Analisi Matematica I
Limiti di successioni

Esercizio 1.1. Verificare, usando la definizione, i seguenti limiti di successioni

(1) $\lim\limits_{n\to\infty} \dfrac{2n+3}{n+1} = 2$

(2) $\lim\limits_{n\to\infty} \dfrac{n^2+3n}{n^2+1} = 1$

(3) $\lim\limits_{n\to\infty} \sqrt{\dfrac{2n+3}{2n+1}} = 1$

(4) $\lim\limits_{n\to\infty} \dfrac{4\sqrt{n}-3}{\sqrt{n}+1} = 4$

(5) $\lim\limits_{n\to\infty} \sqrt{n+1}-\sqrt{n} = 0$

(6) $\lim\limits_{n\to\infty} \sqrt{n^2+n+1} = +\infty$

(7) $\lim\limits_{n\to\infty} \dfrac{n^2+1}{n+1} = +\infty$

(8) $\lim\limits_{n\to\infty} \log_2 \dfrac{1}{n} = -\infty$

(9) $\lim\limits_{n\to\infty} \sqrt{n}-n = -\infty$

Esercizio 1.2. Calcolare il limite per $n \to \infty$ delle seguenti successioni

(1) $a_n = \dfrac{n^3+2n^2-\sqrt{n}}{n^2+3n-1}$

(2) $a_n = \dfrac{n^3+2n^2-\sqrt{n}}{n^2+3n-1}\Big(\dfrac{1}{n}-\dfrac{2}{n^2}\Big)$

(3) $a_n = \dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^3+\sqrt[3]{n+7}}$

(4) $a_n = \dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^2+\sqrt[3]{n+7}}$

(5) $a_n = \dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}}$

(6) $a_n = \Big(\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2\Big)(n^{4/3}+2n+1)$

(7) $a_n = \Big(\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2\Big)^2(n^{3/2}+7n^2+\pi)$

(8) $a_n = \sqrt{\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2}\ (7n+2)$

(9) $a_n = \Big(\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2\Big)\sqrt{7n+2}$

(10) $a_n = \sqrt{n+1} - \sqrt{n}$

(11) $a_n = \sqrt{n+1} - n$

(12) $a_n = \sqrt{n^2+1} - n$

(13) $a_n = \sqrt{n^2+n+1} - \sqrt{n}$

(14) $a_n = \sqrt{n^2+n+1} - n$

(15) $a_n = \sqrt{n^2+\sqrt{n+1}} - n$

(16) $a_n = (\sqrt{n^2+\sqrt{n+1}} - n)^2$

(17) $a_n = (\sqrt{n^2+\sqrt{n+1}} - n)^2\sqrt{n+1}$

(18) $a_n = \sqrt{n^2+n^3+1} - n$

(19) $a_n = \sqrt{n^2+n^{1/3}+1} - n$

(20) $a_n = \sqrt{n^2+n^3+1} - n^{3/2}$

(21) $a_n = \sqrt{n^3+n+1} - n^{3/2}$

(22) $a_n = \dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}}(\sqrt{n^3+n+1} - n^{3/2})$

(23) $a_n = \dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}} + \sqrt{n^3+n+1} - n^{3/2}$

(24) $a_n = \Big(\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}}\Big)^2 + \sqrt{n^3+n+1} - n^{3/2}$

(25) $a_n = \sqrt[3]{n^3+2n^2} - n$

(26) $a_n = \sqrt[3]{n^6-n^4+1} - n^2$

Esercizio 1.3. Calcolare il limite per $n \to \infty$ delle seguenti successioni

(1) $a_n = \dfrac{\sqrt{n+\sqrt{n+3}} - n5^{-\sqrt{n}} + 3}{\big(n^5+3\operatorname{arctg}(n!)+7\big)^{2/7}}$

(2) $a_n = \dfrac{\sqrt{n+\sqrt{20^n+1}} + 5\cdot 2^n\sqrt{n} + 2}{3^n + 8\cdot 5^{-n^2+n} + 1}$

(3) $a_n = \dfrac{n!}{(n+1)! - (n-1)!}$

(4) $a_n = \dfrac{n^{n-3}(n+3)! + n^{n-2}(n+2)!}{n!\cdot n^n}$

(5) $a_n = \dfrac{n!(2n+3\cos n) - (n+1)!}{n!(2n-\log_3 n) + 2^{\log_3(n!)}}$

(6) $a_n = \dfrac{2n! + (2n)!}{n^n + 3n!}$

(7) $a_n = \dfrac{(\sqrt{n+1}+\sqrt{n}\,)n! + 3n^{51} + 5^{n+1}}{(n-1)!\,\big(4n + n^{1/3} + \sin(n^5+3)\big)^{3/2}}$

(8) $a_n = \dfrac{(\sqrt{n+1}-\sqrt{n}\,)n! + 3n^{51} + (n+1)5^{n+1}}{(n-1)!\,\sqrt{4n + 2n^2 + n^{3/2}\sin(n^5+3)}}$

(9) $a_n = \dfrac{n^n + 3^n}{2^{n\log_2 n}}$

(10) $a_n = \dfrac{n^n + n!}{2^{n^2}}$

(11) $a_n = \dfrac{n!(n^7+5n^2+1)2^n}{6^{n^2}}$

(12) $a_n = \dfrac{5^{n\log_5 n} - 5^n}{n^{\log_5 n} + n^{n+\log_5 n}}$

(13) $a_n = \dfrac{n!\cdot 3^{(n+1)!} + 5^{(n+1)!}}{\big((n+1)!\big)^2}$

(14) $a_n = \dfrac{n!\cdot 7^{n!} - 5^{(n+1)!}}{\big((n+1)!\big)^2 + 32^{n^2} + 1}$

(15) $a_n = \dfrac{n!\cdot 7^{n(n+1)} + 4^{(n+1)!}}{(n+1)^n}$

(16) $a_n = \dfrac{(n!)^n\cdot 7^{n(n+1)} + 4^{(n+1)!}}{\big((n-1)!\big)^n}$

(17) $a_n = \dfrac{(n!\cdot 7^n)^{n+1} - 4^{(n+1)!} + 2n^n}{n^{(n-1)!}}$

Esercizio 1.4. Disporre in ordine di infinito/infinitesimo crescente le seguenti successioni

(1) $a_n = n^{150}$, $\quad b_n = \left(\frac{3}{2}\right)^n$, $\quad c_n = \log_{12} n$, $\quad d_n = \frac{n!}{6}$, $\quad e_n = n^n$

(2) $a_n = n^{150} + n!$, $\quad b_n = \left(\frac{3}{2}\right)^n - \log_{12} n$, $\quad c_n = 7^n(n - 8n^7)$, $\quad d_n = 8n^7 + \frac{1}{n^n}$

(3) $a_n = \log_{12} n - \left(\frac{2}{3}\right)^n$, $\quad b_n = n^{150} + \left(\frac{3}{2}\right)^n$, $\quad c_n = \log_{12} n + \sqrt[12]{n}$, $\quad d_n = \frac{n!}{(n-1)!} - 3$

(4) $a_n = n^{3/2} + 3$, $\quad b_n = n^{1/2}(1 + n^{1/4})$, $\quad c_n = \log_3 n^2$, $\quad d_n = n + \left(\frac{1}{2}\right)^n$

(5) $a_n = 2^n$, $\quad b_n = n^2 + \log_5 n$, $\quad c_n = \frac{(n+1)!-(n-1)!}{n!}$, $\quad d_n = \frac{1}{\sqrt{n+1}-\sqrt{n}}$

(6) $a_n = \frac{2^n}{n^2}$, $\quad b_n = n\log_5 n$, $\quad c_n = \frac{n^2}{\log_5 n}$, $\quad d_n = n$

(7) $a_n = \frac{1}{n^{150}}$, $\quad b_n = \left(\frac{2}{3}\right)^n$, $\quad c_n = \frac{1}{\log_3 n}$, $\quad d_n = \frac{1}{n!}$, $\quad e_n = \frac{1}{n^n}$

(8) $a_n = \frac{1}{\log_3 n} + \frac{1}{n^{150}}$, $\quad b_n = \left(\frac{2}{3}\right)^n + \frac{4}{n!}$, $\quad c_n = \frac{1}{n^n} - \frac{1}{n!}$, $\quad d_n = \frac{1}{\sqrt{n}} - \left(\frac{2}{3}\right)^n$

(9) $a_n = \frac{\sqrt{n}}{n!(n^{3/2}+1)}$, $\quad b_n = \frac{n^{3/2}}{n!(n^{3/2}+1)}$, $\quad c_n = \frac{n^{3/2}}{(n!)^2(\sqrt{n}+1)}$, $\quad d_n = \frac{n!}{n^{3n}}$

(10) $a_n = \frac{1}{(n^2+1)}$, $\quad b_n = \frac{\sqrt{n}+n^{7/3}}{n^5+n^3+8}$, $\quad c_n = \frac{1}{1+2^n}$, $\quad d_n = 2 - \frac{2n^2}{n^2+n}$

(11) $a_n = \frac{\sqrt{n}}{n^2+1}$, $\quad b_n = \frac{1}{n\log_3 n}$, $\quad c_n = \frac{(\log_3 n)^2}{n}$, $\quad d_n = \frac{n!}{(n+1)!-(n-1)!}$

(12) $a_n = \frac{\log\sqrt{n}}{n^2+\log(n^{12})}$, $\quad b_n = \frac{(\log_3(n^2+1))^4}{n\log_3 n}$, $\quad c_n = \frac{(\log_3 n)^2}{\log_3(3^n+1)}$, $\quad d_n = \frac{\log(n^2+\log n)}{(n^2+\sqrt{n})\log(n+(\log n)^2)}$

Esercizio 1.5. Determinare estremo superiore e inferiore dei seguenti insiemi

(1) $\left\{\dfrac{n-1}{n} : n \in \mathbb{N}\right\}$

(2) $\left\{\dfrac{n^2+4}{5n^2} : n \in \mathbb{N}\right\}$

(3) $\{\sqrt{n+1}-\sqrt{n} : n \in \mathbb{N}\}$

(4) $\left\{\dfrac{n^2+1}{n} : n \in \mathbb{N}\right\}$

(5) $\left\{\dfrac{n}{n^2+1} : n \in \mathbb{N}\right\}$

(6) $\left\{\dfrac{(-1)^n}{n^2+1} : n \in \mathbb{N}\right\}$

(7) $\left\{(-1)^n\dfrac{(n^2+4)}{5n^2} : n \in \mathbb{N}\right\}$

(8) $\left\{(-1)^n\sqrt{1-\dfrac{1}{n}} : n \in \mathbb{N}\right\}$

(9) $\left\{\dfrac{|n-5|}{n+2} : n \in \mathbb{N}\right\}$

(10) $\left\{\dfrac{n-1}{n^2-3} : n \in \mathbb{N}\right\}$

Esercizio 1.6. Calcolare il limite per $n \to \infty$ delle seguenti successioni

(1) $a_n = \left(1+\dfrac{1}{2n}\right)^n$

(2) $a_n = \left(1+\dfrac{1}{2n+1}\right)^n$

(3) $a_n = \left(1+\dfrac{1}{n}\right)^{2n}$

(4) $a_n = \dfrac{(n+1)^n}{n^n+3}$

(5) $a_n = \dfrac{(n+1)^n}{n^n+n^2}$

(6) $a_n = \dfrac{(2n+1)^n}{(2n)^n+n^4}$

(7) $a_n = \dfrac{(2n)^n+2^n}{(2n+1)^n}$

(8) $a_n = \dfrac{(2n+1)^n}{2n^n+1}$

(9) $a_n = (n+1)^n - n!$

(10) $a_n = (n+1)^{n+1} - n^{n+1}$

(11) $a_n = (n+1)^{n!} - n^{2n}$

(12) $a_n = \dfrac{(n+1)^n + n!}{n^n + 5^n - n!}$

(13) $a_n = \dfrac{(2n+1)^n + n! + 1}{(2n+2)^n - n! + n^2}$

(14) $a_n = \dfrac{(2n+1)^n + (2n)^n}{(2n+2)^n - (2n+1)^n}$

(15) $a_n = \left(\sqrt{1+e^{-n}} - 1\right)\left(e^n - 3^n + n^2\right)$

(16) $a_n = \dfrac{(e^n+1)(n+\log n)}{(e^n+2^n)(2+\log n^6)}$

(17) $a_n = \sqrt[n]{2n}$

(18) $a_n = \sqrt[n]{n^3}$

(19) $a_n = \sqrt[2n+1]{-n}$

(20) $a_n = \sqrt[n]{n^2+3}$

(21) $a_n = \sqrt[n]{2^n+n^2}$

(22) $a_n = \left(\sqrt{n+\sqrt{n}} - \sqrt{n}\right)\sqrt[n]{2^{n+1}+n^2}$

(23) $a_n = \dfrac{(n^2+5n+7)e^{1/n}}{(n+1)(\sqrt{n+3}-\sqrt{n})}$

(24) $a_n = n^2(2n+\sqrt{n}\,)^{1/n} - \cos(n^3)$

(25) $a_n = \left(4n^n - (n+1)^n\right)^{1/n}$

(26) $a_n = \left((n+1)^{n+1} - n^{n+1}\right)^{1/n}$

Esercizio 1.7. Calcolare il limite per $n \to \infty$ delle seguenti successioni, usando la formula di Stirling

(1) $a_n = \dfrac{n!}{n^{n/2}}$

(2) $a_n = \sqrt[n]{n!}$

(3) $a_n = \dfrac{\sqrt[n]{n!}}{n}$

(4) $a_n = \dfrac{\sqrt[2n]{n!}}{n}$

(5) $a_n = \dfrac{1}{n}\sqrt[n]{\dfrac{(2n)!}{n!}}$

(6) $a_n = \sqrt[n]{\dbinom{2n}{n}}$

(7) $a_n = \sqrt[n]{\dbinom{4n}{2n}}$

Analisi Matematica I
Limiti di successioni (svolgimenti)

Svolgimento esercizio 1.1

(1) Infatti, per ogni $\varepsilon > 0$ si ha $\left|\frac{2n+3}{n+1} - 2\right| < \varepsilon \iff \frac{1}{n+1} < \varepsilon \iff n > \frac{1}{\varepsilon} - 1$, e quindi basta prendere $n_\varepsilon := \left[\frac{1}{\varepsilon}\right] - 1$.

(2) Infatti, per ogni $\varepsilon > 0$ si ha $\left|\frac{n^2+3n}{n^2+1} - 1\right| < \varepsilon \iff \frac{3n-1}{n^2+1} < \varepsilon \iff \varepsilon n^2 - 3n + \varepsilon - 1 > 0$, ed è sufficiente prendere $n > \frac{3+\sqrt{9+4\varepsilon-4\varepsilon^2}}{2\varepsilon}$, e quindi basta prendere $n_\varepsilon := \left[\frac{3+\sqrt{9+4\varepsilon-4\varepsilon^2}}{2\varepsilon}\right]$. Più semplicemente, poiché $\frac{3n-1}{n^2+1} < \frac{3n}{n^2} = \frac{3}{n}$, è sufficiente prendere $n > \frac{3}{\varepsilon}$, e quindi $n_\varepsilon := \left[\frac{3}{\varepsilon}\right]$.

(3) Infatti, per ogni $\varepsilon > 0$ si ha $\left|\sqrt{\frac{2n+3}{2n+1}} - 1\right| < \varepsilon \iff \frac{\frac{2n+3}{2n+1}-1}{\sqrt{\frac{2n+3}{2n+1}}+1} < \varepsilon \iff \frac{\frac{2}{2n+1}}{\sqrt{\frac{2n+3}{2n+1}}+1} < \varepsilon$. Poiché $\frac{\frac{2}{2n+1}}{\sqrt{\frac{2n+3}{2n+1}}+1} < \frac{2}{2n+1} < \frac{1}{n}$, è sufficiente prendere $n > \frac{1}{\varepsilon}$, e quindi $n_\varepsilon := \left[\frac{1}{\varepsilon}\right]$.

(4) Infatti, per ogni $\varepsilon > 0$ si ha $\left|\frac{4\sqrt{n}-3}{\sqrt{n}+1} - 4\right| < \varepsilon \iff \frac{7}{\sqrt{n}+1} < \varepsilon \iff n > \left(\frac{7}{\varepsilon} - 1\right)^2$, e quindi basta prendere $n_\varepsilon := \left[\left(\frac{7}{\varepsilon} - 1\right)^2\right]$, ma basterebbe anche $n_\varepsilon := \left[\frac{49}{\varepsilon^2}\right]$.

(5) Infatti, per ogni $\varepsilon > 0$ si ha $\left|\sqrt{n+1} - \sqrt{n}\right| < \varepsilon \iff \frac{1}{\sqrt{n+1}+\sqrt{n}} < \varepsilon$. Poiché $\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{2}{\sqrt{n}}$, è sufficiente prendere $n > \frac{4}{\varepsilon^2}$, e quindi $n_\varepsilon := \left[\frac{4}{\varepsilon^2}\right]$.

(6) Infatti, per ogni $M > 0$ si ha $\sqrt{n^2+n+1} > M \iff n^2 + n - (M^2 - 1) > 0$, ed è sufficiente prendere $n > \frac{-1+\sqrt{4M^2-3}}{2}$, e quindi $n_M := \left[\frac{\sqrt{4M^2-3}-1}{2}\right]$. Più semplicemente, poiché $\sqrt{n^2+n+1} > \sqrt{n^2} = n$, è sufficiente prendere $n > M$, e quindi $n_M := \left[M\right]$.

(7) Infatti, per ogni $M > 0$ si ha $\frac{n^2+1}{n+1} > M \iff \frac{n^2-Mn-(M-1)}{n+1} > 0 \iff n^2 - Mn - (M-1) > 0$, ed è sufficiente prendere $n > \frac{M+\sqrt{M^2+4M-4}}{2}$, e quindi $n_M := \left[\frac{M+\sqrt{M^2+4M-4}}{2}\right]$. Più semplicemente, poiché $\frac{n^2+1}{n+1} > \frac{n^2}{2n} = \frac{n}{2}$, è sufficiente prendere $n > 2M$, e quindi $n_M := \left[2M\right]$.

(8) Infatti, per ogni $M > 0$ si ha $\log_2 \frac{1}{n} < -M \iff \log_2 n > M \iff n > 2^M$, e quindi basta prendere $n_M := \left[2^M\right]$.

(9) Infatti, per ogni $M > 0$ si ha $\sqrt{n} - n < -M \iff n - \sqrt{n} - M > 0$, e quindi è sufficiente prendere $n > \left(\frac{1+\sqrt{1+4M}}{2}\right)^2$, e quindi $n_M := \left[\left(\frac{1+\sqrt{1+4M}}{2}\right)^2\right]$.

$\square$

Svolgimento esercizio 1.2

(1) Si ha $\dfrac{n^3+2n^2-\sqrt{n}}{n^2+3n-1} = \dfrac{n^3(1+o(1))}{n^2(1+o(1))} = n(1+o(1)) \to +\infty.$

(2) Si ha $\dfrac{n^3+2n^2-\sqrt{n}}{n^2+3n-1}\left(\dfrac{1}{n} - \dfrac{2}{n^2}\right) = \dfrac{n^3(1+o(1))}{n^2(1+o(1))}\dfrac{1}{n}(1+o(1)) = 1 + o(1) \to 1.$

(3) Si ha $\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^3+\sqrt[3]{n+7}} = \dfrac{n^2(1+o(1))}{5n^3(1+o(1))} = \dfrac{1}{5n}(1+o(1)) \to 0.$

(4) Si ha $\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^2+\sqrt[3]{n+7}} = \dfrac{n^2(1+o(1))}{5n^2(1+o(1))} = \dfrac{1}{5}(1+o(1)) \to \dfrac{1}{5}.$

(5) Si ha $\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}}=\dfrac{n^2(1+o(1))}{6n^{1/3}(1+o(1))}=\dfrac{n^{5/6}}{6}(1+o(1))\to+\infty.$

(6) Si ha $\big(\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2\big)(n^{4/3}+2n+1)=\dfrac{\sqrt{n}+1}{n^2+1}\,(n^{4/3}+2n+1)=\dfrac{\sqrt{n}(1+o(1))}{n^2(1+o(1))}\,n^{4/3}(1+o(1))=\dfrac{1}{n^{1/6}}(1+o(1))\to 0.$

(7) Si ha $\big(\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2\big)(n^{3/2}+7n^2+\pi)=\dfrac{\sqrt{n}+1}{n^2+1}\,(n^{3/2}+7n^2+\pi)=\dfrac{\sqrt{n}(1+o(1))}{n^2(1+o(1))}\,7n^2(1+o(1))=7\sqrt{n}(1+o(1))\to+\infty.$

(8) Si ha $\sqrt{\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2}\;(7n+2)=\sqrt{\dfrac{\sqrt{n}+1}{n^2+1}}\;(7n+2)=\sqrt{\dfrac{\sqrt{n}(1+o(1))}{n^2(1+o(1))}}\;7n(1+o(1))=7\sqrt[4]{n}(1+o(1))\to+\infty.$

(9) Si ha $\big(\dfrac{2n^2+3+\sqrt{n}}{n^2+1}-2\big)\sqrt{7n+2}=\dfrac{\sqrt{n}+1}{n^2+1}\sqrt{7n+2}=\dfrac{\sqrt{n}(1+o(1))}{n^2(1+o(1))}\sqrt{7n}(1+o(1))=\dfrac{\sqrt{7}}{n}(1+o(1))\to 0.$

(10) Si ha $\sqrt{n+1}-\sqrt{n}=\dfrac{1}{\sqrt{n+1}+\sqrt{n}}=\dfrac{1}{2\sqrt{n}(1+o(1))}\to 0.$

(11) Si ha $\sqrt{n+1}-n=-n(1+o(1))\to-\infty.$

(12) Si ha $\sqrt{n^2+1}-n=\dfrac{1}{\sqrt{n^2+1}+n}=\dfrac{1}{2n(1+o(1))}\to 0.$

(13) Si ha $\sqrt{n^2+n+1}-\sqrt{n}=n(1+o(1))\to+\infty.$

(14) Si ha $\sqrt{n^2+n+1}-n=\dfrac{n+1}{\sqrt{n^2+n+1}+n}=\dfrac{n(1+o(1))}{2n(1+o(1))}=\dfrac{1}{2}(1+o(1))\to\dfrac{1}{2}.$

(15) Si ha $\sqrt{n^2+\sqrt{n+1}}-n=\dfrac{\sqrt{n+1}}{\sqrt{n^2+n+1}+n}=\dfrac{\sqrt{n}(1+o(1))}{2n(1+o(1))}=\dfrac{1}{2\sqrt{n}(1+o(1))}\to 0.$

(16) Si ha $\big(\sqrt{n^2+\sqrt{n+1}}-n\big)^2=\Big(\dfrac{\sqrt{n+1}}{\sqrt{n^2+n+1}+n}\Big)^2=\Big(\dfrac{\sqrt{n}(1+o(1))}{2n(1+o(1))}\Big)^2=\Big(\dfrac{1}{2\sqrt{n}(1+o(1))}\Big)^2=\dfrac{1}{4n(1+o(1))}\to 0.$

(17) Si ha $\big(\sqrt{n^2+\sqrt{n+1}}-n\big)^2\sqrt{n+1}=\Big(\dfrac{\sqrt{n+1}}{\sqrt{n^2+n+1}+n}\Big)^2\sqrt{n+1}=\Big(\dfrac{\sqrt{n}(1+o(1))}{2n(1+o(1))}\Big)^2\sqrt{n}(1+o(1))=\Big(\dfrac{1}{2\sqrt{n}(1+o(1))}\Big)^2\sqrt{n}(1+o(1))=\dfrac{\sqrt{n}(1+o(1))}{4n(1+o(1))}=\dfrac{1}{4\sqrt{n}(1+o(1))}\to 0.$

(18) Si ha $\sqrt{n^2+n^3+1}-n=\dfrac{n^3+1}{\sqrt{n^2+n^3+1}+n}=\dfrac{n^3(1+o(1))}{n^{3/2}(1+o(1))}=n^{3/2}(1+o(1))\to+\infty.$

(19) Si ha $\sqrt{n^2+n^{1/3}+1}-n=\dfrac{n^{1/3}+1}{\sqrt{n^2+n^{1/3}+1}+n}=\dfrac{n^{1/3}(1+o(1))}{n^{3/2}(1+o(1))}=\dfrac{1}{n^{7/6}}(1+o(1))\to 0.$

(20) Si ha $\sqrt{n^2+n^3+1}-n^{3/2}=\dfrac{n^2+1}{\sqrt{n^2+n^3+1}+n^{3/2}}=\dfrac{n^2(1+o(1))}{2n^{3/2}(1+o(1))}=\dfrac{n^{1/2}}{2}(1+o(1))\to+\infty.$

(21) Si ha $\sqrt{n^3+n+1}-n^{3/2}=\dfrac{n+1}{\sqrt{n^3+n+1}+n^{3/2}}=\dfrac{n(1+o(1))}{2n^{3/2}(1+o(1))}=\dfrac{1}{2n^{1/2}}(1+o(1))\to 0.$

(22) Usando il risultato dell'esercizio (21), si ha $\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}}\,(\sqrt{n^3+n+1}-n^{3/2})=$ $\dfrac{n^2(1+o(1))}{6n^{1/3}(1+o(1))}\,\dfrac{1}{2n^{1/2}(1+o(1))}=\dfrac{n^{7/6}}{12}(1+o(1))\to+\infty.$

(23) Usando il risultato dell'esercizio (21), si ha $\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}}+\sqrt{n^3+n+1}-n^{3/2}=$ $\dfrac{n^2(1+o(1))}{6n^{1/3}(1+o(1))}+\dfrac{1}{2n^{1/2}(1+o(1))}=\dfrac{n^{5/3}}{6}(1+o(1))\to+\infty.$

(24) Usando il risultato dell'esercizio (21), si ha $\Big(\dfrac{n^2+3n^{3/2}+\sqrt{n+3}+1}{5n^{1/3}+\sqrt[3]{n+7}}\Big)^2+\sqrt{n^3+n+1}-n^{3/2}=$ $\Big(\dfrac{n^2(1+o(1))}{6n^{1/3}(1+o(1))}\Big)^2+\dfrac{1}{2n^{1/2}(1+o(1))}=\dfrac{n^{10/3}}{36}(1+o(1))\to+\infty.$

(25) Si ha $\sqrt[3]{n^3+2n^2}-n=\dfrac{2n^2}{(n^3+2n^2)^{2/3}+n(n^3+2n^2)^{1/3}+n^2}=\dfrac{2n^2}{3n^2(1+o(1))}\to\dfrac{2}{3}.$

(26) Si ha $\sqrt[3]{n^6-n^4+1}-n^2=\dfrac{-n^4+1}{(n^6-n^4+1)^{2/3}+n^2(n^6-n^4+1)^{1/3}+n^4}=\dfrac{-n^4(1+o(1))}{3n^4(1+o(1))}\to-\dfrac{1}{3}.$

$\square$

Svolgimento esercizio 1.3

(1) Si ha $\dfrac{\sqrt{n+\sqrt{n+3}}-n5^{-\sqrt{n}}+3}{\big(n^5+3\,\mathrm{arctg}(n!)+7\big)^{2/7}}=\dfrac{\sqrt{n(1+o(1))}+3+o(1)}{\big(n^5(1+o(1))\big)^{2/7}}=\dfrac{\sqrt{n}(1+o(1))}{n^{10/7}(1+o(1))}\to 0.$

(2) Si ha $\dfrac{\sqrt{n+\sqrt{20^n+1}}+5\cdot 2^n\sqrt{n}+2}{3^n+8\cdot 5^{-n^2+n}+1}=\dfrac{(\sqrt[4]{20})^n(1+o(1))+5\sqrt{n}\cdot 2^n(1+o(1))}{3^n(1+o(1))}=\dfrac{(\sqrt[4]{20})^n(1+o(1))}{3^n(1+o(1))}\to$ 0, perché $\dfrac{\sqrt{n}\cdot 2^n}{(\sqrt[4]{20})^n}=\dfrac{\sqrt{n}}{(\sqrt[4]{\frac{5}{4}})^n}\to 0.$

(3) Si ha $\dfrac{n!}{(n+1)!-(n-1)!}=\dfrac{(n-1)!n}{(n-1)!n(n+1)-(n-1)!}=\dfrac{n}{n^2+n-1}=\dfrac{n}{n^2(1+o(1))}=\dfrac{1}{n}(1+$ $o(1))\to 0.$

(4) Si ha $\dfrac{n^{n-3}(n+3)!+n^{n-2}(n+2)!}{n!\cdot n^n}=\dfrac{(n+3)!}{n!n^3}+\dfrac{(n+2)!}{n!n^2}=\dfrac{(n+3)(n+2)(n+1)}{n^3}+\dfrac{(n+2)(n+1)}{n^2}=$ $2(1+o(1))\to 2.$

(5) Si ha $\dfrac{n!(2n+3\cos n)-(n+1)!}{n!(2n-\log_3 n)+2^{\log_3(n!)}}=\dfrac{n!(2n+3\cos n-(n+1))}{n!\cdot 2n(1+o(1))+(n!)^{\log_3 2}}=\dfrac{n!\cdot n(1+o(1))}{n!\cdot 2n(1+o(1))}\to\dfrac{1}{2}.$

(6) Si ha $\dfrac{2n!+(2n)!}{n^n+3n!}=\dfrac{(2n)!(1+o(1))}{n^n(1+o(1))}=\dfrac{2n}{n}\dfrac{2n-1}{n}\cdots\dfrac{n+1}{n}n!(1+o(1))\geq n!\to+\infty.$

(7) Si ha $\dfrac{(\sqrt{n+1}+\sqrt{n})n!+3n^{51}+5^{n+1}}{(n-1)!\,\big(4n+n^{1/3}+\sin(n^5+3)\big)^{3/2}}=\dfrac{2\sqrt{n}\cdot n!(1+o(1))}{(n-1)!(4n)^{3/2}(1+o(1))}=\dfrac{2n\sqrt{n}(1+o(1))}{8n^{3/2}(1+o(1))}\to\dfrac{1}{4}.$

(8) Si ha $\dfrac{(\sqrt{n+1}-\sqrt{n}\,)n!+3n^{51}+(n+1)5^{n+1}}{(n-1)!\,\sqrt{4n+2n^2+n^{3/2}\sin(n^5+3)}} = \dfrac{\frac{1}{2\sqrt{n}}n!(1+o(1))}{(n-1)!\cdot n\sqrt{2}(1+o(1))} = \dfrac{1}{2\sqrt{2n}}(1+o(1)) \to 0.$

(9) Si ha $\dfrac{n^n+3^n}{2^{n\log_2 n}} = \dfrac{n^n(1+o(1))}{n^n} \to 1.$

(10) Si ha $\dfrac{n^n+n!}{2^{n^2}} = \dfrac{2^{n\log_2 n}(1+o(1))}{2^{n^2}} \to 0.$

(11) Si ha $\dfrac{n!(n^7+5n^2+1)2^n}{6^{n^2}} = \dfrac{n!\cdot n^7(1+o(1))}{6^{n^2-n\log_6 2}} = \dfrac{n!\cdot n^7(1+o(1))}{6^{n^2}(1+o(1))} \to 0.$

(12) Si ha $\dfrac{5^{n\log_5 n}-5^n}{n^{\log_5 n}+n^{n+\log_5 n}} = \dfrac{n^n(1+o(1))}{n^{n+\log_5 n}(1+o(1))} = \dfrac{1}{5^{(\log_5 n)^2}}(1+o(1)) \to 0.$

(13) Si ha $\dfrac{n!\cdot 3^{(n+1)!}+5^{(n+1)!}}{\big((n+1)!\big)^2} = \dfrac{5^{(n+1)!}(1+o(1))}{((n+1)!)^2} \to +\infty$, perché $\dfrac{n!\cdot 3^{(n+1)!}}{5^{(n+1)!}} = \dfrac{1}{n+1}\,\dfrac{(n+1)!}{(\frac{5}{3})^{(n+1)!}} \to 0.$

(14) Si ha $\dfrac{n!\cdot 7^{n!}-5^{(n+1)!}}{\big((n+1)!\big)^2+32^{n^2}+1} = -\dfrac{5^{(n+1)!}(1+o(1))}{32^{n^2}(1+o(1))} \to -\infty$, perché $\dfrac{5^{(n+1)!}}{n!\cdot 7^{n!}} = 5^{(n+1)!-\log_5(n!)-n!\log_7 5} = 5^{(n+1)!(1+o(1))} \to +\infty$, e $0 \le \dfrac{\big((n+1)!\big)^2}{32^{n^2}} \le (n+1)^2\dfrac{n^{2n}}{32^{n^2}} = \dfrac{n^2(1+o(1))}{32^{n^2-2n\log_{32} n}} \to 0.$

(15) Si ha $\dfrac{n!\cdot 7^{n(n+1)}+4^{(n+1)!}}{(n+1)^n} = \dfrac{4^{(n+1)!}(1+o(1))}{(n+1)^n} \to +\infty.$

(16) Si ha $\dfrac{(n!)^n\cdot 7^{n(n+1)}+4^{(n+1)!}}{\big((n-1)!\big)^n} = \dfrac{4^{(n+1)!}(1+o(1))}{4^{n\log_4(n-1)!}} \to +\infty.$

(17) Si ha $\dfrac{(n!\cdot 7^n)^{n+1}-4^{(n+1)!}+2n^n}{n^{(n-1)!}} = \dfrac{-4^{(n+1)!}(1+o(1))}{4^{(n-1)!\log_4 n}} \to -\infty.$

□

Svolgimento esercizio 1.4

(1) Poiché si ha $\frac{c_n}{a_n}\to 0$, $\frac{a_n}{b_n}\to 0$, $\frac{b_n}{d_n}\to 0$, $\frac{d_n}{e_n}\to 0$, l'ordinamento degli infiniti è c_n, a_n, b_n, d_n, e_n.

(2) Poiché si ha $a_n = n!(1+o(1))$, $b_n = (\frac{3}{2})^n(1+o(1))$, $c_n = -8n^7\cdot 7^n(1+o(1))$, $d_n = 8n^7(1+o(1))$, ne segue che $\frac{d_n}{b_n}\to 0$, $\frac{b_n}{c_n} = \frac{(3/2)^n(1+o(1))}{-8n^7\cdot 7^n(1+o(1))} = -\frac{1}{8}\,(\frac{3}{14})^n(1+o(1)) \to 0$, $\frac{c_n}{a_n}\to 0$, e l'ordinamento degli infiniti è d_n, b_n, c_n, a_n.

(3) Poiché si ha $a_n = \log_{12} n(1+o(1))$, $b_n = (\frac{3}{2})^n(1+o(1))$, $c_n = \sqrt[12]{n}(1+o(1))$, $d_n = n(1+o(1))$, ne segue che $\frac{a_n}{c_n}\to 0$, $\frac{c_n}{d_n}\to 0$, $\frac{d_n}{b_n}\to 0$, e l'ordinamento degli infiniti è a_n, c_n, d_n, b_n.

(4) Poiché si ha $a_n = n^{3/2}(1+o(1))$, $b_n = n^{3/4}(1+o(1))$, $c_n = 2\log_3 n$, $d_n = n(1+o(1))$, ne segue che $\frac{c_n}{b_n}\to 0$, $\frac{b_n}{d_n}\to 0$, $\frac{d_n}{a_n}\to 0$, e l'ordinamento degli infiniti è b_n, c_n, d_n, a_n.

(5) Poiché si ha $a_n = 2^n$, $b_n = n^2(1+o(1))$, $c_n = \frac{n(n+1)-1}{n} = n(1+o(1))$, $d_n = \sqrt{n+1}+\sqrt{n} = 2\sqrt{n}(1+o(1))$, ne segue che $\frac{d_n}{c_n}\to 0$, $\frac{c_n}{b_n}\to 0$, $\frac{b_n}{a_n}\to 0$, e l'ordinamento degli infiniti è d_n, c_n, b_n, a_n.

(6) Si ha $\frac{d_n}{b_n}\to 0$, $\frac{b_n}{c_n}\to 0$, $\frac{c_n}{a_n}\to 0$, e l'ordinamento degli infiniti è d_n, b_n, c_n, a_n.

(7) Poiché si ha $\frac{a_n}{c_n}\to 0$, $\frac{b_n}{a_n}\to 0$, $\frac{d_n}{b_n}\to 0$, $\frac{e_n}{d_n}\to 0$, l'ordinamento degli infinitesimi è c_n, a_n, b_n, d_n, e_n.

(8) Poiché si ha $a_n = \frac{1}{\log_3 n}(1+o(1))$, $b_n = (\frac{2}{3})^n(1+o(1))$, $c_n = -\frac{1}{n!}(1+o(1))$, $d_n = \frac{1}{\sqrt{n}}(1+o(1))$, ne segue che $\frac{d_n}{a_n} \to 0$, $\frac{b_n}{d_n} \to 0$, $\frac{c_n}{b_n} \to 0$, e l'ordinamento degli infinitesimi è a_n, d_n, b_n, c_n.

(9) Poiché si ha $a_n = \frac{1}{n!\cdot n}(1+o(1))$, $b_n = \frac{1}{n!}(1+o(1))$, $c_n = \frac{n}{n!}(1+o(1))$, $d_n = \frac{n!}{n^{3n}}$, ne segue che $\frac{b_n}{c_n} \to 0$, $\frac{a_n}{b_n} \to 0$, $\frac{d_n}{a_n} = \frac{(n!)^2\cdot n}{n^{3n}}(1+o(1)) = (\frac{n!}{n^n})^2 \frac{n}{n^n} \to 0$, e l'ordinamento degli infinitesimi è c_n, b_n, a_n, d_n.

(10) Poiché si ha $a_n = \frac{1}{n^2}(1+o(1))$, $b_n = \frac{1}{n^{8/3}}(1+o(1))$, $c_n = \frac{1}{2^n}(1+o(1))$, $d_n = \frac{2n}{n^2+n} = \frac{2}{n}(1+o(1))$, ne segue che $\frac{a_n}{d_n} \to 0$, $\frac{b_n}{a_n} \to 0$, $\frac{c_n}{b_n} \to 0$, e l'ordinamento degli infinitesimi è d_n, a_n, b_n, c_n.

(11) Poiché si ha $a_n = \frac{1}{n^{3/2}}(1+o(1))$, $b_n = \frac{1}{n\log_3 n}$, $c_n = \frac{(\log_3 n)^2}{n}$, $d_n = \frac{n}{n^2+n-1} = \frac{1}{n}(1+o(1))$, ne segue che $\frac{d_n}{c_n} \to 0$, $\frac{b_n}{d_n} \to 0$, $\frac{a_n}{b_n} \to 0$, e l'ordinamento degli infinitesimi è c_n, d_n, b_n, a_n.

(12) Poiché si ha $a_n = \frac{\log n}{2n^2}(1+o(1))$, $b_n = \frac{16(\log_3 n)^3}{n}(1+o(1))$, $c_n = \frac{(\log_3 n)^2}{n}(1+o(1))$, $d_n = \frac{2\log n(1+o(1))}{n^2\log n(1+o(1))} = \frac{2}{n^2}(1+o(1))$, ne segue che $\frac{c_n}{b_n} \to 0$, $\frac{a_n}{c_n} \to 0$, $\frac{d_n}{a_n} \to 0$, e l'ordinamento degli infinitesimi è b_n, c_n, a_n, d_n.

□

Svolgimento esercizio 1.5

(1) Si ha $a_{n+1} - a_n = \frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2-(n^2-1)}{n(n+1)} = \frac{1}{n(n+1)} > 0$, per ogni $n \in \mathbb{N}$, e quindi la successione è crescente. Alternativamente, osserviamo che $a_n = 1 - \frac{1}{n}$, e quindi la successione è crescente. Allora, $\min A = a_1 = 0$, $\sup A = \lim_{n\to\infty} a_n = 1$.

(2) Si ha $a_{n+1} - a_n = \frac{(n+1)^2+4}{5(n+1)^2} - \frac{n^2+4}{5n^2} = \frac{-4(2n+1)}{5n^2(n+1)^2} < 0$, per ogni $n \in \mathbb{N}$, e quindi la successione è decrescente. Alternativamente, osserviamo che $a_n = \frac{1}{5} + \frac{4}{5n^2}$, e quindi la successione è decrescente. Allora, $\max A = a_1 = 1$, $\inf A = \lim_{n\to\infty} a_n = \frac{1}{5}$.

(3) Si ha $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, e quindi la successione è decrescente. Allora, $\max A = a_1 = \sqrt{2} - 1$, $\inf A = \lim_{n\to\infty} a_n = 0$.

(4) Si ha $a_{n+1} - a_n = \frac{(n+1)^2+1}{n+1} - \frac{n^2+1}{n} = \frac{n\big((n+1)^2+1\big)-(n^2+1)(n+1)}{n(n+1)} = \frac{n^3+2n^2+2n-n^3-n^2-n-1}{n(n+1)} = \frac{n^2+n-1}{n(n+1)} > 0$, per ogni $n \in \mathbb{N}$, in quanto $x^2+x-1 = 0 \iff x = \frac{-1\pm\sqrt{5}}{2}$. Poiché la successione è crescente, $\min A = a_1 = 2$, $\sup A = \lim_{n\to\infty} a_n = +\infty$.

(5) Si ha $a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{(n+1)(n^2+1)-n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)} = \frac{n^3+n^2+n+1-n^3-2n^2-2n}{(n^2+2n+2)(n^2+1)} = \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} > 0$, per nessun $n \in \mathbb{N}$, in quanto $x^2+x-1 = 0 \iff x = \frac{-1\pm\sqrt{5}}{2}$. Poiché la successione è decrescente, $\max A = a_1 = \frac{1}{2}$, $\inf A = \lim_{n\to\infty} a_n = +\infty$.

(6) Si ha

(a) $a_{2n} = \frac{1}{(2n)^2+1}$ è decrescente, per cui $\begin{cases} \max\{a_{2n} : n \in \mathbb{N}\} = a_2 = \frac{1}{5}, \\ \inf\{a_{2n} : n \in \mathbb{N}\} = \lim_{n\to\infty} \frac{1}{(2n)^2+1} = 0. \end{cases}$

(b) $a_{2n-1} = -\frac{1}{(2n-1)^2+1}$ è crescente, per cui $\begin{cases} \min\{a_{2n-1} : n \in \mathbb{N}\} = a_1 = -\frac{1}{2}, \\ \sup\{a_{2n-1} : n \in \mathbb{N}\} = \lim_{n\to\infty} -\frac{1}{(2n-1)^2+1} = 0. \end{cases}$

Allora, $\min A = -\frac{1}{2}$, $\max A = \frac{1}{5}$.

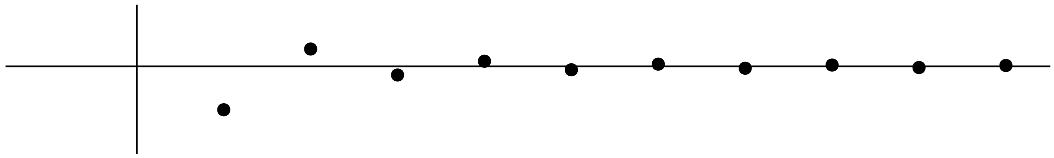

Figura 1: Grafico per l'esercizio (6)

(7) Si ha

(a) $a_{2n} = \frac{(2n)^2+4}{5(2n)^2} = \frac{1}{5} + \frac{1}{5n^2}$ è decrescente, per cui $\begin{cases} \max\{a_{2n} : n \in \mathbb{N}\} = a_2 = \frac{2}{5}, \\ \inf\{a_{2n} : n \in \mathbb{N}\} = \lim_{n\to\infty} \frac{(2n)^2+4}{5(2n)^2} = \frac{1}{5}. \end{cases}$

(b) $a_{2n-1} = -\frac{(2n-1)^2+4}{5(2n-1)^2} = -\frac{1}{5} - \frac{4}{5(2n-1)^2}$ è crescente, per cui $\begin{cases} \min\{a_{2n-1} : n \in \mathbb{N}\} = a_1 = -1, \\ \sup\{a_{2n-1} : n \in \mathbb{N}\} = \lim_{n\to\infty} -\frac{(2n-1)^2+4}{5(2n-1)^2} = -\frac{1}{5}. \end{cases}$

Allora, $\min A = -1$, $\max A = \frac{2}{5}$.

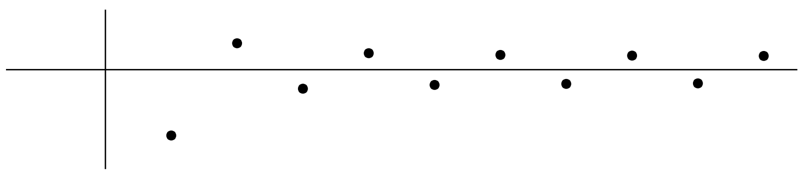

Figura 2: Grafico per l'esercizio (7)

(8) Si ha

(a) $a_{2n} = \sqrt{1 - \frac{1}{2n}}$ è crescente, per cui $\begin{cases} \min\{a_{2n} : n \in \mathbb{N}\} = a_2 = \frac{1}{\sqrt{2}}, \\ \sup\{a_{2n} : n \in \mathbb{N}\} = \lim_{n\to\infty} \sqrt{1 - \frac{1}{2n}} = 1. \end{cases}$

(b) $a_{2n-1} = -\sqrt{1 - \frac{1}{2n-1}}$ è decrescente, per cui $\begin{cases} \max\{a_{2n-1} : n \in \mathbb{N}\} = a_1 = 0, \\ \inf\{a_{2n-1} : n \in \mathbb{N}\} = \lim_{n\to\infty} -\sqrt{1 - \frac{1}{2n-1}} = -1. \end{cases}$

Allora, $\inf A = -1$, $\sup A = 1$.

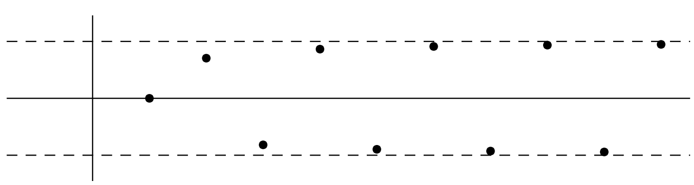

Figura 3: Grafico per l'esercizio (8)

(9) Si ha

(a) $n < 5$. Allora $a_n = \frac{5-n}{n+2} = -1 + \frac{7}{n+2}$ è decrescente, per cui $\begin{cases} \min\{a_n : n \in \mathbb{N}, n < 5\} = a_4 = \frac{1}{6}, \\ \max\{a_n : n \in \mathbb{N}, n < 5\} = a_1 = \frac{4}{3}. \end{cases}$

(b) $n \geq 5$. Allora $a_n = \frac{n-5}{n+2} = 1 - \frac{7}{n+2}$ è crescente, per cui $\begin{cases} \min\{a_n : n \in \mathbb{N}, n \geq 5\} = a_5 = 0, \\ \sup\{a_n : n \in \mathbb{N}, n \geq 5\} = \lim_{n\to\infty} \frac{n-5}{n+2} = 1. \end{cases}$
Allora, $\min A = 0$, $\max A = \frac{4}{3}$.

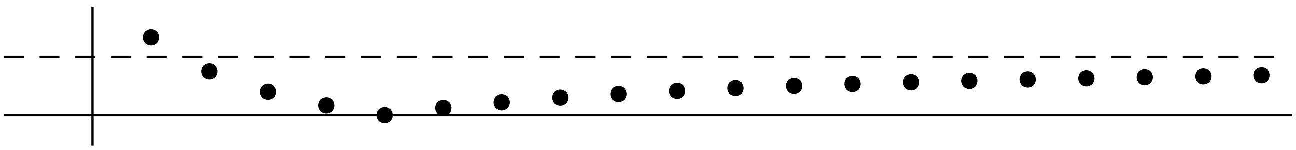

Figura 4: Grafico per l'esercizio (9)

(10) Si ha $a_{n+1} - a_n = \frac{n}{(n+1)^2-3} - \frac{n-1}{n^2-3} = \frac{n(n^2-3)-(n-1)(n^2+2n-2)}{(n^2+2n-2)(n^2-3)} = \frac{-(n^2-n+2)}{(n^2+2n-2)(n^2-3)} < 0$, per ogni $n \in \mathbb{N}$ tale che $n > 1$. Quindi $a_2 - a_1 > 0$, mentre $a_{n+1} - a_n < 0$, per ogni $n > 1$. Allora, $\max A = a_2 = 1$, ed essendo $a_1 = 0 = \lim_{n\to\infty} a_n$ si ha $\min A = a_1 = 0$.

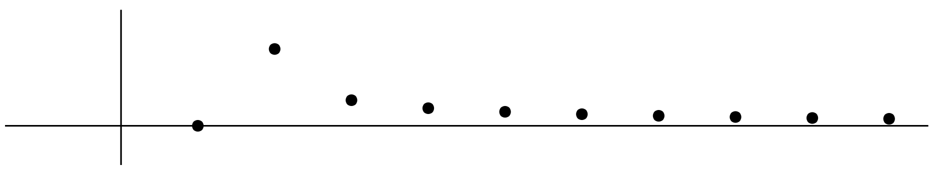

Figura 5: Grafico per l'esercizio (10)

□

Svolgimento esercizio 1.6

(1) Si ha $\left(1 + \dfrac{1}{2n}\right)^n = \sqrt{\left(1 + \dfrac{1}{2n}\right)^{2n}} \to \sqrt{e}$, sfruttando la continuità di $x \to \sqrt{x}$.

(2) Si ha $\left(1 + \dfrac{1}{2n+1}\right)^n = \sqrt{\dfrac{\left(1 + \frac{1}{2n+1}\right)^{2n+1}}{1 + \frac{1}{2n+1}}} \to \sqrt{e}$, sfruttando la continuità di $x \to \sqrt{x}$.

(3) Si ha $\left(1 + \dfrac{1}{n}\right)^{2n} = \left(\left(1 + \dfrac{1}{n}\right)^n\right)^2 \to e^2$.

(4) Si ha $\dfrac{(n+1)^n}{n^n+3} = \dfrac{(n+1)^n}{n^n(1+o(1))} = \left(\dfrac{n+1}{n}\right)^n(1+o(1)) = \left(1 + \dfrac{1}{n}\right)^n(1+o(1)) \to e$.

(5) Si ha $\dfrac{(n+1)^n}{n^n+n^2} = \dfrac{(n+1)^n}{n^n(1+o(1))} = \left(1 + \dfrac{1}{n}\right)^n(1+o(1)) \to e$.

(6) Si ha $\dfrac{(2n+1)^n}{(2n)^n+n^4} = \dfrac{(2n+1)^n}{(2n)^n(1+o(1))} = \left(\dfrac{2n+1}{2n}\right)^n(1+o(1)) = \left(1+\dfrac{1}{2n}\right)^n(1+o(1)) \to \sqrt{e}$, usando il risultato dell'esercizio (1).

(7) Si ha $\dfrac{(2n)^n + 2^n}{(2n+1)^n} = \dfrac{(2n)^n(1+o(1))}{(2n+1)^n} \to \dfrac{1}{\sqrt{e}}$, usando il risultato dell'esercizio (6).

(8) Si ha $\dfrac{(2n+1)^n}{2n^n+1} = \dfrac{(2n+1)^n}{2n^n(1+o(1))} = \dfrac{1}{2}\Big(\dfrac{2n+1}{n}\Big)^n(1+o(1)) = \dfrac{1}{2}\Big(2+\dfrac{1}{n}\Big)^n(1+o(1)) \to +\infty$, perché $\Big(2+\frac{1}{n}\Big)^n \geq 2^n \to +\infty$.

(9) Si ha $(n+1)^n - n! = n^n\Big(\dfrac{n+1}{n}\Big)^n - n! = en^n(1+o(1)) - n! = en^n(1+o(1)) \to +\infty$.

(10) Si ha $(n+1)^{n+1} - n^{n+1} = n^{n+1}\Big\{\Big(\dfrac{n+1}{n}\Big)^{n+1} - 1\Big\} = n^{n+1}\big(e + o(1) - 1\big) = (e-1)n^{n+1}(1+o(1)) \to +\infty$.

(11) Si ha $(n+1)^{n!} - n^{2n} = (n+1)^{n!}\Big\{1 - \Big(\dfrac{n^2}{(n+1)^{(n-1)!}}\Big)^n\Big\} = (n+1)^{n!}(1+o(1)) \to +\infty$, poiché $\frac{n^2}{(n+1)^{(n-1)!}} \to 0$.

(12) Si ha $\dfrac{(n+1)^n + n!}{n^n + 5^n - n!} = \dfrac{(n+1)^n(1+o(1))}{n^n(1+o(1))} = \Big(\dfrac{n+1}{n}\Big)^n(1+o(1)) \to e$.

(13) Si ha $\dfrac{(2n+1)^n + n! + 1}{(2n+2)^n - n! + n^2} = \dfrac{(2n+1)^n(1+o(1))}{(2n+2)^n(1+o(1))} = \Big(\Big(\dfrac{2n+2}{2n+1}\Big)^n\Big)^{-1}(1+o(1)) = \Big(\Big(1+\dfrac{1}{2n+1}\Big)^n\Big)^{-1}(1+o(1)) \to \dfrac{1}{\sqrt{e}}$, usando il risultato dell'esercizio (2).

(14) Si ha $\dfrac{(2n+1)^n + (2n)^n}{(2n+2)^n - (2n+1)^n} = \dfrac{1 + \frac{(2n)^n}{(2n+1)^n}}{\frac{(2n+2)^n}{(2n+1)^n} + 1} = \dfrac{1 + \big((\frac{2n+1}{2n})^n\big)^{-1}}{(\frac{2n+2}{2n+1})^n + 1} = \dfrac{1 + \big((1+\frac{1}{2n})^n\big)^{-1}}{(1+\frac{1}{2n+1})^n + 1} \to \dfrac{1 + \frac{1}{\sqrt{e}}}{\sqrt{e}+1} = \dfrac{1}{\sqrt{e}}$, usando i risultati degli esercizi (1) e (2).

(15) Si ha $\big(\sqrt{1+e^{-n}} - 1\big)\big(e^n - 3^n + n^2\big) = \dfrac{e^{-n}}{\sqrt{1+e^{-n}}+1}\big(e^n - 3^n + n^2\big) = \dfrac{e^{-n}}{2(1+o(1))}e^n(1+o(1)) = \dfrac{1}{2}(1+o(1)) \to \dfrac{1}{2}$.

(16) Si ha $\dfrac{(e^n+1)(n+\log n)}{(e^n+2^n)(2+\log n^6)} = \dfrac{e^n(1+o(1))n(1+o(1))}{e^n(1+o(1))6\log n(1+o(1))} = \dfrac{n}{6\log n}(1+o(1)) \to +\infty$.

(17) Si ha $\sqrt[n]{2n} = \sqrt[n]{2}\cdot\sqrt[n]{n} \to 1$.

(18) Si ha $\sqrt[n]{n^3} = \big(\sqrt[n]{n}\big)^3 \to 1$.

(19) Intanto $\sqrt[2n+1]{-n} = -\sqrt[2n+1]{n}$. Poi $1 \leq \sqrt[2n+1]{n} \leq \sqrt[2n+1]{2n+1} \to 1$. Per il teorema dei due carabinieri, si ha $\sqrt[2n+1]{n} \to 1$, e quindi $\sqrt[2n+1]{-n} \to -1$.

(20) Intanto $\sqrt[n]{n^2+3} = \sqrt[n]{n^2\Big(1+\frac{3}{n^2}\Big)} = \big(\sqrt[n]{n}\big)^2\sqrt[n]{1+\frac{3}{n^2}}$. Poi, definitivamente si ha $1 \leq \sqrt[n]{1+\frac{3}{n^2}} \leq \sqrt[n]{2} \to 1$. Per il teorema dei due carabinieri, si ha $\sqrt[n]{1+\dfrac{3}{n^2}} \to 1$, e quindi $\sqrt[n]{n^2+3} = \big(\sqrt[n]{n}\big)^2\sqrt[n]{1+\frac{3}{n^2}} \to 1$.

(21) Intanto $\sqrt[n]{2^n+n^2} = \sqrt[n]{2^n\Big(1+\frac{n^2}{2^n}\Big)} = 2\sqrt[n]{1+\frac{n^2}{2^n}}$. Poi, definitivamente si ha $1 \leq \sqrt[n]{1+\frac{n^2}{2^n}} \leq$

$\sqrt[n]{2} \to 1$. Per il teorema dei due carabinieri, si ha $\sqrt[n]{1+\dfrac{n^2}{2^n}} \to 1$, e quindi $\sqrt[n]{2^n+n^2} = 2\sqrt[n]{1+\frac{n^2}{2^n}} \to 2$.

(22) Intanto $\sqrt{n+\sqrt{n}}-\sqrt{n} = \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} = \frac{\sqrt{n}}{2\sqrt{n}(1+o(1))} \to \frac{1}{2}$. Inoltre, $\sqrt[n]{2^{n+1}+n^2} = \sqrt[n]{2^{n+1}\Big(1+\frac{n^2}{2^{n+1}}\Big)} = 2\sqrt[n]{2}\sqrt[n]{1+\frac{n^2}{2^{n+1}}}$. Poi, definitivamente si ha $1 \le \sqrt[n]{1+\frac{n^2}{2^{n+1}}} \le \sqrt[n]{2} \to 1$. Per il teorema dei due carabinieri, si ha $\sqrt[n]{1+\dfrac{n^2}{2^{n+1}}} \to 1$, e quindi $\big(\sqrt{n+\sqrt{n}}-\sqrt{n}\big)\sqrt[n]{2^{n+1}+n^2} \to 1$.

(23) Si ha $\dfrac{(n^2+5n+7)e^{1/n}}{(n+1)(\sqrt{n+3}-\sqrt{n})} = \dfrac{n^2(1+o(1))}{n(1+o(1))}\dfrac{\sqrt{n+3}+\sqrt{n}}{3} = n(1+o(1))\cdot 2\sqrt{n}(1+o(1)) = 2n^{3/2}(1+o(1)) \to +\infty$.

(24) Si ha $n^2(2n+\sqrt{n})^{1/n} - \cos(n^3) = n^2\sqrt[n]{2n}\Big(1+\dfrac{1}{2\sqrt{n}}\Big)^{1/n} - \cos(n^3) = n^2(1+o(1)) - \cos(n^3) = n^2(1+o(1)) \to +\infty$, poiché $1 \le (1+\frac{1}{2\sqrt{n}})^{1/n} \le 2^{1/n} \to 1$.

(25) Si ha $\big(4n^n-(n+1)^n\big)^{1/n} = n\Big\{4-\Big(\dfrac{n+1}{n}\Big)^n\Big\}^{1/n} = n(4-e+o(1))^{1/n} = n(1+o(1)) \to +\infty$, poiché $1 \le (4-e+o(1))^{1/n} \le 4^{1/n} \to 1$.

(26) Si ha $\big((n+1)^{n+1}-n^{n+1}\big)^{1/n} = n\sqrt[n]{n}\Big\{\Big(\dfrac{n+1}{n}\Big)^{n+1}-1\Big\}^{1/n} = n(1+o(1))(e+o(1)-1)^{1/n} = n(1+o(1)) \to +\infty$, poiché $1 \le (e-1+o(1))^{1/n} \le e^{1/n} \to 1$.

□

Svolgimento esercizio 1.7

(1) Si ha $\dfrac{n!}{n^{n/2}} = \dfrac{n^n e^{-n}\sqrt{2\pi n}(1+o(1))}{n^{n/2}} = n^{n/2}e^{-n}\sqrt{2\pi n}(1+o(1)) = \sqrt{\dfrac{n^n}{e^{2n}}}\sqrt{2\pi n}(1+o(1)) \to +\infty$.

(2) Si ha $\sqrt[n]{n!} = \sqrt[n]{n^n e^{-n}\sqrt{2\pi n}(1+o(1))} = \dfrac{n}{e}\big(\sqrt[n]{2\pi n}\big)^{1/2}(1+o(1)) \to +\infty$, in quanto $\sqrt[n]{2\pi n} \to 1$, e $1 \le \sqrt[n]{1+o(1)} \le \sqrt[n]{2} \to 1$.

(3) Si ha $\dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{n}\sqrt[n]{n^n e^{-n}\sqrt{2\pi n}(1+o(1))} = \dfrac{1}{e}\big(\sqrt[n]{2\pi n}\big)^{1/2}(1+o(1)) \to \dfrac{1}{e}$.

(4) Si ha $\dfrac{\sqrt[2n]{n!}}{n} = \dfrac{1}{n}\Big(\sqrt[n]{n^n e^{-n}\sqrt{2\pi n}(1+o(1))}\Big)^{1/2} = \dfrac{1}{n}\sqrt{\dfrac{n}{e}}\big(\sqrt[n]{2\pi n}\big)^{1/4}(1+o(1)) \to 0$.

(5) Si ha $\dfrac{1}{n}\sqrt[n]{\dfrac{(2n)!}{n!}} = \dfrac{1}{n}\sqrt[n]{\dfrac{(2n)^{2n}e^{-2n}\sqrt{2\pi\cdot 2n}(1+o(1))}{n^n e^{-n}\sqrt{2\pi n}(1+o(1))}} = \dfrac{1}{n}\dfrac{4n}{e}\sqrt[n]{\sqrt{2}}(1+o(1)) \to \dfrac{4}{e}$.

(6) $\sqrt[n]{\dbinom{2n}{n}} = \sqrt[n]{\dfrac{(2n)!}{n!\cdot n!}} = \sqrt[n]{\dfrac{(2n)^{2n}e^{-2n}\sqrt{2\pi\cdot 2n}(1+o(1))}{n^{2n}e^{-2n}2\pi n(1+o(1))}} = \dfrac{4}{(\sqrt[n]{\pi n})^{1/2}}(1+o(1)) = 4$.

(7) $\sqrt[n]{\dbinom{4n}{2n}} = \sqrt[n]{\dfrac{(4n)!}{(2n)!\cdot(2n)!}} = \sqrt[n]{\dfrac{(4n)^{4n}e^{-4n}\sqrt{2\pi\cdot 4n}(1+o(1))}{(2n)^{4n}e^{-4n}2\pi\cdot 2n(1+o(1))}} = \dfrac{16}{(\sqrt[n]{2\pi n})^{1/2}}(1+o(1)) = 16$.

□