Name: Matriculation number:

June 18, 2019, L.A.G. exam

Solve the following exercises, explaining clearly each passage:

1) Compute the rank of the matrix $\begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}$ depending on the parameter k. When is the matrix invertible?

Answer: The determinant of the matrix is $(k-1)^2(k+2)$ and so it vanishes for $k = 1$ and $k = -2$. So for $k \neq 1$ and $k \neq -2$ the rank is 3 and the matrix is invertible. By row reduction it is easy to see that for $k = 1$ the rank is 1, and for $k = -2$ the rank is 2.

2) Consider the linear transformation $f : \mathbb{R}^4 \to \mathbb{R}^2$ defined by multiplication by the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 3 \end{bmatrix}$

(a) Determine the dimension and a basis of $\mathrm{Ker}(f)$ and of $\mathrm{Im}(f)$.

(b) Say if f is injective and if it is surjective.

(c) Determine all the vectors $v \in \mathbb{R}^4$ such that $f(v) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Answer: The rank of A is 2. So $dim(Im(f)) = 2$ and by the dimension theorem $dim(Ker(f)) = 4 - 2 = 2$. By solving the homogenous linear system $Av = 0$ we obtain that a basis of $Ker(f)$ is $\{v_1, v_2\}$ with $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$. Since $dim(\mathbb{R}^2) = 2 = rank(A)$ we have that f is surjective, i.e. $Im(f) = \mathbb{R}^2$. A basis of $Im(f)$ is for example $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\}$ (since the pivots appear in the first and fourth column of a reduction), or simply the canonical basis of $\mathbb{R}^2$. Since $dim(\mathbb{R}^4) = 4 > rank(A)$ the map f is not injective. The linear system for point c) $Av = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gives the solution $v = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + sv_1 + tv_2$.

3) In the 3-dimensional euclidean space consider the line r passing through the point $P = (3, 1, 4)$, parallel to the vector $v = (1, 1, 1)$, and the line s, with cartesian equation

$$\begin{cases} x = 0 \\ z + y = -1 \end{cases}$$

(a) Find a parametric equation of s.

(b) Determine the mutual position of r and s.

(c) Determine the cartesian equation of the plane π parallel to r and s and passing through the point $(1, 0, 1)$.

(d) Determine the distance between the plane π and the origin.

Answer: a parametric equation of r is $\begin{cases} x = 3 + t \\ y = 1 + t \\ z = 4 + t \end{cases}$ and a parametric equation for s is $\begin{cases} x = 0 \\ y = -1 - t' \\ z = t' \end{cases}$ the respective direction vectors v and $(0, -1, 1)$ are not proportional and so r, s are not parallel. So they are either incident or skew. The system $\begin{cases} 3 + t = 0 \\ 5 + 2t = -1 \end{cases}$ has the solution $t = -3$ and so the lines are incident, intersecting in the point $(0, -2, 1)$. The plane π is orthogonal to the cross product of $(1, 1, 1) \times (0, -1, 1) \sim (2, -1, -1)$ and so we can choose $a = 2, b = -1, c = -1$ in the cartesian equation $ax + by + cz + d = 0$ of π. Imposing passage through $(1, 0, 1)$ gives $a + c + d = 0$ and so $d = -1$. The equation is $2x - y - z = 1$. The standard formula gives that the distance of π from the origin $1/\sqrt{6}$.

4) Consider the matrix $M = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

(a) Calculate the real eigenvalues of M, and their algebraic and geometric multiplicities.

(b) Determine whether M is diagonalizable.

(c) For each eigenvalue find an orthonormal basis of its eigenspace.

(d) Is there an orthogonal matrix C such that $C^T MC$ is a diagonal matrix?

Answer: The characteristic polynomial is $det(M - \lambda I) = (\lambda - 2)^2(\lambda - 1)$ so the eigenvalues are $\lambda = 2$ with algebraic multiplicity 2, and $\lambda = 1$ with algebraic multiplicity 1. the rank of $M - 2I$ is 1 and so the geometric multiplicity of 2 is $3 - 1 = 2$. The eigenspace $V_2 = Ker(M - 2I)$ has a basis $\{v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$. Clearly the geometric multiplicity of 1 is 1 (since the algebraic multiplicity is 1). The eigenspace $V_1 = Ker(M - I)$ has a basis $\{v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\}$. An orthonormal basis of V_2 is $\{\frac{1}{\sqrt{2}}v_1, v_2\}$ since $v_1 \cdot v_2 = 0$. An orthonormal basis of V_1 is $\{\frac{1}{\sqrt{2}}v_3\}$. The matrix is diagonalizable since geometric and algebraic multiplicity coincide, for each eigenvalue. There is no orthonormal basis of eigenvectors, because there are eigenvectors in V_1 and V_2 that are not orthogonal, for example $v_1 \cdot v_3 \neq 0$. This cannot be fixed by the Gram-Schmidt algorithm! So there is no C as required in (d).

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Solve the following exercises, explaining clearly each passage:

1) Compute the rank of the matrix

$$\begin{bmatrix} 1 & k & 1 \\ k & 1 & k \\ 1 & k & 1 \end{bmatrix}$$

depending on the parameter k. When is the matrix invertible?

Answer: the determinant of the matrix is always 0 so the matrix is not invertible for any k, and the rank is either 1 or 2. The square submatrices 2×2 have determinant either $k^2 - 1$ (up to a sign) or 0. So for $k = \pm 1$ the rank is 1, and for $k \neq \pm 1$ the rank is 2.

2) Consider the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^4$ defined by multiplication by the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 3 \end{bmatrix}^T$

(a) Determine the dimension and a basis of $\text{Ker}(f)$ and of $\text{Im}(f)$.

(b) Say if f is injective and if it is surjective.

(c) Determine all the vectors $v \in \mathbb{R}^2$ such that $f(v) = \begin{bmatrix} 0, 1, 0, 0 \end{bmatrix}^T$

Answer: the rank of A is 2. So the dimension of $Im(f)$ is 2, and the dimension of $Ker(f)$ is $2 - 2 = 0$. So the function is injective, and not surjective. A basis of $Im(f)$ is given by the two columns of A, and $Ker(f)$ has empty basis. The system $AX = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ has no solution since the augmented matrix $\bar{A}$ has $rank(\bar{A}) = 3 > rank(A)$ and there is no such v satisfying (c).

3) In the 3-dimensional euclidean space consider the line r passing through the point $P = (3, 1, 4)$, parallel to the vector $v = (1, 1, 1)$, and the line s, with cartesian equation

$$\begin{cases} y = 0 \\ x + z = -1 \end{cases}$$

(a) Find a parametric equation of s.

(b) Determine the mutual position of r and s.

(c) Determine the cartesian equation of the plane π parallel to r and s and passing through the point $(1, 0, 1)$.

(d) Determine the distance between the plane π and the origin.

Answer: a parametric equation of r is $\begin{cases} x = 3 + t \\ y = 1 + t \\ z = 4 + t \end{cases}$ and one of s is $\begin{cases} x = -1 - t \\ y = 0 \\ z = t \end{cases}$.

The direction vectors v and $(-1, 0, 1)$ are not proportional and so r and s are not parallel. They are either incident or skew. Substituting the parametric equation of r in the equation of s we get the system $\begin{cases} 1 + t = 0 \\ 7 + 2t = -1 \end{cases}$ that is inconsistent, and so the lines are skew. The plane is orthogonal to the vector $(1, -2, 1)$, that is proportional to the cross product of the directions of r and s. So it has the form $x - 2y + z = d$, and imposing the passage through the given point yields $d = 2$. Its distance from the origin is $2/\sqrt{6}$.

4) Consider the matrix $M = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

(a) Calculate the real eigenvalues of M, and their algebraic and geometric multiplicities.

(b) Determine whether M is diagonalizable.

(c) For each eigenvalue find an orthonormal basis of its eigenspace.

(d) Is there an orthogonal matrix C such that $C^T MC$ is a diagonal matrix?

Answer: the characteristic polynomial is $det(M - \lambda I) = (2 - \lambda)(-1 - \lambda)^2$ and so the eigenvalues are $\lambda = 2$ with algebraic multiplicity 1 (and so also geometric multiplicity 1), and $\lambda = -1$ with algebraic multiplicity 2. The rank of $M - (-1)I$ is 2, and so the geometric multiplicity of -1 is $3 - 2 = 1$, and the matrix M is not diagonalizable. A basis of V_2 is given by $\{v_1 = (-9, -2, 3)\}$ and a basis of V_{-1} is $\{v_2 = (0, 1, 0)\}$. The latter is an orthonormal basis. An orthonormal basis of V_2 is $\frac{1}{\sqrt{94}} v_1$. The matric C does not exist since M is not diagonalizable, let alone by an o.n. basis !