitehead product of the embeddings of the factors [5]. □

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