~~(29) $f(x) = \arcsin\left(\log_5(4|x| - x^2)\right)$~~

~~(30) $f(x) = \arcsin\left(\log_{10}\left(\dfrac{x+2}{x+1}\right)\right)$~~

~~**Esercizio 3.** Determinare il dominio delle seguenti funzioni. Verificare se sono iniettive, dove sono crescenti/decrescenti. Determinare l'insieme immagine e la funzione inversa (se esiste).~~

~~(1) $f(x) = \sin(\arcsin x)$~~

~~(2) $f(x) = \arcsin(2x)$~~

~~(3) $f(x) = \arccos\left(\dfrac{x+1}{x}\right)$~~

~~(4) $f(x) = \operatorname{arctg}\left(\dfrac{3x}{x-4}\right)$~~

~~(5) $f(x) = \sqrt{\operatorname{arctg}\left(\dfrac{x+2}{x}\right)}$~~

~~(6) $f(x) = \arcsin\sqrt{\log_5(x+2)}$~~

~~(7) $f(x) = \arcsin\left(\dfrac{x}{\sqrt{x^2+1}}\right)$~~

~~(8) $f(x) = \log_3(x^2-4)$~~

~~(9) $f(x) = \operatorname{arctg}\sqrt{x^2-2x+1}$~~

~~(10) $f(x) = 2^{\sqrt{-x^2+5x-4}}$~~

~~(11) $f(x) = \sqrt{\arcsin(x^2-3)}$~~

~~(12) $f(x) = \log_4\left(\arcsin(x^2-3)\right)$~~

~~(13) $f(x) = \arcsin(\log_2(x^2-3))$~~

~~(14) $f(x) = \arcsin(\sin x)$~~

Esercizio 4. Calcolare i seguenti numeri complessi

(1) $\dfrac{(1+2i)^2-(1-i)^3}{(3+2i)^3-(2+i)^2}$

(2) $(-1+i\sqrt{3})^{60}$

(3) $(2-2i)^7$

(4) $(\sqrt{3}-3i)^6$

(5) $\left(\dfrac{1+i\sqrt{3}}{1-i}\right)^{40}$

(6) $\left(\dfrac{1-i}{1+i}\right)^{8}$

(7) $\dfrac{(1+i)^9}{(1-i)^7}$

(8) $\dfrac{(1+i)(3-3i)}{(\sqrt{2}+i\sqrt{6})^5}$

(9) $\sqrt{i}$

(10) $\sqrt{2-2i\sqrt{3}}$

(11) $\sqrt[3]{i}$

(12) $\sqrt[3]{-1+i}$

(13) $\sqrt[4]{-1}$

(14) $\sqrt[4]{-i}$

(15) $\sqrt[4]{1}$

(16) $\sqrt[4]{1-i}$

(17) $\sqrt[5]{\sqrt{3}+i}$

Esercizio 5. Determinare tutte le soluzioni delle seguenti equazioni

(1) $z^2+4z+5=0$

(2) $z^2+2iz+3=0$

(3) $z^2-2z+1-i=0$

(4) $z^2-2z-i\sqrt{3}=0$

(5) $z^2+2(2\sqrt{3}+i)z+7=0$

(6) $z^3+1=0$

(7) $z^3+z^2+z+1=0$

(8) $z^4-4z^2+8=0$

(9) $z^4-2iz^2-2=0$

(10) $\left(\dfrac{2z+1}{2z-1}\right)^4=1$

(11) $z^5-1=0$

(12) $(z+1)^5-(z-1)^5=0$

(13) $z^6+27=0$

(14) $z^8=\dfrac{1+i}{\sqrt{3}-i}$

~~**Esercizio 6.** Utilizzare il principio di induzione per dimostrare le seguenti affermazioni~~

~~(1) $\sum_{k=1}^n k = 1+2+\ldots+n = \frac{1}{2}n(n+1), \quad \forall n\in\mathbb{N}$~~

~~(2) $\sum_{k=1}^n k^2 = 1+2^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1), \quad \forall n\in\mathbb{N}$~~

~~(3) $\sum_{k=1}^n k^3 = 1+2^3+\ldots+n^3 = \frac{1}{4}n^2(n+1)^2, \quad \forall n\in\mathbb{N}$~~

~~(4) $\sum_{k=0}^n q^n = 1+q+q^2+\ldots+q^n = \frac{q^{n+1}-1}{q-1}, \quad \forall q\neq 1, n\in\mathbb{N}$~~

~~periodica), ed è (strettamente) crescente in $[-\frac{\pi}{2},\frac{\pi}{2}]$ mod 2π, e (strettamente) decrescente in $[\frac{\pi}{2},\frac{3\pi}{2}]$ mod 2π. Determiniamo, infine, l'immagine di f. Si ha $y \in \operatorname{im} f \iff \exists x \in \operatorname{dom} f$ tale che $y = f(x) = \arcsin\sin x \iff \begin{cases} y \in [-\frac{\pi}{2},\frac{\pi}{2}] \\ \sin y = \sin x \end{cases} \iff \begin{cases} y \in [-\frac{\pi}{2},\frac{\pi}{2}] \\ x = \arcsin\sin y + 2k\pi \ \vee \ x = -\arcsin\sin y + (2k+1)\pi, \end{cases}$ e quindi l'equazione $y = f(x)$ ha soluzione $\iff y \in [-\frac{\pi}{2},\frac{\pi}{2}]$. Quindi $\operatorname{im} f = [-\frac{\pi}{2},\frac{\pi}{2}]$.~~

$\square$

Svolgimento esercizio 4

(1) Si ha $\frac{(1+2i)^2-(1-i)^3}{(3+2i)^3-(2+i)^2} = \frac{1+4i-4-1+3i+3-i}{27+54i-36-8i-4-4i+1} = \frac{-1+6i}{-12+42i} = \frac{(-1+6i)(-1-7i)}{6\cdot 50} = \frac{43+i}{300}$.

(2) Poiché $|-1+i\sqrt{3}| = 2$, $\operatorname{Arg}(-1+i\sqrt{3}) = \frac{2\pi}{3}$, si ha $(-1+i\sqrt{3})^{60} = 2^{60}\big(\cos(60\frac{2\pi}{3})+i\sin(60\frac{2\pi}{3})\big) = 2^{60}$.

(3) Poiché $|2-2i| = 2\sqrt{2}$, $\operatorname{Arg}(2-2i) = -\frac{\pi}{4}$, si ha $(2-2i)^7 = 2^{10}\sqrt{2}\big(\cos(-\frac{7\pi}{4}) + i\sin(-\frac{7\pi}{4})\big) = 2^{10}\sqrt{2}\big(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\big)$.

(4) Poiché $|\sqrt{3}-3i| = 2\sqrt{3}$, $\operatorname{Arg}(\sqrt{3}-3i) = -\frac{\pi}{3}$, si ha $(\sqrt{3}-3i)^6 = 2^6\cdot 3^3\big(\cos(-\frac{6\pi}{3})+i\sin(-\frac{6\pi}{3})\big) = 2^6\cdot 3^3$.

(5) Poiché $\left|\frac{1+i\sqrt{3}}{1-i}\right| = \frac{2}{\sqrt{2}} = \sqrt{2}$, $\operatorname{Arg}\left(\frac{1+i\sqrt{3}}{1-i}\right) = \frac{\pi}{3}+\frac{\pi}{4} = \frac{7\pi}{12}$, si ha $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{40} = 2^{20}\big(\cos(40\frac{7\pi}{12}) + i\sin(40\frac{7\pi}{12})\big) = -2^{20}\big(\cos(\frac{\pi}{3})+i\sin(\frac{\pi}{3})\big)$.

(6) Poiché $\left|\frac{1-i}{1+i}\right| = 1$, $\operatorname{Arg}\left(\frac{1-i}{1+i}\right) = -\frac{\pi}{4}-\frac{\pi}{4} = -\frac{\pi}{2}$, si ha $\left(\frac{1-i}{1+i}\right)^8 = \big(\cos(-\frac{8\pi}{2})+i\sin(-\frac{8\pi}{2})\big) = 1$.

(7) Poiché $|1+i| = |1-i| = \sqrt{2}$, $\operatorname{Arg}(1+i) = \frac{\pi}{4}$, $\operatorname{Arg}(1-i) = -\frac{\pi}{4}$, si ha $(1+i)^9 = 2^{9/2}\big(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\big) = 2^{9/2}\big(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\big)$, e $(1-i)^7 = 2^{7/2}\big(\cos(-\frac{7\pi}{4})+i\sin(-\frac{7\pi}{4})\big) = 2^{7/2}\big(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\big)$, per cui $\frac{(1+i)^9}{(1-i)^7} = 2$.

(8) Poiché $|\sqrt{2}+i\sqrt{6}| = 2\sqrt{2}$, $\operatorname{Arg}(\sqrt{2}+i\sqrt{6}) = \frac{\pi}{6}$, si ha $\frac{(1+i)(3-3i)}{(\sqrt{2}+i\sqrt{6})^5} = \frac{6}{2^7\sqrt{2}\big(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\big)} = \frac{3\sqrt{2}}{2^7}\big(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\big) = -\frac{3\sqrt{2}}{2^7}\big(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\big)$.

(9) Dalla formula di de Moivre si ha $\sqrt{i} = \{\cos(\frac{\pi}{4}+k\pi)+i\sin(\frac{\pi}{4}+k\pi) : k = 0,1\} = \{\pm\big(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\big)\} = \{\pm\big(\frac{1}{2}+i\frac{\sqrt{2}}{2}\big)\}$.

(10) Poiché $|2-2i\sqrt{3}| = 4$, $\operatorname{Arg}(2-2i\sqrt{3}) = -\frac{\pi}{3}$, dalla formula di de Moivre si ha $\sqrt{2-2i\sqrt{3}} = \{2\big(\cos(-\frac{\pi}{6}+k\pi)+i\sin(-\frac{\pi}{6}+k\pi)\big) : k = 0,1\} = \{\pm 2\big(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\big)\} = \{\pm\big(\sqrt{3}-i\big)\}$.

(11) Dalla formula di de Moivre si ha $\sqrt[3]{i} = \{\cos(\frac{\pi}{6}+\frac{2k\pi}{3})+i\sin(\frac{\pi}{6}+\frac{2k\pi}{3}) : k = 0,1,2\} = \{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\,, \cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\,, -i\}$.

(12) Poiché $|-1+i| = \sqrt{2}$, $\operatorname{Arg}(-1+i) = \frac{3\pi}{4}$, dalla formula di de Moivre si ha $\sqrt[3]{-1+i} = \{\sqrt[6]{2}\big(\cos(\frac{\pi}{4}+\frac{2k\pi}{3})+i\sin(\frac{\pi}{4}+\frac{2k\pi}{3})\big) : k = 0,1,2\} = \{\sqrt[6]{2}\big(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\big), \sqrt[6]{2}\big(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\big), \sqrt[6]{2}\big(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\big)\}$.

(13) Dalla formula di de Moivre si ha $\sqrt[4]{-1} = \{\cos(\frac{\pi}{4}+\frac{k\pi}{2})+i\sin(\frac{\pi}{4}+\frac{k\pi}{2}) : k = 0,1,2,3\} = \{\pm\big(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\big), \pm\big(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\big)\} = \{\frac{\sqrt{2}}{2}(\pm 1 \pm i)\}$.

(14) Dalla formula di de Moivre si ha $\sqrt[4]{-i} = \{\cos(\frac{3\pi}{8}+\frac{k\pi}{2})+i\sin(\frac{3\pi}{8}+\frac{k\pi}{2}) : k = 0,1,2,3\} = \{\pm\big(\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}\big), \pm\big(\cos\frac{7\pi}{8}+i\sin\frac{7\pi}{8}\big)\}$.

(15) Dalla formula di de Moivre si ha $\sqrt[4]{1} = \{\cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2} : k = 0, 1, 2, 3\} = \{\pm 1, \pm i\}$.

(16) Poiché $|1-i| = \sqrt{2}$, $\mathrm{Arg}(1-i) = -\frac{\pi}{4}$, dalla formula di de Moivre si ha $\sqrt[4]{1-i} = \{\sqrt[8]{2}\big(\cos(-\frac{\pi}{16} + \frac{k\pi}{2}) + i\sin(-\frac{\pi}{16} + \frac{k\pi}{2})\big) : k = 0, 1, 2, 3\} = \{\pm\sqrt[8]{2}\big(\cos\frac{\pi}{16} - i\sin\frac{\pi}{16}\big), \pm\sqrt[8]{2}\big(\cos\frac{7\pi}{16} + i\sin\frac{7\pi}{16}\big)\}$.

(17) Poiché $|\sqrt{3}+i| = 2$, $\mathrm{Arg}(\sqrt{3}+i) = \frac{\pi}{6}$, dalla formula di de Moivre si ha $\sqrt[4]{\sqrt{3}+i} = \{\sqrt[5]{2}\big(\cos(\frac{\pi}{30} + \frac{2k\pi}{5}) + i\sin(\frac{\pi}{30} + \frac{2k\pi}{5})\big) : k = 0, 1, 2, 3, 4\} = \{\sqrt[5]{2}\big(\cos\frac{\pi}{30} + i\sin\frac{\pi}{30}\big), \sqrt[5]{2}\big(\cos\frac{13\pi}{30} + i\sin\frac{13\pi}{30}\big), \sqrt[5]{2}\big(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\big), \sqrt[5]{2}\big(\cos\frac{37\pi}{30} + i\sin\frac{37\pi}{30}\big), \sqrt[5]{2}\big(\cos\frac{49\pi}{30} + i\sin\frac{49\pi}{30}\big)\}$.

$\square$

Svolgimento esercizio 5

(1) Si ha $z = -2 + \sqrt{-1} = -2 \pm i$.

(2) Si ha $z = -i + \sqrt{-4} = -i \pm 2i = \begin{cases} -3i, \\ i. \end{cases}$

(3) Si ha $z = 1 + \sqrt{i} = 1 \pm \big(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\big) = \begin{cases} 1 + \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \\ 1 - \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}. \end{cases}$

(4) Si ha $z = 1 + \sqrt{1 + i\sqrt{3}}$. Poiché $|1+i\sqrt{3}| = 2$, $\mathrm{Arg}(1+i\sqrt{3}) = \frac{\pi}{3}$, si ha $z = 1 \pm \sqrt{2}\big(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\big) = \begin{cases} 1 + \frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}, \\ 1 - \frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}. \end{cases}$

(5) Si ha $z = -2\sqrt{3} + i + 2\sqrt{1 - i\sqrt{3}}$. Poiché $|1 - i\sqrt{3}| = 2$, $\mathrm{Arg}(1 - i\sqrt{3}) = -\frac{\pi}{3}$, si ha $z = -2\sqrt{3} + i \pm 2\sqrt{2}\big(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}\big) = \begin{cases} -2\sqrt{3} - \sqrt{6} + (1 + \sqrt{2})i, \\ -2\sqrt{3} + \sqrt{6} + (1 - \sqrt{2})i. \end{cases}$

(6) Si ha $z = \sqrt[3]{-1} = \{\cos(\frac{\pi}{3} + \frac{2k\pi}{3}) + i\sin(\frac{\pi}{3} + \frac{2k\pi}{3}) : k = 0, 1, 2\} = \{\frac{1}{2}(1 \pm \sqrt{3}), -1\}$.

(7) Poiché $z^3 + z^2 + z + 1 = (z + 1)(z^2 + 1)$, si ha $z = -1 \vee z = \pm i$.

(8) Si ha $z^4 - 4z^2 + 8 = 0 \iff z^2 = 2 \pm 2i \iff z = \sqrt{2 + 2i} = \pm\sqrt[4]{8}\big(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\big) \vee z = \sqrt{2 - 2i} = \pm\sqrt[4]{8}\big(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\big)$.

(9) Si ha $z^4 - 2iz^2 - 2 = 0 \iff z^2 = i \pm 1 \iff z = \sqrt{i - 1} = \pm\sqrt[4]{2}\big(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\big) \vee z = \sqrt{i + 1} = \pm\sqrt[4]{2}\big(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\big)$.

(10) Si ha $\big(\frac{2z+1}{2z-1}\big)^4 = 1 \iff \frac{2z+1}{2z-1} = \sqrt[4]{1} = \{\pm 1, \pm i\} \iff z = 0 \vee z = \pm\frac{i}{2}$.

(11) Si ha $z = \sqrt[5]{1} = \{\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{3} : k = 0, 1, 2, 3, 4\}$.

(12) Si ha $(z + 1)^5 - (z - 1)^5 = 0 \iff \big(\frac{z+1}{z-1}\big)^5 = 1 \iff \frac{z+1}{z-1} = \sqrt[5]{1} = \{\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{3} : k = 0, 1, 2, 3, 4\} \iff z = \Big\{\frac{\cos\frac{2k\pi}{5} + 1 + i\sin\frac{2k\pi}{5}}{\cos\frac{2k\pi}{5} - 1 + i\sin\frac{2k\pi}{5}} : k = 1, 2, 3, 4\Big\}$.

(13) Si ha $z = \sqrt[6]{-27} = \big\{\sqrt{3}\big(\cos(\frac{\pi}{6} + \frac{k\pi}{3}) + i\sin(\frac{\pi}{6} + \frac{k\pi}{3})\big) : k = 0, 1, 2, 3, 4, 5\big\} \iff z = \pm i\sqrt{3} \vee z = \frac{\pm 3 \pm i\sqrt{3}}{2}$.

(14) Poiché $\frac{1+i}{\sqrt{3}-i} = \frac{1}{\sqrt{2}}\big(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\big)$, si ha $z = \sqrt[8]{\frac{1+i}{\sqrt{3}-i}} = \big\{\frac{1}{\sqrt[16]{2}}\big(\cos(\frac{5\pi}{96} + \frac{k\pi}{4}) + i\sin(\frac{5\pi}{96} + \frac{k\pi}{4})\big) : k = 0, 1, 2, 3, 4, 5, 6, 7\big\}$.

$\square$