Name: Matriculation number:

July 2, 2019, L.A.G. exam

Solve the following exercises, explaining clearly each passage:

1) Consider the matrix $A = \begin{bmatrix} k & 1 \\ -1 & k+2 \end{bmatrix}$

(1a) Compute the rank of A depending on the parameter k.

(1b) For which values of k is the matrix invertible?

(1c) Compute the inverse of A for the values of k from the previous point

Solution: The determinant $det(A) = (k+1)^2$ vanishes for $k = -1$. So for $k \neq -1$ the rank is 2 and the matrix is invertible. For $k = -1$ the rank is 1 and the matrix is not invertible. For $k \neq -1$ $A^{-1} = \frac{1}{(k+1)^2} \begin{bmatrix} k+2 & -1 \\ 1 & k \end{bmatrix}$

2) Consider the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \end{bmatrix}$$

(2a) Determine the dimension and a basis of $\mathrm{Ker}(f)$ and of $\mathrm{Im}(f)$.

(2b) Say if f is injective and if it is surjective.

(2c) Determine all the vectors $v \in \mathbb{R}^3$ such that $f(v) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution: f is the multiplication by the matrix $M = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ that has rank 2. So $dim(Im(f)) = 2 = dim(\mathbb{R}^2)$, and f is surjective. By the dimension theorem $dim(Ker(f)) = dim(\mathbb{R}^3) - 2 = 1 > 0$ and f is not injective. A basis of $Im(f)$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. By solving $Mv = 0$ we find that a basis of the kernel is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. By solving $Mv = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we find that

$$v = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3) Consider the line r of cartesian equation $\begin{cases} x - y - 1 = 0 \\ y - 2z - 1 = 0 \end{cases}$

(3a) Find a parametric equation of r.

(3b) Determine the cartesian equation of the plane π perpendicular to r passing through the point $(0, 0, 1)$

(3c) Determine the cartesian equation of the plane ρ containing r and the origin.

(3d) Find a parametric equation of the line $\pi \cap \rho$.

Solution: Setting z as free variable a parametric equation for r is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

The cartesian equation of π has the form $2(x-0)+2(y-0)+1(z-1)=0$ i.e. $2x+2y+z-1=0$. The cartesian equation of ρ is a linear combination $a(x-y-1)+b(y-2z-1)=0$ passing through the origin so $a+b=0$ obtaining $x-2y+2z=0$. A parametric equation of $\pi \cap \rho$ by solving the system of the equations of π and ρ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/6 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

4) Consider the conic

$$3x^2 - 2xy + 3y^2 = 1$$

(4a) Find a rotation that puts the conic in canonical form

(4b) Recognize the type of conic

(4c) Find the coordinates of the centre, if it exists

(4d) Find the coordinates of the vertices.

The matrix of the conic is $\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ The eigenvalues are 2 and 4. The normalized eigenvectors are $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ The change of coordinates is

$$\begin{cases} x = \frac{1}{\sqrt{2}}(x' + y') \\ y = \frac{1}{\sqrt{2}}(x' - y') \end{cases}$$

a rotation by 45 degrees. So in the new coordinates the equation of the conic is $2x'^2 + 4y'^2 = 1$ and the conic is an ellipse. The center is the point with $x' = 0, y' = 0$ and the 4 vertices are the points $(x', y') = (\pm\frac{1}{\sqrt{2}}, 0)$ and $(x', y') = (0, \pm 1/2)$.

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(1a) Compute the rank of A depending on the parameter k.

(1b) For which values of k is the matrix invertible?

(1c) Compute the inverse of A for the values of k from the previous point

Solution: The determinant $det(A) = k(k+1)$ vanishes for $k = 0$ and $k = -1$. So for $k \neq 0, -1$ the rank is 2 and the matrix is invertible. For $k = 0$ and $k = -1$ the rank is 1 and the matrix is not invertible. The inverse for $k \neq 0, -1$ is $A^{-1} = \frac{1}{k(k+1)} \begin{bmatrix} k+2 & k \\ 1 & k \end{bmatrix}$

2) Consider the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_1 \\ 2x_1 - 2x_2 \end{bmatrix}$$

(2a) Determine the dimension and a basis of $\mathrm{Ker}(f)$ and of $\mathrm{Im}(f)$.

(2b) Say if f is injective and if it is surjective.

(2c) Determine all the vectors $v \in \mathbb{R}^3$ such that $f(v) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Solution: f is the multiplication by the matrix $M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}$ that has rank 1. So $dim(Im(f)) = 1 < 2 = dim(\mathbb{R}^3)$, and f is not surjective. By the dimension theorem $dim(Ker(f)) = dim(\mathbb{R}^2) - 1 = 1 > 0$ and f is not injective. A basis of $Im(f)$ is $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. By solving $Mv = 0$ we find that a basis of the kernel is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

By solving $Mv = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ we find that

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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(3d) Find a parametric equation of the line $\pi \cap \rho$.

Solution: Setting z as free variable a parametric equation for r is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

The cartesian equation of π has the form $2(x-0)-2(y-0)+1(z-1)=0$ i.e. $2x-2y+z-1=0$. The cartesian equation of ρ is a linear combination $a(x+y-1)+b(y+2z-1)=0$ passing through the origin so $a+b=0$ obtaining $x-2z=0$. A parametric equation of $\pi \cap \rho$ by solving the system of the equations of π and ρ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

4) Consider the conic

$$x^2+6xy+y^2=1$$

(4a) Find a rotation that puts the conic in canonical form

(4b) Recognize the type of conic

(4c) Find the coordinates of the centre, if it exists

(4d) Find the coordinates of the vertices.

Solution: The matrix of the conic is $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ The eigenvalues are 4 and -2. The normalized eigenvectors are $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ The change of coordinates is

$$\begin{cases} x = \frac{1}{\sqrt{2}}(x'+y') \\ y = \frac{1}{\sqrt{2}}(x'-y') \end{cases}$$

a rotation by 45 degrees. In the new coordinates the equation of the conic is $4x'^2-2y'^2=1$ and the conic is a hyperbola. The center is the point with $x'=0, y'=0$ and the 2 vertices are the points $(x',y')=(\pm 1/2, 0)$.