Name:

Matriculation number:

April 29, 2019

Midterm LAG exam

Solve the following exercises, explaining clearly each passage:

1) Consider the linear system

$$\begin{cases} x - y + z = 1 \\ x - z = 2 \\ 3x - 2y + z = k \end{cases}$$

For which values of the parameter k is the system consistent? For any such k find the solutions of the system. Which geometric object do they form?

Solution: the system is consistent if and only if $k = 4$. In this case the solutions form the line

$$\begin{cases} x = 2 + t \\ y = 1 + 2t \\ z = t \end{cases}$$

2) Compute the rank of the following matrices. For each matrix say if its linear transformation is injective, surjective, bijective. Determine which matrices are invertible and compute their inverses.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 4 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

Solution: $rank(A) = 3$, $rank(B) = 2$, $rank(C) = 2$. The transf. of A is bijective, that of B is surjective, that of C is neither injective nor surjective. Only A is invertible and

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1/2 \end{bmatrix}$$

3) Consider in $\mathbb{R}^3$ the linear space

$$U = Span\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}$$

and the linear space V of solutions of the homogeneous system

$$\begin{cases} x_2 - x_3 = 0 \\ 2x_2 - 2x_3 = 0 \end{cases}$$

Find a basis and the dimension for each of the linear spaces U, V, $U \cap V$, $U + V$.

Solution: $dim(U) = 2$, $dim(V) = 2$, $dim(U \cap V) = 1$, $dim(U + V) = 3$.

A basis of U is given by the first two vectors, a basis of V is $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}$, a basis of $U \cap V$ is $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$, and a basis of $U + V = \mathbb{R}^3$ is the canonical basis.

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Solve the following exercises, explaining clearly each passage:

1) Consider the linear system

$$\begin{cases} x - y + z = 1 \\ x - z = k \\ 3x - 2y + z = 3 \end{cases}$$

For which values of the parameter k is the system consistent? For any such k find the solutions of the system. Which geometric object do they form?

Solution: the system is consistent if and only if $k = 1$. In this case the solutions form the line

$$\begin{cases} x = 1 + t \\ y = 2t \\ z = t \end{cases}$$

2) Compute the rank of the following matrices. For each matrix say if its linear transformation is injective, surjective, bijective. Determine which matrices are invertible and compute their inverses.

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 & -2 \\ 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Solution: $rank(A) = 2$, $rank(B) = 1$, $rank(C) = 3$. The transf. of A is neither injective nor surjective, that of B is neither injective nor surjective, that of C is bijective. Only C is invertible and

$$C^{-1} = \begin{bmatrix} -1/2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

3) Consider in $\mathbb{R}^3$ the linear space $U = Span\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \}$ and the linear space V of solutions of the homogeneous system

$$\begin{cases} x_2 - x_3 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

Find a basis and the dimension for each of the linear spaces U, V, $U \cap V$, $U + V$.

Solution: $dim(U) = 2$, $dim(V) = 1$, $dim(U \cap V) = 0$, $dim(U + V) = 3$.

A basis of U is given by the first two vectors, a basis of V is $\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \}$, the basis of $U \cap V$ is empty, and a basis of $U + V = \mathbb{R}^3$ is the canonical basis.