Analisi Matematica II

Integrali doppi e tripli (svolgimenti)

Svolgimento esercizio 1

(1) Si ha $\iint_D x^3y\,dxdy = \int_0^1 x^3\,dx \cdot \int_{-1}^0 y\,dy = \big[\frac{x^4}{4}\big]_0^1 \cdot \big[\frac{1}{2}\,y^2\big]_{-1}^0 = -\frac{1}{8}$.

(2) Si ha $\iint_D \frac{x^2}{(y+1)^2}\,dxdy = \int_0^1 x^2\,dx \cdot \int_1^2 \frac{dy}{(y+1)^2} = \big[\frac{x^3}{3}\big]_0^1 \cdot \big[-\frac{1}{y+1}\big]_1^2 = \frac{1}{18}$.

(3) Si ha $\iint_D \sin x\cos y\,dxdy = \int_0^\pi \sin x\,dx \cdot \int_0^\pi \cos y\,dy = \big[-\cos x\big]_0^\pi \cdot \big[\sin y\big]_0^\pi = 0$.

(4) Si ha $\iint_D e^{x+y}\,dxdy = \int_0^{\frac{\pi}{2}} e^x\,dx \cdot \int_0^\pi e^y\,dy = \big[e^x\big]_0^{\frac{\pi}{2}} \cdot \big[e^y\big]_0^\pi = (e^{\frac{\pi}{2}}-1)(e^\pi-1)$.

(5) Si ha $\iint_D \frac{1}{1-x-y+xy}\,dxdy = \iint_D \frac{1}{(1-x)(1-y)}\,dxdy = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{1-x} \cdot \int_0^{\frac{1}{2}} \frac{dy}{1-y} = \big[-\log|1-x|\big]_{-\frac{1}{2}}^{\frac{1}{2}} \cdot \big[-\log|1-y|\big]_0^{\frac{1}{2}} = \log 2\cdot\log 3$.

(6) Si ha $\iint_D \frac{1}{(1+x+y)^2}\,dxdy = \int_1^2 \big[-\frac{1}{1+x+y}\big]_{y=x-1}^{y=3-x}\,dx = \int_1^2 \big(\frac{1}{2x}-\frac{1}{4}\big)\,dx = \big[\frac{1}{2}\,\log x-\frac{1}{4}\,x\big]_1^2 = \frac{1}{2}\,\log 2-\frac{1}{4}$.

(7) Si ha $\iint_D x\big(y+\sin(\pi y)\big)\,dxdy = \int_1^2 x\big[\frac{1}{2}\,y^2-\frac{1}{\pi}\,\cos(\pi y)\big]_{y=x-1}^{y=3-x}\,dx = \int_1^2 x\big(\frac{1}{2}\,(3-x)^2-\frac{1}{\pi}\,\cos(3\pi-\pi x)-\frac{1}{2}\,(x-1)^2+\frac{1}{\pi}\,\cos(\pi x-\pi)\big)\,dx = 2\,\int_1^2(2x-x^2)\,dx = 2\big[x^2-\frac{1}{3}\,x^3\big]_1^2 = \frac{4}{3}$.

(8) Si ha $\iint_D x^2e^{xy}\,dxdy = \int_0^1 \big(\int_0^x x^2e^{xy}\,dy\big)\,dx \overset{(a)}{=} \int_0^1 x\big(\int_0^{x^2} e^t\,dt\big)\,dx = \int_0^1 x\big[e^t\big]_0^{x^2}\,dx = \int_0^1 x\big(e^{x^2}-1\big)\,dx = \big[\frac{1}{2}\,e^{x^2}-\frac{1}{2}\,x^2\big]_0^1 = \frac{e}{2}-1$, dove in (a) si è usato il cambio di variabile $xy=t \implies xdy=dt$.

(9) Si ha $\iint_D x^2e^{xy}\,dxdy = 2\int_0^1 \big(\int_{2x}^2 x^2e^{xy}\,dy\big)\,dx \overset{(a)}{=} 2\int_0^1 x\big(\int_{2x^2}^{2x} e^t\,dt\big)\,dx = 2\int_0^1 x\big[e^t\big]_{2x^2}^{2x}\,dx = 2\int_0^1 x\big(e^{2x}-e^{2x^2}\big)\,dx = 2\big[\frac{1}{2}\,xe^{2x}-\frac{1}{4}\,e^{2x}-\frac{1}{4}\,e^{2x^2}\big]_0^1 = 1$, dove in (a) si è usato il cambio di variabile $xy=t \implies xdy=dt$.

(10) Si ha $\iint_D e^{y^2}\,dxdy = \int_0^2 e^{y^2}\big[x\big]_{-y/2}^{y/2}\,dy = \int_0^2 ye^{y^2}\,dy \overset{(a)}{=} \frac{1}{2}\int_0^4 e^t\,dt = \frac{1}{2}\big[e^t\big]_0^4 = \frac{1}{2}(e^4-1)$, dove in (a) si è usato il cambio di variabile $y^2=t \implies 2ydy=dt$.

(11) Si ha $\iint_D xy\,dxdy = \int_{-1}^0 \big(\int_{-x^2}^{1+x} xy\,dy\big)\,dx = \frac{1}{2}\,\int_{-1}^0 x\big[y^2\big]_{y=-x^2}^{y=1+x}\,dx = \frac{1}{2}\,\int_{-1}^0 \big(x+2x^2+x^3-x^5\big)\,dx = \frac{1}{2}\,\big[\frac{1}{2}\,x^2+\frac{2}{3}\,x^3+\frac{1}{4}\,x^4-\frac{1}{6}\,x^6\big]_{-1}^0 = -\frac{1}{8}$.

(12) Si ha $\iint_D(x^2+y)\,dxdy = \int_{-1}^1 \big(\int_0^{1-x^2}(x^2+y)\,dy\big)\,dx = \int_{-1}^1 \big[x^2y+\frac{1}{2}\,y^2\big]_{y=0}^{y=1-x^2}\,dx = \int_{-1}^1(1-x^4)\,dx = \big[x-\frac{1}{5}\,x^5\big]_{-1}^1 = \frac{4}{5}$.

(13) Si ha $\iint_D e^{-(x-y)}\,dxdy = \int_0^{2/3} e^{-x}\,\big(\int_x^{2-2x} e^y\,dy\big)\,dx = \int_0^{2/3} e^{-x}\big[e^y\big]_{y=x}^{y=2-2x}\,dx = \int_0^{2/3} \big(e^{2-3x}-1\big)\,dx = \big[-\frac{1}{3}\,e^{2-3x}-x\big]_0^{2/3} = \frac{1}{3}\,e^2-1$.

(14) Si ha $\iint_D x^2\,dxdy = \int_0^{2\pi} x^2\,\big(\int_{\sin x}^{1+\sin x}\,dy\big)\,dx = \int_0^{2\pi} x^2\,dx = \big[\frac{1}{3}\,x^3\big]_0^{2\pi} = \frac{8}{3}\,\pi^3$.

(15) Si ha $\iint_D y^2\,dxdy = \int_0^{2\pi}\,\big[\frac{1}{3}\,y^3\big]_{\sin x}^{1+\sin x}\,dx = \frac{1}{3}\,\int_0^{2\pi}(1+3\sin x+3\sin^2 x)\,dx = \frac{1}{3}\,\big[x-3\cos x+\frac{3}{2}\,x-\frac{3}{4}\sin 2x\big]_0^{2\pi} = \frac{5}{3}\,\pi$.

(16) Si ha $\iint_D \sin(x+y)\,dxdy = \int_0^{\pi/2}\big(\int_0^x \sin(x+y)\,dy\big)\,dx = \int_0^{\pi/2}\big[-\cos(x+y)\big]_{y=0}^{y=x}\,dx = \int_0^{\pi/2}\big(\cos x-\cos(2x)\big)\,dx = \big[\sin x-\frac{1}{2}\,\sin(2x)\big]_0^{\pi/2} = 1$.

(17) Si ha $\iint_D y\cos x\,dxdy = \int_{-\sqrt{\pi}}^{2\sqrt{\pi}} y\big[\sin x\big]_{x=0}^{x=y^2+\pi} dy = -\int_{-\sqrt{\pi}}^{2\sqrt{\pi}} y\sin(y^2)\,dy \overset{(a)}{=} -\frac{1}{2}\int_{-\pi}^{4\pi}\sin t\,dt = \frac{1}{2}\big[\cos t\big]_{-\pi}^{4\pi} = 1$, dove in (a) si è usato il cambio di variabile $y^2 = t \implies 2ydy = dt$.

$\square$

Svolgimento esercizio 2

(1) Si ha $\iint_D \frac{|x|}{\sqrt{1+y^2}}\,dxdy = \int_{-1}^{1}|x|\,dx \cdot \int_0^1 \frac{dy}{\sqrt{1+y^2}} \overset{(a)}{=} 2\int_0^1 x\,dx \cdot \int_1^{3+\sqrt{10}} \frac{dt}{t} = \big[x^2\big]_0^1 \cdot \big[\log t\big]_1^{3+\sqrt{10}} = \log(3+\sqrt{10})$, dove in (a) si è usato il cambio di variabile $\sqrt{1+y^2} = t - y \implies \sqrt{1+y^2} = \frac{t^2+1}{2t}$, $dy = \frac{t^2+1}{2t^2}\,dt$.

(2) Si ha $\iint_D \frac{1}{(x-y+6)^2}\,dxdy = \int_0^1 \big[\frac{1}{x-y+6}\big]_{y=1}^{y=2} dx = \int_0^1 \big(\frac{1}{x+4} - \frac{1}{x+5}\big)\,dx = \big[\log\frac{x+4}{x+5}\big]_0^1 = \log\frac{25}{24}$.

(3) Si ha $\iint_D \frac{1-3x^2}{(x-y+6)^2}\,dxdy = \int_0^1 (1-3x^2)\big[\frac{1}{x-y+6}\big]_{y=1}^{y=2} dx = \int_0^1 (1-3x^2)\big(\frac{1}{x+4} - \frac{1}{x+5}\big)\,dx = -\int_0^1 \frac{3x^2-1}{(x+4)(x+5)}\,dx = -\int_0^1 \big(3 - \frac{47}{x+4} + \frac{20}{x+5}\big)\,dx = \big[-3x + 47\log(x+4) - 20\log(x+5)\big]_0^1 = -3 - 47\log 4 + 67\log 5 - 20\log 6$.

(4) Si ha $\iint_D \frac{1-3x^2}{x-y+6}\,dxdy = \int_0^1 (1-3x^2)\big[-\log|x-y+6|\big]_{y=1}^{y=2} dx = \int_0^1 (3x^2-1)\big(\log(x+4) - \log(x+5)\big)\,dx = \big[(x^3-x)\big(\log(x+4) - \log(x+5)\big)\big]_0^1 - \int_0^1 (x^3-x)\big(\frac{1}{x+4} - \frac{1}{x+5}\big)\,dx = -\int_0^1 \frac{x^3-x}{(x+4)(x+5)}\,dx = -\int_0^1 \big(x - 9 - \frac{104}{x+4} + \frac{175}{x+5}\big)\,dx = \big[-\frac{1}{2}x^2 + 9x + 104\log(x+4) - 175\log(x+5)\big]_0^1 = \frac{17}{2} - 104\log 4 + 279\log 5 - 175\log 6$.

(5) Si ha $\iint_D \frac{1}{2x+y+1}\,dxdy = \int_0^1 \big[\log(2x+y+1)\big]_{y=0}^{y=1} dx = \int_0^1 \big(\log(2x+2) - \log(2x+1)\big)\,dx \overset{(a)}{=} \frac{1}{2}\int_4^6 \log t\,dt - \frac{1}{2}\int_3^5 \log t\,dt = \frac{1}{2}\big[t\log t - t\big]_4^6 - \frac{1}{2}\big[t\log t - t\big]_3^5 = 3\log 6 - 2\log 4 - \frac{5}{2}\log 5 + \frac{3}{2}\log 3$, dove in (a) si è eseguito il cambio di variabile $2x+2 = t$, risp. $2x+1 = t$, negli integrali.

(6) Si ha $\iint_D \frac{x}{2x+y+1}\,dxdy = \int_0^1 x\big[\log(2x+y+1)\big]_{y=0}^{y=1} dx = \int_0^1 x\big(\log(2x+2) - \log(2x+1)\big)\,dx \overset{(a)}{=} \frac{1}{4}\int_4^6 (t-2)\log t\,dt - \frac{1}{4}\int_3^5 (t-1)\log t\,dt = \frac{1}{4}\big[\frac{1}{2}(t-2)^2\log t - \frac{1}{2}\int \frac{(t-2)^2}{t}\,dt\big]_4^6 - \frac{1}{4}\big[\frac{1}{2}(t-1)^2\log t - \frac{1}{2}\int \frac{(t-1)^2}{t}\,dt\big]_3^5 = 2\log 6 - \frac{1}{2}\log 4 - \frac{1}{8}\big[\frac{1}{2}t^2 - 4t + 4\log t\big]_4^6 - 2\log 5 + \frac{1}{8}\log 3 + \frac{1}{8}\big[\frac{1}{2}t^2 - 2t + \log t\big]_3^5 = \frac{1}{4} + \frac{3}{2}\log 6 - \frac{15}{8}\log 5 + \frac{3}{8}\log 3$, dove in (a) si è eseguito il cambio di variabile $2x+2 = t$, risp. $2x+1 = t$, negli integrali.

(7) Si ha $\iint_D \frac{1}{y^2+2x+1}\,dxdy = \int_0^1 \big[\frac{1}{2}\log(y^2+2x+1)\big]_{x=1}^{x=2} dy = \frac{1}{2}\int_0^1 \big(\log(y^2+5) - \log(y^2+3)\big)\,dx = \frac{1}{2}\big[y\log(y^2+5) - y\log(y^2+3)\big]_0^1 - \frac{1}{2}\int_0^1 \big(\frac{2y^2}{y^2+5} - \frac{2y^2}{y^2+3}\big)\,dy = \frac{1}{2}\log\frac{6}{4} + \int_0^1 \big(1 + \frac{5}{y^2+5} - 1 - \frac{3}{y^2+3}\big)\,dy \overset{(a)}{=} \frac{1}{2}\log\frac{3}{2} + \big[\sqrt{5}\,\mathrm{arctg}\,\frac{y}{\sqrt{5}} - \sqrt{3}\,\mathrm{arctg}\,\frac{y}{\sqrt{3}}\big]_0^1 = \frac{1}{2}\log\frac{3}{2} + \sqrt{5}\,\mathrm{arctg}\,\frac{1}{\sqrt{5}} - \sqrt{3}\,\mathrm{arctg}\,\frac{1}{\sqrt{3}}$, dove in (a) si è usato il fatto che $\int \frac{dt}{t^2+c} = \frac{1}{\sqrt{c}}\,\mathrm{arctg}\,\frac{t}{\sqrt{c}}$.

(8) Si ha $\iint_D \frac{1}{y^2+2x+1}\,dxdy = \frac{1}{2}\int_1^2 \big(\int_0^1 \big(1 - \frac{y^2+1}{y^2+2x+1}\big)\,dx\big)\,dy = \frac{1}{2} - \frac{1}{4}\int_0^1 (y^2+1)\big[\log(y^2+2x+1)\big]_{x=1}^{x=2} dy = \frac{1}{2} - \frac{1}{4}\int_0^1 (y^2+1)\big(\log(y^2+5) - \log(y^2+3)\big)\,dx \overset{(a)}{=} \frac{1}{2} - \frac{1}{4}\big[\big(\frac{y^3}{3}+y\big)\log(y^2+5) - \frac{2}{3}\big(\frac{y^3}{3} - 5y\big) - 2y - \frac{20}{3\sqrt{5}}\,\mathrm{arctg}\,\frac{y}{\sqrt{5}}\big]_0^1 - \frac{1}{4}\big[\big(\frac{y^3}{3}+y\big)\log(y^2+3) - \frac{2}{3}\big(\frac{y^3}{3} - 3y\big) - 2y\big]_0^1 = \frac{1}{2} - \frac{1}{4}\big[\big(\frac{y^3}{3}+y\big)\log\frac{y^2+5}{y^2+3} + \frac{4}{3}y - \frac{20}{3\sqrt{5}}\,\mathrm{arctg}\,\frac{y}{\sqrt{5}}\big]_0^1 = \frac{1}{2} - \frac{1}{3}\log\frac{3}{2} - \frac{1}{3} + \frac{5}{3\sqrt{5}}\,\mathrm{arctg}\,\frac{1}{\sqrt{5}} = \frac{1}{6} - \frac{1}{3}\log\frac{3}{2} + \frac{\sqrt{5}}{3}\,\mathrm{arctg}\,\frac{1}{\sqrt{5}}$, dove in (a) si è usato il fatto che $\int (y^2+1)\log(y^2+c)\,dy = \big(\frac{y^3}{3}+y\big)\log(y^2+c) - \int \big(\frac{y^3}{3}+y\big)\frac{2y}{y^2+c}\,dy = \big(\frac{y^3}{3}+y\big)\log(y^2+c) - \int \big(\frac{2}{3}\frac{y^4-c^2}{y^2+c} + 2 + \frac{\frac{2}{3}c^2-2c}{y^2+c}\big)\,dy = \big(\frac{y^3}{3}+y\big)\log(y^2+c) - \frac{2}{3}\big(\frac{y^3}{3} - cy\big) - 2y - \big(\frac{2}{3}c^2 - 2c\big)\frac{1}{\sqrt{c}}\,\mathrm{arctg}\,\frac{y}{\sqrt{c}}$.

(9) Si ha $\iint_D \log(x+y+5)\,dxdy = \int_{-2}^{1}\big(\int_x^1 \log(x+y+5)\,dy\big)\,dx = \int_{-2}^{1}\big[(x+y+5)\log(x+y+5) - y\big]_{y=x}^{y=1}\,dx = \int_{-2}^{1}\big((x+6)\log(x+6) - 1 - (2x+5)\log(2x+5) + x\big)\,dx \overset{(a)}{=} \big[\frac{1}{2}\,x^2 - x\big]_{-2}^{1} + \int_4^7 t\log t\,dt - \frac{1}{2}\int_1^7 t\log t\,dt \overset{(b)}{=} -\frac{9}{2} + \big[\frac{1}{2}\,t^2\log t - \frac{1}{4}\,t^2\big]_4^7 - \frac{1}{2}\big[\frac{1}{2}\,t^2\log t - \frac{1}{4}\,t^2\big]_1^7 = -\frac{27}{4} + \frac{49}{4}\log 7 - 8\log 4$, dove si sono usati in (a) i cambi di variabile $x+6=t$, $2x+5=t$, e in (b) il risultato $\int t\log t\,dt = \frac{1}{2}\,t^2\log t - \frac{1}{4}\,t^2$.

(10) Si ha $\iint_D \frac{y^2+1}{y^2+2x+1}\,dxdy = 2\int_0^1\big(\int_y^1 \frac{y^2+1}{y^2+2x+1}\,dx\big)\,dy = \int_0^1 (y^2+1)\big[\log(y^2+2x+1)\big]_{x=y}^{x=1}\,dy = \int_0^1 (y^2+1)\big(\log(y^2+3) - 2\log(y+1)\big)\,dy = \big[\big(\frac{y^3}{3}+y\big)\big(\log(y^2+3) - 2\log(y+1)\big)\big]_0^1 - \int_0^1 \big(\frac{y^3}{3}+y\big)\big(\frac{2y}{y^2+3} - \frac{2}{y+1}\big) = -\frac{2}{3}\int_0^1 \big(y^2 - \frac{y(y^2+3)}{y+1}\big)\,dy = -\frac{2}{3}\int_0^1 \big(y - 4 + \frac{4}{y+1}\big)\,dy = -\frac{2}{3}\,\big[\frac{1}{2}\,y^2 - 4y + 4\log(y+1)\big]_0^1 = \frac{7}{3} - \frac{8}{3}\,\log 2$.

(11) Si ha $\iint_D \frac{1}{(1+x+y)^2}\,dxdy = \int_0^1\big(\int_0^{2\sqrt{x}} \frac{1}{(1+x+y)^2}\,dy\big)\,dx = \int_0^1 \big[-\frac{1}{1+x+y}\big]_0^{2\sqrt{x}}\,dx = \int_0^1 \big(\frac{1}{1+x} - \frac{1}{1+x+2\sqrt{x}}\big)\,dx \overset{(a)}{=} \big[\log(1+x)\big]_0^1 - \int_0^1 \frac{2t\,dt}{1+2t+t^2} = \log 2 - \int_0^1 \big(\frac{2}{t+1} - \frac{2}{(t+1)^2}\big)\,dt = \log 2 - 2\big[\log(t+1) + \frac{1}{t+1}\big]_0^1 = 1 - \log 2$, dove in (a) si è usato il cambio di variabile $x = t^2$.

(12) Si ha $\iint_D x\big(y+\sin(\pi y)\big)\,dxdy = \int_0^1 x\,\big(\int_0^{2\sqrt{x}} (y+\sin\pi y)\,dy\big)\,dx = \int_0^1 x\big[\frac{1}{2}\,y^2 - \frac{1}{\pi}\,\cos\pi y\big]_0^{2\sqrt{x}}\,dx = \int_0^1 \big(2x^2 + \frac{1}{\pi}\,x - \frac{1}{\pi}\,x\cos(2\pi\sqrt{x})\big)\,dx \overset{(a)}{=} \big[\frac{2}{3}\,x^3 + \frac{1}{2\pi}\,x^2\big]_0^1 - \frac{1}{\pi}\int_0^1 t^2\cos(2\pi t)\cdot 2t\,dt \overset{(b)}{=} \frac{2}{3} + \frac{1}{2\pi} - \frac{2}{\pi}\big[\frac{1}{2\pi}\,t^3\sin(2\pi t) + \frac{3}{(2\pi)^2}\,t^2\cos(2\pi t) - \frac{6}{(2\pi)^3}\,t\sin(2\pi t) - \frac{6}{(2\pi)^4}\cos(2\pi t)\big]_0^1 = \frac{2}{3} + \frac{1}{2\pi} - \frac{3}{2\pi^3}$, dove si sono usati in (a) il cambio di variabile $x = t^2$, e in (b) il risultato $\int t^3\cos(2\pi t)\,dt = \frac{1}{2\pi}\,t^3\sin(2\pi t) + \frac{3}{(2\pi)^2}\,t^2\cos(2\pi t) - \frac{6}{(2\pi)^3}\,t\sin(2\pi t) - \frac{6}{(2\pi)^4}\cos(2\pi t)$.

(13) Si ha $\iint_D |x|y\,dxdy = 2\int_0^{2\pi} y\,\big(\int_0^{1-\cos y} x\,dx\big)\,dy = \int_0^{2\pi} y\big[x^2\big]_0^{1-\cos y}\,dy = \int_0^{2\pi} y(1-\cos y)^2\,dy = \int_0^{2\pi} y\big(\frac{3}{2} - 2\cos y + \frac{1}{2}\,\cos 2y\big)\,dy \overset{(a)}{=} \big[\frac{3}{4}\,y^2 - 2y\sin y - 2\cos y + \frac{1}{4}\,y\sin(2y) + \frac{1}{8}\cos(2y)\big]_0^{2\pi} = 3\pi^2$, dove in (a) si è usato il risultato $\int y\cos(cy)\,dy = \frac{1}{c}\,y\sin(cy) + \frac{1}{c^2}\cos(cy)$.

(14) Posto $D' := \big\{(x,y)\in\mathbb{R}^2 : 1 \le x+y \le 2, x\ge 0, y\ge 0\big\}$, si ha $\iint_D (x-1)|y|\,dxdy = -\iint_D |y|\,dxdy = -4\iint_{D'} y\,dxdy = 2\int_0^1 \big[y^2\big]_0^{1-x}\,dx - 2\int_0^1 \big[y^2\big]_0^{2-x}\,dx = 2\int_0^1 (1-x)^2\,dx - 2\int_0^1 (2-x)^2\,dx = 2\big[x - x^2 + \frac{1}{3}\,x^3\big]_0^1 - 2\big[4x - 2x^2 + \frac{1}{3}\,x^3\big]_0^2 = -\frac{14}{3}$.

(15) Si ha $\int_0^1\big(\int_x^{x^2} \frac{1}{2x+y+1}\,dy\big)dx = \int_0^1 \big[\log(2x+y+1)\big]_{y=x}^{x^2}\,dx = \int_0^1 \big(2\log(x+1) - \log(3x+1)\big)\,dx = \big[2x\log(x+1) - x\log(3x+1)\big]_0^1 - \int_0^1 \big(\frac{2x}{x+1} - \frac{3x}{3x+1}\big)\,dx = \int_0^1 \big(-1 + \frac{2}{x+1} - \frac{1}{3x+1}\big)\,dx = \big[-x + 2\log(x+1) - \frac{1}{3}\,\log(3x+1)\big]_0^1 = \frac{4}{3}\log 2 - 1$.

(16) Si ha $\int_0^1\big(\int_x^{x^2} \frac{x}{2x+y+1}\,dy\big)dx = \int_0^1 x\big[\log(2x+y+1)\big]_{y=x}^{x^2}\,dx = \int_0^1 x\big(2\log(x+1) - \log(3x+1)\big)\,dx = \big[x^2\log(x+1) - \frac{1}{2}\,x^2\log(3x+1)\big]_0^1 - \int_0^1 \big(\frac{x^2}{x+1} - \frac{3}{2}\,\frac{x^2}{3x+1}\big)\,dx = -\int_0^1 \big(x - 1 + \frac{1}{x+1} - \frac{1}{6}\,(3x-1) - \frac{1}{6}\,\frac{1}{3x+1}\big)\,dx = -\big[\frac{1}{4}\,x^2 - \frac{5}{6}\,x + \log(x+1) - \frac{1}{18}\log(3x+1)\big]_0^1 = \frac{7}{12} - \frac{8}{9}\log 2$.

(17) Si ha $\int_0^1\big(\int_x^{\sqrt{x}} \frac{1}{y^2+2x+1}\,dy\big)dx = \int_0^1\big(\int_{y^2}^{y} \frac{1}{y^2+2x+1}\,dx\big)dy = \frac{1}{2}\int_0^1 \big[\log(y^2+2x+1)\big]_{x=y^2}^{x=y}\,dy = \frac{1}{2}\int_0^1 \big(2\log(y+1) - \log(2y^2+1)\big)\,dy = \big[y\log(y+1) - \frac{1}{2}\,y\log(2y^2+1)\big]_0^1 - \int_0^1 \big(\frac{y}{y+1} - \frac{2y^2}{2y^2+1}\big)\,dy = \log 2 - \frac{1}{2}\log 3 - \int_0^1 \big(-\frac{1}{y+1} + \frac{1}{2y^2+1}\big)\,dy = \log 2 - \frac{1}{2}\log 3 + \big[\log(y+1) - \frac{1}{\sqrt{2}}\,\mathrm{arctg}(y\sqrt{2})\big]_0^1 = 2\log 2 - \frac{1}{2}\log 3 - \frac{1}{\sqrt{2}}\,\mathrm{arctg}\,\sqrt{2}$.

(18) Si ha $\iint_D \frac{x^2}{(y+1)^2}\,dxdy = \int_{-1}^{3} x^2\big(\int_{x^2}^{2x+3} \frac{1}{(y+1)^2}\,dy\big)dx = \int_{-1}^{3}\big[-\frac{1}{y+1}\big]_{y=x^2}^{2x+3}\,dx = \int_{-1}^{3} x^2\big(\frac{1}{x^2+1} - \frac{1}{2x+4}\big)\,dx = \int_{-1}^{3}\big(1 - \frac{1}{x^2+1} - \frac{1}{2}(x-2) - \frac{2}{x+2}\big)\,dx = \big[2x - \frac{1}{4}\,x^2 - 2\log(x+2) - \mathrm{arctg}\,x\big]_{-1}^{3} = 6 - \log 5 - \mathrm{arctg}\,3 - \frac{\pi}{4}$.

(19) Si ha $\iint_D \frac{1}{2x+y+2}\,dxdy = \int_{-1}^{3}\big(\int_{x^2}^{2x+3}\frac{1}{2x+y+2}\,dy\big)dx = \int_{-1}^{3}\big[\log(2x+y+2)\big]_{y=x^2}^{2x+3}dx = \int_{-1}^{3}\big(\log(4x+5)-\log(x^2+2x+2)\big)\,dx = \big[x\log(4x+5)-x\log(x^2+2x+2)\big]_{-1}^{3} - \int_{-1}^{3}\big(\frac{4x}{4x+5}-\frac{x(2x+2)}{x^2+2x+2}\big)\,dx = \int_{-1}^{3}\big(1+\frac{5}{4x+5}-\frac{2x+2}{x^2+2x+2}-\frac{2}{(x+1)^2+1}\big)\,dx = \big[x+\frac{5}{4}\log(4x+5)-\log(x^2+2x+2)-2\,\mathrm{arctg}(x+1)\big]_{-1}^{3} = 4+\frac{1}{4}\log 17 - 2\,\mathrm{arctg}\,4$.

(20) Si ha $\iint_D \frac{1}{(2x-y+8)^2}\,dxdy = \int_{-1}^{3}\big(\int_{x^2}^{2x+3}\frac{1}{(2x-y+8)^2}\,dy\big)dx = \int_{-1}^{3}\big[\frac{1}{2x-y+8}\big]_{y=x^2}^{2x+3}dx = \int_{-1}^{3}\big(\frac{1}{5}+\frac{1}{x^2-2x-8}\big)\,dx = \int_{-1}^{3}\big(\frac{1}{5}+\frac{1}{6}\,\frac{1}{x-4}-\frac{1}{6}\,\frac{1}{x+2}\big)\,dx = \big[\frac{1}{5}\,x+\frac{1}{6}\,\log\frac{4-x}{x+2}\big]_{-1}^{3} = \frac{4}{5}-\frac{1}{3}\log 5$.

(21) Si ha $\iint_D \frac{1-3x^2}{(2x-y+8)^2}\,dxdy = \int_{-1}^{3}(1-3x^2)\big(\int_{x^2}^{2x+3}\frac{1}{(2x-y+8)^2}\,dy\big)dx = \int_{-1}^{3}(1-3x^2)\big[\frac{1}{2x-y+8}\big]_{y=x^2}^{2x+3}dx = \int_{-1}^{3}(1-3x^2)\big(\frac{1}{5}+\frac{1}{x^2-2x-8}\big)\,dx = \frac{1}{5}\,\big[x-x^3\big]_{-1}^{3} - \int_{-1}^{3}\big(3+\frac{47}{6}\,\frac{1}{x-4}-\frac{11}{6}\,\frac{1}{x+2}\big)\,dx = -\frac{24}{5}-\big[3x+\frac{47}{6}\log(4-x)-\frac{11}{6}\log(x+2)\big]_{-1}^{3} = -\frac{84}{5}+\frac{29}{3}\log 5$.

(22) Si ha $\iint_D \frac{1-3x^2}{2x-y+8}\,dxdy = \int_{-1}^{3}(1-3x^2)\big(\int_{x^2}^{2x+3}\frac{1}{2x-y+8}\,dy\big)dx = \int_{-1}^{3}(3x^2-1)\big[\log|2x-y+8|\big]_{y=x^2}^{2x+3}dx = \int_{-1}^{3}(3x^2-1)(\log 5-\log|x^2-2x-8|)\,dx = \big[(x^3-x)(\log 5-\log|x^2-2x-8|)\big]_{-1}^{3} + \int_{-1}^{3}\frac{(x^3-x)(2x-2)}{x^2-2x-8}\,dx = 2\int_{-1}^{3}\big(x^2+x+9+\frac{30}{x-4}-\frac{3}{x+2}\big)\,dx = 2\big[\frac{1}{3}\,x^3+\frac{1}{2}\,x^2+9x+30\log(4-x)-3\log(x+2)\big]_{-1}^{3} = \frac{296}{3}-66\log 5$.

□

Svolgimento esercizio 3

(1) Si ha $\iint_D \sqrt{x^2+y^2}\,dxdy \overset{(a)}{=} \int_0^{2\pi}\big(\int_0^3 \varrho\,\varrho\,d\varrho\big)\,d\vartheta = 2\pi\big[\frac{1}{3}\,\varrho^3\big]_0^3 = 18\pi$, dove in (a) si è usato il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$.

(2) Si ha $\iint_D 2xy\,dxdy \overset{(a)}{=} \int_{\pi/4}^{7\pi/4}\big(\int_0^3 2\varrho^2\cos\vartheta\sin\vartheta\,\varrho\,d\varrho\big)\,d\vartheta = \big[\sin^2\vartheta\big]_{\pi/4}^{7\pi/4}\cdot\big[\frac{1}{4}\,\varrho^4\big]_0^3 = -\frac{81}{8}$, dove in (a) si è usato il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$.

(3) Si ha $\iint_D \big|\frac{x}{y}\big|\,dxdy \overset{(a)}{=} 2\int_{\pi/4}^{\pi/2}\big(\int_1^4 \frac{\cos\vartheta}{\sin\vartheta}\,\varrho\,d\varrho\big)\,d\vartheta = \big[\log|\sin\vartheta|\big]_{\pi/4}^{\pi/2}\cdot\big[\varrho^2\big]_1^4 = \frac{15}{2}\,\log 2$, dove in (a) si sono usate le simmetrie del dominio e della funzione, e il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$.

(4) Si ha $\iint_D \frac{1}{x}\,dxdy \overset{(a)}{=} 2\int_0^{\pi/4}\big(\int_1^2 \frac{1}{\varrho\cos\vartheta}\,\varrho\,d\varrho\big)\,d\vartheta = 2\int_0^{\pi/4}\frac{\cos\vartheta}{1-\sin^2\vartheta}\,d\vartheta\cdot\big[\varrho\big]_1^2 \overset{(b)}{=} 2\int_0^{\sqrt{2}/2}\frac{dt}{1-t^2} = 2\int_0^{\sqrt{2}/2}\big(\frac{1}{2}\,\frac{1}{t+1}-\frac{1}{2}\,\frac{1}{t-1}\big)\,dt = \big[\log\frac{1+t}{1-t}\big]_0^{\sqrt{2}/2} = 2\log(2+\sqrt{2})-\log 2$, dove in (a) si sono usate le simmetrie del dominio e della funzione, e il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$, e in (b) il cambio di variabile $\sin\vartheta = t \implies \cos\vartheta\,d\vartheta = dt$.

(5) Si ha $\iint_D (x-y)^2\,dxdy \overset{(a)}{=} \int_{-2}^{2}\big(\int_{-2}^{2}(x-y)^2\,dy\big)\,dx - \int_0^{2\pi}\big(\int_0^3 \varrho^2(\cos\vartheta-\sin\vartheta)^2\,\varrho\,d\varrho\big)\,d\vartheta = \frac{1}{3}\int_{-2}^{2}\big[(y-x)^3\big]_{y=-2}^{y=2}dx - \int_0^{2\pi}(1-2\cos\vartheta\sin\vartheta)\,d\vartheta\cdot\big[\frac{1}{4}\,\varrho^4\big]_0^1 = \frac{16}{3}\int_{-2}^{2}(x+1)\,dx - \frac{1}{4}\,\big[\vartheta-\sin^2\vartheta\big]_0^{2\pi} = \frac{64}{3}-\frac{\pi}{2}$, dove in (a) si è usato il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$.

(6) Si ha $\int_0^4\big(\int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}}\sqrt{y^2+x^2}\,dy\big)dx \overset{(a)}{=} \int_{-\pi/2}^{\pi/2}\big(\int_0^{4\cos\vartheta}\varrho\,\varrho\,d\varrho\big)\,d\vartheta = \int_{-\pi/2}^{\pi/2}\big[\frac{1}{3}\,\varrho^3\big]_0^{4\cos\vartheta}d\vartheta = \frac{64}{3}\int_{-\pi/2}^{\pi/2}\cos^3\vartheta\,d\vartheta = \frac{64}{3}\int_{-\pi/2}^{\pi/2}(1-\sin^2\vartheta)\cos\vartheta\,d\vartheta = \frac{64}{3}\,\big[\sin\vartheta-\frac{1}{3}\,\sin^3\vartheta\big]_{-\pi/2}^{\pi/2} = \frac{256}{9}$, dove in (a) si è usato il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$.

(7) Si ha $\iint_D \frac{x^2}{x^2+y^2}\,dxdy \overset{(a)}{=} \int_0^{2\pi}\big(\int_1^3\cos^2\vartheta\,\varrho\,d\varrho\big)\,d\vartheta = \int_0^{2\pi}\frac{1+\cos 2\vartheta}{2}\,d\vartheta\cdot\big[\frac{1}{2}\,\varrho^2\big]_1^3 = 2\big[\vartheta+\frac{1}{2}\sin 2\vartheta\big]_0^{2\pi} = 4\pi$, dove in (a) si è usato il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$.

(8) Si ha $\iint_D \frac{x^2}{x^2+y^2}\,dxdy \overset{(a)}{=} \iint_{D_{\varrho,\vartheta}} \cos^2\vartheta\,\varrho\,d\varrho d\vartheta = \int_{\pi/6}^{5\pi/6}\cos^2\vartheta\big(\int_2^{4\sin\vartheta}\varrho\,d\varrho\big)\,d\vartheta = \int_{\pi/6}^{5\pi/6}\cos^2\vartheta\big[\frac{1}{2}\,\varrho^2\big]_2^{4\sin\vartheta}\,d\vartheta = \frac{1}{2}\int_{\pi/6}^{5\pi/6}\cos^2\vartheta(16\sin^2\vartheta-4)\,d\vartheta = 2\int_{\pi/6}^{5\pi/6}(\sin^2 2\vartheta-\cos^2\vartheta)\,d\vartheta = \int_{\pi/6}^{5\pi/6}(1-\cos 4\vartheta-1-\cos 2\vartheta)\,d\vartheta = -\big[\frac{1}{4}\sin 4\vartheta+\frac{1}{2}\sin 2\vartheta\big]_{\pi/6}^{5\pi/6} = \frac{3\sqrt{3}}{4}$, dove in (a) si è usato il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$, per cui $D_{\varrho\vartheta}=\{(\varrho,\vartheta)\in\mathbb{R}^2 : 2\le\varrho\le 4\sin\vartheta, \vartheta\in[\frac{\pi}{6},\frac{5\pi}{6}]\}$.

(9) Si ha $\iint_D |xy|\,dxdy \overset{(a)}{=} \int_0^{2\pi}\big(\int_1^3\varrho^2|\cos\vartheta\sin\vartheta|\,\varrho\,d\varrho\big)\,d\vartheta = \frac{1}{2}\int_0^{2\pi}|\sin 2\vartheta|\,d\vartheta\cdot\big[\frac{1}{4}\,\varrho^4\big]_1^3 = 40\int_0^{\pi/2}\sin 2\vartheta\,d\vartheta = 40\big[-\frac{1}{2}\cos 2\vartheta\big]_0^{\pi/2} = 40$, dove in (a) si è usato il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$.

(10) Usando il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$, per cui $D\rightsquigarrow D_{\varrho\vartheta}=\{(\varrho,\vartheta)\in\mathbb{R}^2 : 0\le\varrho\le 6\sin\vartheta, \vartheta\in[0,\pi]\}$, si ha $\iint_D|xy|\,dxdy = \iint_{D_{\varrho,\vartheta}}\varrho^2|\cos\vartheta\sin\vartheta|\,\varrho\,d\varrho d\vartheta = \int_0^\pi|\cos\vartheta\sin\vartheta|\big(\int_0^{6\sin\vartheta}\varrho^3\,d\varrho\big)\,d\vartheta = \int_0^\pi|\cos\vartheta\sin\vartheta|\big[\frac{1}{4}\,\varrho^4\big]_0^{6\sin\vartheta}\,d\vartheta = 324\int_0^\pi|\cos\vartheta\sin\vartheta|\sin^4\vartheta\,d\vartheta = 648\int_0^{\pi/2}\sin^5\vartheta\cos\vartheta\,d\vartheta = 108\big[\sin^6\vartheta\big]_0^{\pi/2} = 108$.

(11) Usando il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$, per cui $D\rightsquigarrow D_{\varrho\vartheta}=\{(\varrho,\vartheta)\in\mathbb{R}^2 : 2\le\varrho\le 4\sin\vartheta, \vartheta\in[\frac{\pi}{6},\frac{5\pi}{6}]\}$, si ha $\iint_D|xy|\,dxdy = \iint_{D_{\varrho,\vartheta}}\varrho^2|\cos\vartheta\sin\vartheta|\,\varrho\,d\varrho d\vartheta = \int_{\pi/6}^{5\pi/6}|\cos\vartheta\sin\vartheta|\big(\int_2^{4\sin\vartheta}\varrho^3\,d\varrho\big)\,d\vartheta = \int_{\pi/6}^{5\pi/6}|\cos\vartheta\sin\vartheta|\big[\frac{1}{4}\,\varrho^4\big]_2^{4\sin\vartheta}\,d\vartheta = \int_{\pi/6}^{5\pi/6}|\cos\vartheta\sin\vartheta|(16\sin^4\vartheta-4)\,d\vartheta = 8\int_{\pi/6}^{\pi/2}(4\sin^5\vartheta-\sin\vartheta)\cos\vartheta\,d\vartheta = 8\big[\frac{2}{3}\sin^6\vartheta-\frac{1}{2}\sin^2\vartheta\big]_{\pi/6}^{\pi/2} = 18$.

(12) Si ha $\iint_D \frac{|xy|e^{x^2+y^2}}{x^2+y^2}\,dxdy \overset{(a)}{=} \int_0^{2\pi}\big(\int_1^3 e^{\varrho^2}|\cos\vartheta\sin\vartheta|\,\varrho\,d\varrho\big)\,d\vartheta = \frac{1}{2}\int_0^{2\pi}|\sin 2\vartheta|\,d\vartheta\cdot\big[\frac{1}{2}\,e^{\varrho^2}\big]_1^3 = e(e^8-1)\int_0^{\pi/2}\sin 2\vartheta\,d\vartheta = e(e^8-1)\big[-\frac{1}{2}\cos 2\vartheta\big]_0^{\pi/2} = \frac{e}{2}(e^8-1)$, dove in (a) si è usato il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$.

(13) Si ha $\iint_D \frac{|xy|e^{x^2+y^2}}{x^2+y^2}\,dxdy \overset{(a)}{=} \iint_{D_{\varrho,\vartheta}} e^{\varrho^2}|\cos\vartheta\sin\vartheta|\,\varrho\,d\varrho d\vartheta = \int_{\pi/6}^{5\pi/6}|\cos\vartheta\sin\vartheta|\big(\int_2^{4\sin\vartheta}e^{\varrho^2}\varrho\,d\varrho\big)\,d\vartheta = \int_{\pi/6}^{5\pi/6}|\cos\vartheta\sin\vartheta|\big[\frac{1}{2}\,e^{\varrho^2}\big]_2^{4\sin\vartheta}\,d\vartheta = \frac{1}{2}\int_{\pi/6}^{5\pi/6}|\cos\vartheta\sin\vartheta|(e^{16\sin^4\vartheta}-e^4)\,d\vartheta = \int_{\pi/6}^{\pi/2}(e^{16\sin^2\vartheta}-e^4)\sin\vartheta\cos\vartheta\,d\vartheta \overset{(b)}{=} \frac{1}{32}\int_4^{16}e^t\,dt - e^4\big[\frac{1}{2}\sin^2\vartheta\big]_{\pi/6}^{\pi/2} = \frac{1}{32}\big(e^{16}-e^4\big)-\frac{3}{8}\,e^4 = \frac{1}{32}\,e^{16}-\frac{13}{32}\,e^4$, dove si è usato in (a) il cambio di variabile $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$, per cui $D_{\varrho\vartheta}=\{(\varrho,\vartheta)\in\mathbb{R}^2 : 2\le\varrho\le 4\sin\vartheta, \vartheta\in[\frac{\pi}{6},\frac{5\pi}{6}]\}$, e in (b) il cambio di variabile $t=16\sin^2\vartheta \implies dt=32\sin\vartheta\cos\vartheta\,d\vartheta$.

(14) Si ha $\iint_D x^3y\,dxdy \overset{(a)}{=} \int_{\pi/4}^{5\pi/4}\big(\int_0^1 2\sqrt{2}\varrho^3\cos^3\vartheta\cdot 2\sqrt{2}\varrho\sin\vartheta\,4\varrho\,d\varrho\big)\,d\vartheta = 32\big[-\frac{1}{4}\cos^4\vartheta\big]_{\pi/4}^{5\pi/4}\cdot\big[\frac{1}{6}\,\varrho^6\big]_0^1 = 0$, dove in (a) si è usato il cambio di variabile $x=\sqrt{2}\varrho\cos\vartheta$, $y=2\sqrt{2}\varrho\sin\vartheta$.

(15) Si ha $\iint_D|y|\,dxdy \overset{(a)}{=} \int_{\pi/4}^{5\pi/4}\big(\int_0^1 2\sqrt{2}\varrho|\sin\vartheta|\,4\varrho\,d\varrho\big)\,d\vartheta = 8\sqrt{2}\,\big(\int_{\pi/4}^{\pi}\sin\vartheta\,d\vartheta - \int_\pi^{5\pi/4}\sin\vartheta\,d\vartheta\big)\cdot\big[\frac{1}{3}\,\varrho^3\big]_0^1 = \frac{8\sqrt{2}}{3}\big(\big[-\cos\vartheta\big]_{\pi/4}^{\pi}+\big[\cos\vartheta\big]_\pi^{5\pi/4}\big) = \frac{16\sqrt{2}}{3}$, dove in (a) si è usato il cambio di variabile $x=\sqrt{2}\varrho\cos\vartheta$, $y=2\sqrt{2}\varrho\sin\vartheta$.

(16) Usando il cambio di variabili $\begin{cases}u=x+y\\ v=x-y\end{cases} \iff \begin{cases}x=\frac{u+v}{2}\\ y=\frac{u-v}{2}\end{cases}$ si ha $D_{uv}=\{(u,v)\in\mathbb{R}^2 : u-4\le v\le 0,\ 0\le u\le 4\}$, e $\iint_D(x-y)\sin(x+y)\,dxdy = \frac{1}{2}\int_{D_{uv}}v\sin u\,dudv = \frac{1}{2}\int_0^4\sin u\,du\int_{u-4}^0 v\,dv = \frac{1}{4}\int_0^4\sin u\big[v^2\big]_{u-4}^0\,du = -\frac{1}{4}\int_0^4(u-4)^2\sin u\,du \overset{(a)}{=} -\frac{1}{4}\big[-(u-4)^2\cos u+2(u-4)\sin u+2\cos u\big]_0^4 = -\frac{1}{4}\big(2\cos 4+16-2\big) = -\frac{1}{2}\big(7+\cos 4\big)$, dove in (a) si è usato il risultato $\int(u-4)^2\sin u\,du = -(u-4)^2\cos u+\int 2(u-4)\cos u\,du = -(u-4)^2\cos u+2(u-4)\sin u-\int 2\sin u\,du = -(u-4)^2\cos u+2(u-4)\sin u+2\cos u$.

(17) Usando il cambio di variabili $\begin{cases} u = x + y \\ v = x - y \end{cases} \iff \begin{cases} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \end{cases}$ si ha $D_{uv} = \{(u,v) \in \mathbb{R}^2 : 0 \le u \le 2,\ 0 \le v \le 2\}$, e $\iint_D (x^2 - y^2)\sin(x+y)\,dx\,dy = \frac{1}{2}\int_{D_{uv}} vu\sin u\,du\,dv = \frac{1}{2}\int_0^2 v\,dv \int_0^2 u\sin u\,du = \int_0^2 u\sin u\,du = \big[-u\cos u + \sin u\big]_0^2 = \sin 2 - 2\cos 2.$

(18) Usando il cambio di variabili $\begin{cases} 3y - x = s \\ y - 2x = t \end{cases} \iff \begin{cases} x = \frac{1}{5}\,s - \frac{3}{5}\,t \\ y = \frac{2}{5}\,s - \frac{1}{5}\,t \end{cases}$ si ha $D_{st} = \{(s,t) \in \mathbb{R}^2 : s \in [0,5], t \in [-5,0]\}$, e $\iint_D \sqrt{x+y}\,dxdy = \frac{1}{5\sqrt{5}}\int_{D_{st}} \sqrt{3s-4t}\,dsdt = \frac{1}{5\sqrt{5}}\int_{-5}^0 \big[\frac{2}{9}(3s - 4t)^{3/2}\big]_{s=0}^{s=5} dt = \frac{2}{45\sqrt{5}}\int_{-5}^0 \big((15-4t)^{3/2} - (-4t)^{3/2}\big)\,dt = \frac{2}{45\sqrt{5}}\big[-\frac{1}{10}(15-4t)^{5/2} + \frac{1}{10}(-4t)^{5/2}\big]_{-5}^0 = \frac{49\sqrt{7}}{9} - \sqrt{3} - \frac{32}{9}.$

□

Svolgimento esercizio 4

(1) Si ha area $D = \int_{-1}^1 \big(\int_{x^2}^1 dy\big)\,dx = \int_{-1}^1 (1 - x^2)\,dx = \big[x - \frac{1}{3}\,x^3\big]_{-1}^1 = \frac{4}{3}.$

(2) Si ha area $D = \int_{-1/2}^2 \big(\int_{x^2}^{x+2} dy\big)\,dx = \int_{-1/2}^2 (x + 2 - x^2)\,dx = \big[2x + \frac{1}{2}\,x^2 - \frac{1}{3}\,x^3\big]_{-1/2}^2 = \frac{13}{3}.$

(3) Si ha area $D = 1 + \int_{1/2}^2 \big(\int_0^{1/x} dy\big)\,dx = 1 + \int_{1/2}^2 \frac{1}{x}\,dx = 1 + \big[\log|x|\big]_{1/2}^2 = 1 + 2\log 2.$

(4) Si ha area $D = \int_0^8 \big(\int_{y^2/4}^{2y} dx\big)\,dy = \int_0^8 \big(2y - \frac{1}{4}\,y^2\big)\,dy = \big[y^2 - \frac{1}{12}\,y^3\big]_0^8 = \frac{64}{3}.$

(5) Poiché $D = \{(x,y) \in \mathbb{R}^2 : (x^2+y^2)^3 \le 16x^2\} = \{(x,y) \in \mathbb{R}^2 : |y| \le \sqrt{2^{4/3}x^{2/3} - x^2}\}$, si ha area $D = 4\int_0^2 \big(\int_0^{\sqrt{2^{4/3}x^{2/3}-x^2}} dy\big)\,dx = 4\int_0^2 \sqrt{2^{4/3}x^{2/3} - x^2}\,dx = 4\int_0^2 x^{1/3}\sqrt{2^{4/3} - x^{4/3}}\,dx \overset{(a)}{=} 3\int_0^{2^{4/3}} \sqrt{t}\,dt = \big[2t^{3/2}\big]_0^{2^{4/3}} = 8$, dove in (a) si è usato il cambio di variabile $t = 2^{4/3} - x^{4/3} \implies dt = -\frac{4}{3}\,x^{1/3}$.

(6) Usando il cambio di coordinate $x = \varrho\cos\vartheta, y = \varrho\sin\vartheta$, per cui $D \rightsquigarrow D_{\varrho\vartheta} = \{(\varrho,\vartheta) \in \mathbb{R}^2 : 3 \le \varrho \le 8\sin\vartheta, \vartheta \in [\arcsin\frac{3}{8}, \pi - \arcsin\frac{3}{8}]\}$, si ha area $D = \int_{\arcsin 3/8}^{\pi - \arcsin 3/8} \big(\int_3^{8\sin\vartheta} \varrho\,d\varrho\big)\,d\vartheta = \int_{\arcsin 3/8}^{\pi-\arcsin 3/8} \big[\frac{1}{2}\,\varrho^2\big]_3^{8\sin\vartheta} d\vartheta = \frac{1}{2}\int_{\arcsin 3/8}^{\pi-\arcsin 3/8} (64\sin^2\vartheta - 9)\,d\vartheta = \frac{1}{2}\int_{\arcsin 3/8}^{\pi-\arcsin 3/8} (23 - 32\cos 2\vartheta)\,d\vartheta = \frac{1}{2}\big[23\vartheta - 16\sin 2\vartheta\big]_{\arcsin 3/8}^{\pi-\arcsin 3/8} = \frac{23}{2}\,\pi - 23\arcsin\frac{3}{8} - \frac{3}{2}\sqrt{55}.$

(7) Usando il cambio di coordinate $x = \sqrt{2}\varrho\cos\vartheta, y = 2\sqrt{2}\varrho\sin\vartheta$, per cui $D \rightsquigarrow D_{\varrho\vartheta} = \{(\varrho,\vartheta) \in \mathbb{R}^2 : \varrho \in [0,1], \vartheta \in [\frac{\pi}{4}, \frac{5\pi}{4}]\}$, si ha area $D = \int_{D_{\varrho\vartheta}} 4\varrho\,d\varrho d\vartheta = \int_{\pi/4}^{5\pi/4} \big(\int_0^1 4\varrho\,d\varrho\big)\,d\vartheta = \pi\big[2\varrho^2\big]_0^1 = 2\pi.$

□

Svolgimento esercizio 5

(1) Si ha $\iiint_D \frac{x^2y}{z}\,dxdydz = \int_0^1 x^2\,dx \cdot \int_0^1 y\,dy \cdot \int_1^2 \frac{1}{z}\,dz = \big[\frac{1}{3}x^3\big]_0^1 \cdot \big[\frac{1}{2}\,y^2\big]_0^1 \cdot \big[\log z\big]_1^2 = \frac{1}{6}\log 2.$

(2) Si ha $\iiint_D x\sin^2 y\cos z\,dxdydz = \int_0^1 x\,dx \cdot \int_0^\pi \sin^2 y\,dy \cdot \int_0^{\pi/2} \cos z\,dz = \big[\frac{1}{2}x^2\big]_0^1 \cdot \big[\frac{1}{2}\,y - \frac{1}{4}\,\sin 2y\big]_0^\pi \cdot \big[\sin z\big]_0^{\pi/2} = \frac{\pi}{4}.$

(3) Si ha $\iiint_D 2(x+y+z)\,dxdydz = 2\int_0^1 x\,dx + 2\int_0^1 y\,dy + 2\int_0^1 z\,dz = 6\big[\frac{1}{2}x^2\big]_0^1 = 3.$

(4) Posto $A_z = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le z^2\}$, si ha $\iiint_D (x+y+z)\,dxdydz = \iiint_D z\,dxdydz = \int_{-1}^2 z\big(\int_{A_z} dxdy\big)\,dz = \pi\int_{-1}^2 z^3\,dz = \pi\big[\frac14 z^4\big]_{-1}^2 = \frac{15}{4}\pi$.

(5) Si ha $\iiint_D (x^3+y^3+z^3)\,dxdydz = 3\int_0^1 x^3\big(\int_0^{1-x}\big(\int_0^{1-x-y} dz\big)\,dy\big)\,dx = 3\int_0^1 x^3\big(\int_0^{1-x}(1-x-y)\,dy\big)\,dx = 3\int_0^1 x^3\big[(1-x)y-\frac12 y^2\big]_0^{1-x}\,dx = \frac32\int_0^1 x^3(1-x)^2\,dx = \frac32\big[\frac16 x^6-\frac25 x^5+\frac14 x^4\big]_0^1 = \frac{1}{40}$.

(6) Si ha $\iiint_D \frac{1}{\sqrt{1-z^2}}\,dxdydz = \int_0^{1/2}\frac{1}{\sqrt{1-z^2}}\big(\int_0^z\big(\int_0^y dx\big)\,dy\big)\,dz = \int_0^{1/2}\frac{1}{\sqrt{1-z^2}}\big(\int_0^z y\,dy\big)\,dz = \frac12\int_0^{1/2}\frac{1}{\sqrt{1-z^2}}\cdot\big[y^2\big]_0^z\,dz = \frac12\int_0^{1/2}\frac{z^2}{\sqrt{1-z^2}}\,dz \overset{(a)}{=} \frac12\int_0^{\pi/6}\sin^2 t\,dt = \frac12\big[\frac12 t-\frac14\sin 2t\big]_0^{\pi/6} = \frac{\pi}{24}-\frac{\sqrt3}{16}$, dove in (a) si è usato il cambio di variabili $z=\sin t \implies dz=\cos t\,dt$.

(7) Si ha $\iiint_D xyz^2\,dxdydz = \int_0^1 x\big(\int_{-x}^x z^2\big(\int_{x+z}^4 y\,dy\big)\,dz\big)\,dx = \frac12\int_0^1 x\big(\int_{-x}^x z^2\big(4-(x+z)^2\big)\,dz\big)\,dx = \frac12\int_0^1 x\big(\int_{-x}^x z^2\big((4-x^2)-2xz-z^2\big)\,dz\big)\,dx = \int_0^1 x\big[\frac13(4-x^2)z^3-\frac15 z^5\big]_0^x\,dx = \int_0^1\big(\frac13(4-x^2)x^4-\frac15 x^6\big)\,dx = \big[\frac{4}{3\cdot5}x^5-\frac{8}{15\cdot7}x^7\big]_0^1 = \frac{4}{21}$.

(8) Si ha $\iiint_D (x^2+y^2+z^2)\,dxdydz = \int_0^1\big(\int_0^{1-x}\big(\int_0^{2(1-x-y)}(x^2+y^2+z^2)\,dz\big)\,dy\big)\,dx = \int_0^1\big(\int_0^{1-x}\big[x^2z+y^2z+\frac13 z^3\big]_{z=0}^{z=2(1-x-y)}\,dy\big)\,dx = \int_0^1\big(\int_0^{1-x}\big(2(x^2+y^2)(1-x-y)+\frac83(1-x-y)^3\big)\,dy\big)\,dx = \int_0^1\big(\int_0^{1-x}\big(2x^2(1-x)-2x^2y+2(1-x)y^2-2y^3+\frac83(1-x-y)^3\big)\,dy\big)\,dx = \int_0^1\big[2x^2(1-x)y-x^2y^2+\frac23(1-x)y^3-\frac12 y^4-\frac83\cdot\frac14(1-x-y)^4\big]_{y=0}^{y=1-x}\,dx = \int_0^1\big(2x^2(1-x)^2-x^2(1-x)^2+\frac23(1-x)^4-\frac12(1-x)^4+\frac23(1-x)^4\big)\,dx = \int_0^1\big(x^2(1-x)^2+\frac56(1-x)^4\big)\,dx = \big[\frac13 x^3-\frac12 x^4+\frac15 x^5+\frac16(x-1)^5\big]_0^1 = \frac13-\frac12+\frac15+\frac16 = \frac15$.

(9) Posto $A = \{(x,y) : 0 \le y \le x;\ \frac{1}{16} \le x^2+y^2 \le 1\}$ si ottiene $\iiint_D \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy\,dz = \int_A\big(\int_0^1 \frac{x}{\sqrt{x^2+y^2}}\,dz\big)\,dx\,dy = \int_A \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy \overset{(a)}{=} \int_0^{\pi/4}\int_{1/4}^1 \cos\vartheta\,\varrho\,d\varrho\,d\vartheta = \big[\sin\vartheta\big]_0^{\pi/4}\big[\frac12\varrho^2\big]_{1/4}^1 = \frac{\sqrt2}{2}\cdot\frac12\big(1-\frac{1}{16}\big) = \frac{15\sqrt2}{64}$, dove in (a) si è usato il cambiamento di coordinate polari.

(10) Posto $A = \{(x,y) : 0 \le y \le x;\ \frac{1}{16} \le x^2+y^2 \le 1\}$ si ottiene $\iiint_D \frac{2z}{(x^2+y^2)^{3/2}}\cos\big(\frac{\mathrm{arctg}(y/x)}{\sqrt[4]{x^2+y^2}}\big)\,dx\,dy\,dz = \int_A\big(\int_0^1 \frac{1}{(x^2+y^2)^{3/2}}\cos\big(\frac{\arctan(y/x)}{\sqrt[4]{x^2+y^2}}\big)2z\,dz\big)\,dx\,dy = \int_A \frac{1}{(x^2+y^2)^{3/2}}\cos\big(\frac{\arctan(y/x)}{\sqrt[4]{x^2+y^2}}\big)\,dx\,dy = (\text{coord. polari}) = \int_0^{\pi/4}\int_{1/4}^1 \frac{1}{\rho^3}\cos\frac{\theta}{\sqrt\rho}\rho\,d\rho\,d\theta = \int_{1/4}^1\big(\int_0^{\pi/4}\frac{1}{\rho^2}\cos\frac{\theta}{\sqrt\rho}\big)\,d\theta\,d\rho = \int_{1/4}^1\big(\frac{1}{\rho^{3/2}}\big[\sin\frac{\theta}{\sqrt\rho}\big]_{\theta=0}^{\theta=\pi/4}\big)\,d\rho = \int_{1/4}^1 \frac{1}{\rho^{3/2}}\sin\frac{\pi}{4\sqrt\rho}\,d\rho \overset{(a)}{=} \frac{8}{\pi}\big[\cos\frac{\pi}{4\sqrt\rho}\big]_{1/4}^1 = \frac8\pi\big(\cos\frac\pi4-\cos\frac\pi2\big) = \frac{4\sqrt2}{\pi}$, dove in (a) si è usato il cambiamento di coordinate $s=\frac{\pi}{4\sqrt\varrho} \implies ds = -\frac{\pi}{8\varrho^{3/2}}\,d\varrho$.

(11) Si ha $\iiint_D x^2\,dx\,dy\,dz \overset{(a)}{=} \int_0^\pi\int_0^{2\pi}\int_0^1 \varrho^2\cos^2\vartheta\sin^2\varphi\,\varrho^2\sin\varphi\,d\varrho d\vartheta d\varphi = \int_0^\pi \sin^3\varphi\,d\varphi\cdot\int_0^{2\pi}\cos^2\vartheta\,d\vartheta\cdot\int_0^1\varrho^4\,d\varrho = \int_0^\pi(1-\cos^2\varphi)\sin\varphi\,d\varphi\cdot\int_0^{2\pi}\frac12(1+\cos 2\vartheta)\,d\vartheta\cdot\big[\frac15\varrho^5\big]_0^1 = \frac15\big[\frac13\cos^3\varphi-\cos\varphi\big]_\pi^0\cdot\big[\frac12\vartheta+\frac14\sin 2\vartheta\big]_0^{2\pi} = \frac{4\pi}{15}$, dove si è usato in (a) il cambio di variabile $x=\varrho\sin\varphi\cos\vartheta, y=\varrho\sin\varphi\sin\vartheta, z=\varrho\cos\varphi$, per cui $D \rightsquigarrow D_{\varrho\vartheta\varphi} = \{(\varrho,\vartheta,\varphi)\in\mathbb{R}^3 : 0\le\varrho\le1, 0\le\varphi\le\pi, 0\le\vartheta\le2\pi\}$.

(12) Si ha $\iiint_D (y-3)^2\,dx\,dy\,dz \overset{(a)}{=} \int_0^\pi\int_0^{2\pi}\int_0^2(\varrho\sin\vartheta\sin\varphi-3)^2\,\varrho^2\sin\varphi\,d\varrho d\vartheta d\varphi = \int_0^\pi\int_0^{2\pi}\int_0^2\big(\varrho^4\sin^2\vartheta\sin^3\varphi-6\varrho^3\sin\vartheta\sin^2\varphi+9\varrho^2\sin\varphi\big)\,d\varrho d\vartheta d\varphi = \int_0^\pi\sin^3\varphi\,d\varphi\cdot\int_0^{2\pi}\sin^2\vartheta\,d\vartheta\cdot\int_0^2\varrho^4\,d\varrho+9\int_0^\pi\sin\varphi\,d\varphi\cdot\int_0^{2\pi}d\vartheta\cdot\int_0^2\varrho^2\,d\varrho = \int_0^\pi(1-\cos^2\varphi)\sin\varphi\,d\varphi\cdot\int_0^{2\pi}\frac12(1-\cos2\vartheta)\,d\vartheta\cdot\big[\frac15\varrho^5\big]_0^2+18\pi\big[-\cos\varphi\big]_0^\pi\cdot\big[\frac13\varrho^3\big]_0^2 = \frac{32}{5}\big[\frac13\cos^3\varphi-\cos\varphi\big]_\pi^0\cdot\big[\frac12\vartheta+\frac14\sin2\vartheta\big]_0^{2\pi}+96\pi = \frac{1568\pi}{15}$, dove si è usato in (a) il cambio di variabile $x=\varrho\sin\varphi\cos\vartheta, y=\varrho\sin\varphi\sin\vartheta, z=\varrho\cos\varphi$, per cui $D\rightsquigarrow D_{\varrho\vartheta\varphi} = \{(\varrho,\vartheta,\varphi)\in\mathbb{R}^3 : 0\le\varrho\le2, 0\le\varphi\le\pi, 0\le\vartheta\le2\pi\}$.

(13) Si ha $\iiint_D x^2 y\,dx\,dy\,dz \overset{(a)}{=} \int_0^\pi \int_0^\pi \int_0^1 \varrho^2 \cos^2\vartheta \sin^2\varphi \cdot \varrho \sin\vartheta \sin\varphi\, \varrho^2 \sin\varphi\, d\varrho d\vartheta d\varphi = \int_0^\pi \sin^4\varphi\, d\varphi \cdot \int_0^\pi \cos^2\vartheta \sin\vartheta\, d\vartheta \cdot \int_0^1 \varrho^5\, d\varrho = \int_0^\pi \left(\frac{3}{8} - \frac{1}{2}\cos 2\varphi + \frac{1}{8}\cos 4\varphi\right) d\varphi \cdot \left[-\frac{1}{3}\cos^3\vartheta\right]_0^\pi \cdot \left[\frac{1}{6}\varrho^6\right]_0^1 = \frac{1}{9}\left[\frac{3}{8}\varphi - \frac{1}{4}\sin 2\varphi + \frac{1}{32}\sin 4\varphi\right]_0^\pi = \frac{\pi}{24}$, dove si è usato in (a) il cambio di variabile $x = \varrho \sin\varphi \cos\vartheta, y = \varrho \sin\varphi \sin\vartheta, z = \varrho\cos\varphi$, per cui $D \rightsquigarrow D_{\varrho\vartheta\varphi} = \{(\varrho, \vartheta, \varphi) \in \mathbb{R}^3 : 0 \le \varrho \le 1, 0 \le \varphi \le \pi, 0 \le \vartheta \le \pi\}$.

(14) Si ha $\iiint_D \left(e^z + \frac{2y}{1+x^2+y^2} + x - 1\right) dx\,dy\,dz \overset{(a)}{=} \iiint_D e^z\,dx\,dy\,dz \overset{(b)}{=} \int_0^{2\pi} \int_0^\pi \int_0^1 e^{\varrho\cos\varphi} \varrho^2 \sin\varphi\, d\vartheta\, d\varphi\, d\varrho = 2\pi \int_0^1 \varrho\left(\int_0^\pi e^{\varrho\cos\varphi} \varrho \sin\varphi\, d\varphi\right) d\varrho \overset{(c)}{=} 2\pi \int_0^1 \varrho\left[-e^{\varrho\cos\varphi}\right]_{\varphi=0}^{\varphi=\pi} d\varrho = 2\pi \int_0^1 \varrho(e^\varrho - e^{-\varrho})\, d\varrho \overset{(d)}{=} 2\pi\left[\varrho(e^\varrho + e^{-\varrho}) - e^\varrho + e^{-\varrho}\right]_0^1 = \frac{4\pi}{e}$, dove si è usato in (a) il fatto che D è invariante rispetto alle trasformazioni $y \to -y$ e $x \to 2 - x$, mentre il secondo termine cambia segno sotto la trasformazione $y \to -y$, e il terzo cambia segno sotto la trasformazione $x \to 2 - x$, in (b) il cambio di variabile $x = 1 + \varrho\sin\varphi\cos\vartheta, y = \varrho\sin\varphi\sin\vartheta, z = \varrho\cos\varphi$, per cui $D \rightsquigarrow D_{\varrho\vartheta\varphi} = \{(\varrho, \vartheta, \varphi) \in \mathbb{R}^3 : 0 \le \varrho \le 1, 0 \le \varphi \le \pi, 0 \le \vartheta \le 2\pi\}$, in (c) il cambiamento di variabile $\varrho\cos\varphi = t \implies \varrho\sin\varphi\, d\varphi = -dt$, e in (d) il risultato $\int \varrho(e^\varrho - e^{-\varrho})\, d\varrho = \varrho(e^\varrho + e^{-\varrho}) - \int (e^\varrho + e^{-\varrho})\, d\varrho = \varrho(e^\varrho + e^{-\varrho}) - e^\varrho + e^{-\varrho}$.

(15) Si ha $\iiint_D \left(e^z + z(x-1)^2 + \frac{2z}{1+(x-1)^2+(y/2)^2+z^2}\right) dx\,dy\,dz \overset{(a)}{=} \iiint_D e^z\,dx\,dy\,dz \overset{(b)}{=} \int_0^{2\pi} \int_0^\pi \int_0^1 e^{\varrho\cos\varphi} \cdot 2\varrho^2 \sin\varphi\, d\vartheta\, d\varphi\, d\varrho = 4\pi \int_0^1 \varrho\left(\int_0^\pi e^{\varrho\cos\varphi} \varrho\sin\varphi\, d\varphi\right) d\varrho \overset{(c)}{=} 4\pi \int_0^1 \varrho\left[-e^{\varrho\cos\varphi}\right]_{\varphi=0}^{\varphi=\pi} d\varrho = 2\pi \int_0^1 \varrho(e^\varrho - e^{-\varrho})\, d\varrho \overset{(d)}{=} 4\pi\left[\varrho(e^\varrho + e^{-\varrho}) - e^\varrho + e^{-\varrho}\right]_0^1 = \frac{8\pi}{e}$, dove si è usato in (a) il fatto che D è invariante rispetto alla trasformazione $z \to -z$, mentre il secondo e il terzo termine cambiano segno sotto la medesima trasformazione, in (b) il cambio di coordinate ellittiche $x = 1 + \varrho\sin\varphi\cos\vartheta, y = 2\varrho\sin\varphi\sin\vartheta, z = \varrho\cos\varphi$, per cui $D \rightsquigarrow D_{\varrho\vartheta\varphi} = \{(\varrho, \vartheta\varphi) \in \mathbb{R}^3 : 0 \le \varrho \le 1, 0 \le \varphi \le \pi, 0 \le \vartheta \le 2\pi\}$, e il determinante jacobiano della trasformazione vale $2\varrho^2\sin\varphi$, in (c) si è usato il cambiamento di variabile $\varrho\cos\varphi = t \implies \varrho\sin\varphi\, d\varphi = -dt$, e in (d) si è usato il risultato $\int \varrho(e^\varrho - e^{-\varrho})\, d\varrho = \varrho(e^\varrho + e^{-\varrho}) - \int (e^\varrho + e^{-\varrho})\, d\varrho = \varrho(e^\varrho + e^{-\varrho}) - e^\varrho + e^{-\varrho}$.

□

Svolgimento esercizio 6

(1) Posto $A_z := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 4 - z^2\}$, si ha $\operatorname{vol} D = \int_1^2 \left(\iint_{A_z} dxdy\right) dz = \int_1^2 \pi(4 - z^2)\, dz = \pi\left[4z - \frac{1}{3}z^3\right]_1^2 = \frac{5}{3}\pi$.

(2) Posto $A_z := \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 \le 1 + 9z^2\}$, si ha $\operatorname{vol} D = \int_{-1}^1 \left(\iint_A dxdy\right) dz = \pi \int_{-1}^1 \sqrt{1 + 9z^2} \cdot \frac{1}{2}\sqrt{1 + 9z^2}\, dz = \frac{\pi}{2} \int_{-1}^1 (1 + 9z^2)\, dz = \frac{\pi}{2}\left[z + 3z^3\right]_{-1}^1 = 4\pi$.

(3) Posto $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, si ha $\operatorname{vol} D = \iint_A \left(\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\right) dxdy = 2\int_A \sqrt{4 - x^2 - y^2}\, dxdy = 2\int_0^{2\pi} \left(\int_0^1 \sqrt{4 - \varrho^2}\, \varrho\, d\varrho\right) d\vartheta \overset{(a)}{=} 4\pi \int_3^4 \sqrt{t}\, \frac{1}{2}\, dt = 2\pi\left[\frac{2}{3} t^{3/2}\right]_3^4 = \frac{4\pi}{3}(8 - 3\sqrt{3})$, dove in (a) si è usato il cambio di variabile $t = 4 - \varrho^2 \implies dt = -2\varrho\, d\varrho$.

(4) Posto $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, si ha $\operatorname{vol} D = \iint_A \left(\int_0^{x^2+y^2} dz\right) dxdy = \int_A (x^2 + y^2)\, dxdy = \int_0^{2\pi} \left(\int_0^1 \varrho^2\, \varrho\, d\varrho\right) d\vartheta = 2\pi\left[\frac{1}{4}\varrho^4\right]_0^1 = \frac{\pi}{2}$.

(5) Posto $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, si ha $\operatorname{vol} D = \iint_A \left(\int_{-1}^{x^2-y^2} dz\right) dxdy = \int_A (x^2 - y^2 + 1)\, dxdy = \int_0^{2\pi} \left(\int_0^1 (\varrho^2\cos^2\vartheta - \varrho^2\sin^2\vartheta + 1)\, \varrho\, d\varrho\right) d\vartheta = \int_0^{2\pi} \left(\frac{1}{3}\cos^2\vartheta - \frac{1}{3}\sin^2\vartheta + 1\right) d\vartheta = \int_0^{2\pi} \left(\frac{1}{3}\cos 2\vartheta + 1\right) d\vartheta = \left[\frac{1}{6}\sin 2\vartheta + \vartheta\right]_0^1 = 2\pi$.

(6) Posto $A := \{(x,y) \in \mathbb{R}^2 : x^2+4y^2 \le 1\}$, si ha $\operatorname{vol} D = \iint_A \big(\int_{-\frac13\sqrt{1-x^2-4y^2}}^{\frac13\sqrt{1-x^2-4y^2}} dz \big)\, dxdy = \frac23 \int_A \sqrt{1-x^2-4y^2}\, dxdy \overset{(a)}{=} \frac23 \int_0^{2\pi} \big(\int_0^1 \sqrt{1-\varrho^2}\, \frac12\, \varrho\, d\varrho \big)\, d\vartheta \overset{(b)}{=} \frac{2\pi}{3} \int_0^1 \sqrt{t}\, \frac12\, dt = \frac{\pi}{3} \big[\frac23\, t^{3/2} \big]_0^1 = \frac29\, \pi$, dove si è usato in (a) il cambio di coordinate $x = \varrho\cos\vartheta$, $y = \frac12\, \varrho \sin\vartheta$, e in (b) il cambio di coordinate $t = 1-\varrho^2 \implies dt = -2\varrho\, d\varrho$.

(7) Posto $D_1 := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1, z \ge 0\}$ e $D_2 := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+(z+1)^2 \le 2, z \ge 0\}$, si ha $D = D_1 \setminus D_2^o$, e $\operatorname{vol} D = \operatorname{vol} D_1 - \operatorname{vol} D_2$. Si ha $\operatorname{vol} D_1 = \frac23\, \pi$. Per calcolare $\operatorname{vol} D_2$, poniamo $A_z = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 2-(z+1)^2\}$, per cui $\operatorname{vol} D_2 = \int_0^{\sqrt2-1} \big(\iint_{A_z} dxdy \big)\, dz = \pi \int_0^{\sqrt2-1} (1-2z-z^2)\, dz = \big[z - z^2 - \frac13\, z^3 \big]_0^{\sqrt2-1} = \frac{\pi}{3}\, (4\sqrt2-5)$. Allora $\operatorname{vol} D = \operatorname{vol} D_1 - \operatorname{vol} D_2 = \frac{\pi}{3}\, (7-4\sqrt2)$.

□

Svolgimento esercizio 7

(1) Posto $D^a = \{(x,y) \in \mathbb{R}^2 : y \in [a,1], -y^2 \le x \le y^2\}$, si ha $\iint_D \frac{\operatorname{arctg} y^2}{y}\, dxdy = \lim_{a\to0^+} \iint_{D^a} \frac{\operatorname{arctg} y^2}{y}\, dxdy = \lim_{a\to0^+} \int_a^1 \frac{\operatorname{arctg} y^2}{y} \big(\int_{-y^2}^{y^2} dx \big)\, dy = 2 \lim_{a\to0^+} \int_a^1 y \operatorname{arctg} y^2\, dy = \lim_{a\to0^+} \big[\operatorname{arctg} y^2 \big]_a^1 = \lim_{a\to0^+} \big(\frac{\pi}{4} - \operatorname{arctg} a^2 \big) = \frac{\pi}{4}$.

(2) Posto $D^a = \{(x,y) \in \mathbb{R}^2 : y \in [a,1], -y^2 \le x \le y^2\}$, si ha $\iint_D \frac{\operatorname{arctg} y}{y^2}\, dxdy = \lim_{a\to0^+} \iint_{D^a} \frac{\operatorname{arctg} y}{y^2}\, dxdy = \lim_{a\to0^+} \int_a^1 \frac{\operatorname{arctg} y}{y^2} \big(\int_{-y^2}^{y^2} dx \big)\, dy = 2 \lim_{a\to0^+} \int_a^1 \operatorname{arctg} y\, dy = 2 \lim_{a\to0^+} \big[y \operatorname{arctg} y - \frac12 \log(y^2+1) \big]_a^1 = 2 \lim_{a\to0^+} \big(\frac{\pi}{4} - \frac12 \log 2 - a \operatorname{arctg} a + \frac12 \log(a^2+1) \big) = \frac{\pi}{2} - \log 2$.

(3) Posto $D^a = \{(x,y) \in \mathbb{R}^2 : a \le |y| \le 1, y^2 \le x \le 1\}$, si ha $\iint_D \frac{1}{(y^2+x)^2}\, dxdy = \lim_{a\to0^+} \iint_{D^a} \frac{1}{(y^2+x)^2}\, dxdy = 2 \lim_{a\to0^+} \int_a^1 \big(\int_{y^2}^1 \frac{1}{(y^2+x)^2}\, dx \big)\, dy = -2 \lim_{a\to0^+} \int_a^1 \big[\frac{1}{y^2+x} \big]_{x=y^2}^{x=1}\, dy = -2 \lim_{a\to0^+} \int_a^1 \big(\frac{1}{y^2+1} - \frac{1}{2y^2} \big)\, dy = -2 \lim_{a\to0^+} \big[\operatorname{arctg} y + \frac{1}{2y} \big]_a^1 = -2 \lim_{a\to0^+} \big(\frac{\pi}{4} + \frac12 - \operatorname{arctg} a - \frac{1}{2a} \big) = +\infty$.

(4) Posto $D^a = \{(x,y) \in \mathbb{R}^2 : -\frac12 + a \le x \le 0, |y| \le \sqrt{2x+1}\}$, si ha $\iint_D \frac{(x+2)|y|}{y^2+2x+1}\, dxdy = \lim_{a\to0^+} \iint_{D^a} \frac{(x+2)|y|}{y^2+2x+1}\, dxdy = \lim_{a\to0^+} \int_{-\frac12+a}^0 \big(\int_{-\sqrt{2x+1}}^{\sqrt{2x+1}} \frac{(x+2)|y|}{y^2+2x+1}\, dy \big)\, dx = \lim_{a\to0^+} \int_{-\frac12+a}^0 (x+2) \big[\log(y^2+2x+1) \big]_{y=0}^{y=\sqrt{2x+1}}\, dx = \lim_{a\to0^+} \int_{-\frac12+a}^0 (x+2) \big(\log(4x+2) - \log(2x+1) \big)\, dx = \log 2 \lim_{a\to0^+} \big[\frac12\, x^2 + 2x \big]_{-\frac12+a}^0 = -\log 2 \lim_{a\to0^+} \big(\frac12\, a^2 + \frac32\, a - \frac78 \big)\, dx = \frac78 \log 2$.

(5) Posto $D^a = \{(x,y) \in \mathbb{R}^2 : a \le x \le 1, x \le y \le \sqrt{x}\}$, si ha $\iint_D \frac{(x+2)y}{y^2+2x}\, dxdy = \lim_{a\to0^+} \iint_{D^a} \frac{(x+2)y}{y^2+2x}\, dxdy = \lim_{a\to0^+} \int_a^1 \big(\int_x^{\sqrt{x}} \frac{(x+2)y}{y^2+2x}\, dy \big)\, dx = \frac12 \lim_{a\to0^+} \int_a^1 (x+2) \big[\log(y^2+2x) \big]_{y=x}^{y=\sqrt{x}}\, dx = \frac12 \lim_{a\to0^+} \int_a^1 (x+2) \big(\log(3x) - \log(x^2+2x) \big)\, dx = \frac12 \lim_{a\to0^+} \big[\big(\frac12\, x^2 + 2x \big) \big(\log(3x) - \log(x^2+2x) \big) \big]_a^1 - \frac12 \lim_{a\to0^+} \int_a^1 \big(\frac12\, x^2 + 2x \big) \big(\frac1x - \frac{2x+2}{x^2+2x} \big)\, dx = -\frac12 \lim_{a\to0^+} \int_a^1 \big(-\frac12\, x - 1 + \frac{2}{x+2} \big)\, dx = \frac12 \lim_{a\to0^+} \big[\frac14\, x^2 + x - 2\log(x+2) \big]_a^1 = \frac58 - 2\log\frac32$.

(6) Posto $D^a = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, x \le y \le \sqrt{2x-x^2}, x^2+y^2 \ge a^2\}$, e usando il cambio di coordinate $x = \varrho\cos\vartheta$, $y = \varrho\sin\vartheta$, per cui $D^a_{\varrho\vartheta} = \{(\varrho,\vartheta) \in \mathbb{R}^2 : a \le \varrho \le 2\cos\vartheta, \vartheta \in [\frac{\pi}{4}, \frac{\pi}{2}]\}$, si ha $\iint_D \frac{1}{\sqrt{x^2+y^2}}\, dxdy = \lim_{a\to0^+} \iint_{D^a} \frac{1}{\sqrt{x^2+y^2}}\, dxdy = \lim_{a\to0^+} \iint_{D^a_{\varrho\vartheta}} d\varrho d\vartheta = \lim_{a\to0^+} \int_{\pi/4}^{\pi/2} \big(\int_a^{2\cos\vartheta} d\varrho \big)\, d\vartheta = \lim_{a\to0^+} \int_{\pi/4}^{\pi/2} (2\cos\vartheta - a)\, d\vartheta = \lim_{a\to0^+} \big[2\sin\vartheta - a\vartheta \big]_{\pi/4}^{\pi/2} = \lim_{a\to0^+} \big(2 - \sqrt2 - \frac{\pi}{4}\, a \big) = 2 - \sqrt2$.

(7) Posto $D^a=\{(x,y)\in\mathbb{R}^2 : x^2+(y-3)^2\le 9, x^2+y^2\ge a^2\}$, e usando il cambio di coordinate $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$, per cui $D^a_{\varrho\vartheta}=\{(\varrho,\vartheta)\in\mathbb{R}^2 : a\le\varrho\le 6\sin\vartheta, \vartheta\in[0,\pi]\}$, si ha $\iint_D \frac{x^2}{x^2+y^2}\,dxdy = \lim_{a\to 0^+}\iint_{D^a}\frac{x^2}{x^2+y^2}\,dxdy = \lim_{a\to 0^+}\iint_{D^a_{\varrho\vartheta}}\varrho\cos^2\vartheta\,d\varrho d\vartheta = \lim_{a\to 0^+}\int_0^\pi\big(\int_a^{6\sin\vartheta}\varrho\cos^2\vartheta\,d\varrho\big)\,d\vartheta = \lim_{a\to 0^+}\int_0^\pi\cos^2\vartheta\cdot\big[\frac12\varrho^2\big]_a^{6\sin\vartheta}\,d\vartheta = \frac12\lim_{a\to 0^+}\int_0^\pi(36\sin^2\vartheta - a^2)\cos^2\vartheta\,d\vartheta = \frac12\lim_{a\to 0^+}\int_0^\pi(9\sin^2 2\vartheta - a^2\cos^2\vartheta)\,d\vartheta = \frac14\lim_{a\to 0^+}\big[(9-a^2)\vartheta - 9\cos 4\vartheta - a^2\cos 2\vartheta\big]_0^\pi = \frac14\lim_{a\to 0^+}(9-a^2)\pi = \frac94\,\pi$.

(8) Posto $D^a=\{(x,y)\in\mathbb{R}^2 : x^2+(y-3)^2\le 9, x^2+y^2\ge a^2\}$, e usando il cambio di coordinate $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$, per cui $D^a_{\varrho\vartheta}=\{(\varrho,\vartheta)\in\mathbb{R}^2 : a\le\varrho\le 6\sin\vartheta, \vartheta\in[0,\pi]\}$, si ha $\iint_D \frac{|xy|e^{x^2+y^2}}{x^2+y^2}\,dxdy = \lim_{a\to 0^+}\iint_{D^a}\frac{|xy|e^{x^2+y^2}}{x^2+y^2}\,dxdy = \lim_{a\to 0^+}\iint_{D^a_{\varrho\vartheta}}\varrho e^{\varrho^2}|\cos\vartheta\sin\vartheta|\,d\varrho d\vartheta = 2\lim_{a\to 0^+}\int_0^{\pi/2}\sin\vartheta\cos\vartheta\cdot\big(\int_a^{6\sin\vartheta}\varrho e^{\varrho^2}\,d\varrho\big)\,d\vartheta = 2\lim_{a\to 0^+}\int_0^{\pi/2}\sin\vartheta\cos\vartheta\big[\frac12 e^{\varrho^2}\big]_a^{6\sin\vartheta}\,d\vartheta = \lim_{a\to 0^+}\int_0^{\pi/2}(e^{36\sin^2\vartheta}-e^{a^2})\sin\vartheta\cos\vartheta\,d\vartheta = \lim_{a\to 0^+}\big[\frac{1}{72}e^{36\sin^2\vartheta} - \frac12 a^2\sin^2\vartheta\big]_0^{\pi/2} = \frac{1}{72}(e^{36}-37)$.

(9) Posto $D^a=\{(x,y)\in\mathbb{R}^2 : x^2+(y-1)^2\ge 1, a^2\le x^2+y^2\le 4, |y|\ge|x|\operatorname{tg}a\}$, e usando il cambio di coordinate $x=\varrho\cos\vartheta$, $y=\varrho\sin\vartheta$, per cui $D^a_{\varrho\vartheta}=\{(\varrho,\vartheta)\in\mathbb{R}^2 : \max\{a, 2\sin\vartheta\}\le\varrho\le 2, \vartheta\in[a,\pi-a]\}\cup\{(\varrho,\vartheta)\in\mathbb{R}^2 : a\le\varrho\le 2, \vartheta\in[\pi+a, 2\pi-a]\}$, si ha $\iint_D\sqrt{\frac{x^2}{|y|}}\,dxdy = \lim_{a\to 0^+}\iint_{D^a}\sqrt{\frac{x^2}{|y|}}\,dxdy = \lim_{a\to 0^+}\iint_{D^a_{\varrho\vartheta}}\frac{\varrho|\cos\vartheta|}{\sqrt{\varrho|\sin\vartheta|}}\,\varrho\,d\varrho d\vartheta = 2\lim_{a\to 0^+}\int_{-\pi/2}^{-a}\frac{\cos\vartheta}{\sqrt{-\sin\vartheta}}\big(\int_a^2\varrho^{3/2}\,d\varrho\big)\,d\vartheta + 2\lim_{a\to 0^+}\int_a^{\pi/2}\frac{\cos\vartheta}{\sqrt{\sin\vartheta}}\big(\int_{2\sin\vartheta}^2\varrho^{3/2}\,d\varrho\big)\,d\vartheta = 2\lim_{a\to 0^+}\int_{-\pi/2}^{-a}\frac{\cos\vartheta}{\sqrt{-\sin\vartheta}}\big[\frac25\varrho^{5/2}\big]_a^2\,d\vartheta + 2\lim_{a\to 0^+}\int_a^{\pi/2}\frac{\cos\vartheta}{\sqrt{\sin\vartheta}}\big[\frac25\varrho^{5/2}\big]_{2\sin\vartheta}^2\,d\vartheta = \frac45\lim_{a\to 0^+}\int_{-\pi/2}^{-a}\frac{\cos\vartheta}{\sqrt{-\sin\vartheta}}(4\sqrt2 - a^{5/2})\,d\vartheta + \frac{16\sqrt2}{5}\lim_{a\to 0^+}\int_a^{\pi/2}\frac{\cos\vartheta}{\sqrt{\sin\vartheta}}\big(1-(\sin\vartheta)^{5/2}\big)\,d\vartheta \overset{(a)}{=} \frac45\lim_{a\to 0^+}(4\sqrt2 - a^{5/2})\int_{\sin a}^1\frac{1}{\sqrt t}\,dt + \frac{16\sqrt2}{5}\lim_{a\to 0^+}\int_{\sin a}^1\frac{1}{\sqrt t}\big(1-t^{5/2}\big)\,dt = \frac45\lim_{a\to 0^+}(4\sqrt2 - a^{5/2})\,2(1-\sqrt{\sin a}) + \frac{16\sqrt2}{5}\lim_{a\to 0^+}\big[2\sqrt t - \frac13 t^3\big]_{\sin a}^1 = \frac{32\sqrt2}{5} + \frac{16\sqrt2}{5}\lim_{a\to 0^+}\big(\frac53 - 2\sqrt{\sin a} + \frac13\sin^3 a\big) = \frac{176\sqrt2}{15}$, dove in (a) si sono usati i cambi di variabile $t=-\sin\vartheta$ e $t=\sin\vartheta$ nei due integrali.

(10) Posto $D^a=\{(x,y)\in\mathbb{R}^2 : a^2\le\frac{1}{16}x^2+\frac14 y^2\le 1, 2|y|\ge|x|\operatorname{tg}a\}$, e usando il cambio di coordinate $x=4\varrho\cos\vartheta$, $y=2\varrho\sin\vartheta$, per cui $D^a_{\varrho\vartheta}=\{(\varrho,\vartheta)\in\mathbb{R}^2 : a\le\varrho\le 1, \vartheta\in[a,\pi-a]\cup[\pi+a,2\pi-a]\}$, si ha $\iint_D\big|\frac{x}{y}\big|\,dxdy = \lim_{a\to 0^+}\iint_{D^a}\big|\frac{x}{y}\big|\,dxdy = \lim_{a\to 0^+}\iint_{D^a_{\varrho\vartheta}}\big|\frac{4\varrho\cos\vartheta}{2\varrho\sin\vartheta}\big|\,8\varrho\,d\varrho d\vartheta = 64\lim_{a\to 0^+}\int_a^{\pi/2}\frac{\cos\vartheta}{\sin\vartheta}\,d\vartheta\cdot\int_a^1\varrho\,d\varrho = 32\lim_{a\to 0^+}\big[\log\sin\vartheta\big]_a^{\pi/2}\cdot\big[\varrho^2\big]_a^1 = -32\lim_{a\to 0^+}(1-a^2)\log(\sin a) = +\infty$.

(11) Posto $D^a=\{(x,y)\in\mathbb{R}^2 : x\in[0,1], x-1\le y\le x-a\}$, si ha $\iint_D e^{\frac{x+y}{x-y}}\,dxdy = \lim_{a\to 0^+}\iint_{D^a}e^{\frac{x+y}{x-y}}\,dxdy \overset{(a)}{=} \frac12\lim_{a\to 0^+}\iint_{D^a_{st}}e^{\frac st}\,dsdt = \frac12\lim_{a\to 0^+}\int_a^1\big(\int_{-t}^t e^{\frac st}\,ds\big)\,dt = \frac12\lim_{a\to 0^+}\int_a^1\big[te^{\frac st}\big]_{s=-t}^{s=t}\,dt = \frac12\lim_{a\to 0^+}\int_a^1\big(e-\frac1e\big)t\,dt = \frac14\big(e-\frac1e\big)\lim_{a\to 0^+}\big[t^2\big]_a^1 = \frac14\big(e-\frac1e\big)\lim_{a\to 0^+}(1-a^2) = \frac14\big(e-\frac1e\big)$, dove in (a) si è usato il cambio di variabile

$$\begin{cases} x+y=s \\ x-y=t \end{cases} \iff \begin{cases} x=\frac{s+t}{2} \\ y=\frac{s-t}{2} \end{cases} \quad \text{e } D^a_{st}=\{(s,t)\in\mathbb{R}^2 : t\in[a,1], -t\le s\le t\}.$$

(12) Posto $D^a=\{(x,y)\in\mathbb{R}^2 : x\in[a,1], 0\le y\le x-a\}$, si ha $\iint_D\log\frac{x+y}{x-y}\,dxdy = \lim_{a\to 0^+}\iint_{D^a}\log\frac{x+y}{x-y}\,dxdy \overset{(a)}{=} \frac12\lim_{a\to 0^+}\iint_{D^a_{st}}\log\frac st\,dsdt = \frac12\lim_{a\to 0^+}\int_a^1\big(\int_t^{2-t}(\log s - \log t)\,ds\big)\,dt = \frac12\lim_{a\to 0^+}\int_a^1\big[s(\log s - 1) - s\log t\big]_{s=t}^{s=2-t}\,dt = \frac12\lim_{a\to 0^+}\int_a^1\big((2-t)(\log(2-t)-\log t) + 2(t-1)\big)\,dt = \frac12\lim_{a\to 0^+}\big[(2t-\frac12 t^2)(\log(2-t)-\log t) + (t-1)^2\big]_a^1 -$

$\frac{1}{2}\lim_{a\to 0^+}\int_a^1 (2t-\frac{1}{2}t^2)(\frac{1}{t-2}-\frac{1}{t})\,dt = \frac{1}{2}\lim_{a\to 0^+}\big((2a-\frac{1}{2}a^2)(\log a-\log(2-a))-(a-1)^2\big)+\frac{1}{2}\lim_{a\to 0^+}\int_a^1\big(1-\frac{2}{t-2}\big)\,dt = -\frac{1}{2}+\frac{1}{2}\lim_{a\to 0^+}\big[t-2\log(2-t)\big]_a^1 = -\frac{1}{2}+\frac{1}{2}\lim_{a\to 0^+}\big(1-a+2\log(2-a)\big)=\log 2$, dove in (a) si è usato il cambio di variabile $\begin{cases} x+y=s \\ x-y=t \end{cases} \iff \begin{cases} x=\frac{s+t}{2} \\ y=\frac{s-t}{2} \end{cases}$ e $D^a_{st}=\{(s,t)\in\mathbb{R}^2 : t\in[a,1], t\le s\le 2-t\}$.

(13) Posto $D^-=\{(x,y)\in\mathbb{R}^2 : y\in[-2,0], -2\le x\le y\}$, $D^+=\{(x,y)\in\mathbb{R}^2 : y\in[0,2], -2\le x\le y\}$, $D^{-,b}=\{(x,y)\in\mathbb{R}^2 : y\in[-2,-b], -2\le x\le y\}$, $D^{+,a}=\{(x,y)\in\mathbb{R}^2 : y\in[a,2], -2\le x\le y\}$, si ha $D=D^-\cup D^+$ e $\iint_D \frac{x}{\sqrt{|y|}}\,dxdy=\iint_{D^-}\frac{x}{\sqrt{|y|}}\,dxdy+\iint_{D^+}\frac{x}{\sqrt{|y|}}\,dxdy=\lim_{b\to 0^+}\iint_{D^{-,b}}\frac{x}{\sqrt{|y|}}\,dxdy+\lim_{a\to 0^+}\iint_{D^{+,a}}\frac{x}{\sqrt{|y|}}\,dxdy=\lim_{b\to 0^+}\int_{-2}^{-b}\big(\int_{-2}^{y}\frac{x}{\sqrt{|y|}}\,dx\big)\,dy+\lim_{a\to 0^+}\int_a^2\big(\int_{-2}^{y}\frac{x}{\sqrt{|y|}}\,dx\big)\,dy=\lim_{b\to 0^+}\int_{-2}^{-b}\frac{1}{2\sqrt{|y|}}\big[x^2\big]_{-2}^{y}\,dy+\lim_{a\to 0^+}\int_a^2\frac{1}{2\sqrt{|y|}}\big[x^2\big]_{-2}^{y}\,dy=\lim_{b\to 0^+}\int_b^2\frac{t^2-4}{2\sqrt{t}}\,dt+\lim_{a\to 0^+}\int_a^2\frac{y^2-4}{2\sqrt{y}}\,dy=\lim_{b\to 0^+}\big[\frac{2}{5}t^{5/2}-8\sqrt{t}\big]_b^2+\lim_{a\to 0^+}\big[\frac{2}{5}y^{5/2}-8\sqrt{y}\big]_a^2=\lim_{b\to 0^+}\big(-\frac{32\sqrt{2}}{5}-\frac{2}{5}b^{5/2}+8\sqrt{b}\big)+\lim_{a\to 0^+}\big(-\frac{32\sqrt{2}}{5}-\frac{2}{5}a^{5/2}+8\sqrt{a}\big)=-\frac{64\sqrt{2}}{5}$.

(14) Posto $D^a=\{(x,y)\in\mathbb{R}^2 : x\in[0,a], 0\le y\le e^{-x}\}$, si ha $\iint_D xy\,dxdy=\lim_{a\to+\infty}\iint_{D^a}xy\,dxdy=\lim_{a\to+\infty}\int_0^a x\big(\int_0^{e^{-x}}y\,dy\big)\,dx=\frac{1}{2}\lim_{a\to+\infty}\int_0^a x\big[y^2\big]_{y=0}^{y=e^{-x}}\,dx=\frac{1}{2}\lim_{a\to+\infty}\int_0^a xe^{-2x}\,dx=\frac{1}{2}\lim_{a\to+\infty}\big[-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}\big]_0^a=\frac{1}{2}\lim_{a\to+\infty}\big(\frac{1}{4}-\frac{a}{2}e^{-2a}-\frac{1}{4}e^{-2a}\big)=\frac{1}{8}$.

(15) Posto $D^a=\{(x,y)\in\mathbb{R}^2 : x,y\in[-a,a]\}$, si ha $\iint_D\frac{x^2y^2}{(1+x^6)(1+y^6)}\,dxdy=\lim_{a\to+\infty}\iint_{D^a}\frac{x^2y^2}{(1+x^6)(1+y^6)}\,dxdy=\lim_{a\to+\infty}\int_{-a}^a\frac{x^2}{1+x^6}\,dx\cdot\int_{-a}^a\frac{y^2}{1+y^6}\,dy=\lim_{a\to+\infty}\big(2\int_0^a\frac{x^2}{1+x^6}\,dx\big)^2\overset{(a)}{=}\frac{4}{9}\lim_{a\to+\infty}\big(\int_0^{a^3}\frac{dt}{1+t^2}\big)^2=\frac{4}{9}\lim_{a\to+\infty}\big(\operatorname{arctg}(a^3)\big)^2=\frac{\pi^2}{9}$, dove in (a) si è usato il cambio di variabile $t=x^3$.

(16) Posto $D^a=\{(x,y)\in\mathbb{R}^2 : x,y\ge 0, x^2+y^2\le a^2\}$, si ha $\iint_D\frac{dxdy}{(2+x^2+y^2)\sqrt{x^2+y^2}}=\lim_{a\to+\infty}\iint_{D^a}\frac{dxdy}{(2+x^2+y^2)\sqrt{x^2+y^2}}=\lim_{a\to+\infty}\int_0^{\pi/2}\big(\int_0^a\frac{1}{(2+\varrho^2)\varrho}\,\varrho\,d\varrho\big)\,d\vartheta=\frac{\pi}{2}\lim_{a\to+\infty}\big[\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\varrho}{\sqrt{2}}\big]_0^a=\frac{\pi}{2\sqrt{2}}\lim_{a\to+\infty}\operatorname{arctg}\frac{a}{\sqrt{2}}=\frac{\pi^2}{4\sqrt{2}}$.

(17) Posto $D^a=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le a^2\}$, si ha $\iint_D\frac{x^2y^2e^{-x^2-y^2}}{x^2+y^2}\,dxdy=\lim_{a\to+\infty}\iint_{D^a}\frac{x^2y^2e^{-x^2-y^2}}{x^2+y^2}\,dxdy=\lim_{a\to+\infty}\int_0^{2\pi}\cos^2\vartheta\sin^2\vartheta\,d\vartheta\cdot\int_0^a\varrho^3e^{-\varrho^2}\,d\varrho\overset{(a)}{=}\lim_{a\to+\infty}\int_0^{2\pi}\frac{1}{8}(1-\cos 4\vartheta)\,d\vartheta\cdot\int_0^{a^2}\frac{1}{2}te^{-t}\,dt=\frac{1}{16}\lim_{a\to+\infty}\big[\vartheta-\frac{1}{4}\sin 4\vartheta\big]_0^{2\pi}\cdot\big[-(t+1)e^{-t}\big]_0^{a^2}=\frac{\pi}{8}\lim_{a\to+\infty}\big(1-(a^2+1)e^{-a^2}\big)=\frac{\pi}{8}$, dove in (a) si è usato il cambio di variabile $t=\varrho^2$.

(18) La funzione integranda è positiva, per cui si può calcolare l'integrale usando un'esaustione qualsiasi. Posto, per ogni $a\in(0,1)$, $D^a=\{(x,y,z)\in\mathbb{R}^3 : a\le x\le 1, |y|\le x, 1\le z\le\frac{4}{4x-y^2}\}$, si ha $\iiint_D\frac{dxdydz}{xz^2}=\lim_{a\to 0^+}\int_a^1\frac{dx}{x}\int_{-x}^x dy\int_1^{4/(4x-y^2)}\frac{dz}{z^2}=\lim_{a\to 0^+}\int_a^1\frac{dx}{x}\int_{-x}^x\big[-\frac{1}{z}\big]_1^{4/(4x-y^2)}\,dy=\lim_{a\to 0^+}\int_a^1\frac{dx}{x}\int_{-x}^x(1-x+\frac{y^2}{4})\,dy=\lim_{a\to 0^+}\int_a^1\big[y-xy+\frac{y^3}{12}\big]_{-x}^x\frac{dx}{x}=2\lim_{a\to 0^+}\int_a^1\big(x-x^2+\frac{x^3}{12}\big)\frac{dx}{x}=2\lim_{a\to 0^+}\big[x-\frac{x^2}{2}+\frac{x^3}{36}\big]_a^1=\frac{19}{18}$.

(19) La funzione integranda è positiva, per cui si può calcolare l'integrale usando un'esaustione qualsiasi. Usando il cambiamento di coordinate sferiche $x=\varrho\cos\vartheta\sin\varphi, y=\frac{1}{2}\varrho\sin\vartheta\sin\varphi, z=\varrho\cos\varphi$ si ottiene $D_{\varrho\vartheta\varphi}=\{(\varrho,\vartheta,\varphi)\in\mathbb{R}^3 : 0\le\vartheta\le 2\pi, 0\le\varrho\le 1, 0\le\varphi\le\frac{\pi}{2}\}$. Posto, per ogni

$a \in (0,1)$, $D^a_{\varrho\vartheta\varphi} = \{(\varrho,\vartheta,\varphi) \in \mathbb{R}^3 : 0 \le \vartheta \le 2\pi, 0 \le \varrho \le 1-a, 0 \le \varphi \le \frac{\pi}{2}\}$, si ha

$$\iiint_D \frac{z\,dxdydz}{\sqrt{1-x^2-4y^2-z^2}} = \lim_{a\to 0^+} \iiint_{D^a_{\varrho\vartheta\varphi}} \frac{\varrho\cos\varphi}{\sqrt{1-\varrho^2}} \cdot \frac{1}{2}\varrho^2 \sin\varphi\, d\varrho d\vartheta d\varphi$$
$$= \frac{1}{2}\lim_{a\to 0^+} \int_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi \int_0^{2\pi} d\vartheta \int_0^{1-a} \frac{\varrho^3\, d\varrho}{\sqrt{1-\varrho^2}} \overset{(a)}{=} \pi\Big[\frac{1}{2}\sin^2\varphi\Big]_0^{\pi/2} \lim_{a\to 0^+}\int_{2a-a^2}^1 \frac{1}{2}\Big(\frac{1}{\sqrt{t}} - \sqrt{t}\Big)\, dt$$
$$= \frac{\pi}{4}\lim_{a\to 0^+}\Big[2\sqrt{t} - \frac{2}{3}t^{3/2}\Big]_{2a-a^2}^1 = \frac{\pi}{3}\,,$$

dove in (a) si è usata la sostituzione $1-\varrho^2 = t \implies -2\varrho\, d\varrho = dt$.

(20) La funzione integranda è positiva, per cui si può calcolare l'integrale usando un'esaustione qualsiasi. Passando in coordinate cilindriche, si ha $D_{\varrho\vartheta z} = \{(\varrho,\vartheta,z) \in \mathbb{R}^3 : \varrho^2 \le z, 0 \le \vartheta \le 2\pi\}$. Posto, per ogni $a > 0$, $D^a_{\varrho\vartheta z} := \{(\varrho,\vartheta,z) \in \mathbb{R}^3 : \varrho^2 \le z \le a, 0 \le \vartheta \le 2\pi\}$, si ha $\iiint_D \frac{x^2}{1+z^6}\,dxdydz = \lim_{a\to+\infty}\iiint_{D^a_{\varrho\vartheta z}} \frac{\varrho^2\cos^2\vartheta}{1+z^6}\,\varrho d\varrho d\vartheta = \lim_{a\to+\infty}\int_0^a \frac{dz}{1+z^6}\int_0^{2\pi}\cos^2\vartheta\, d\vartheta \int_0^{\sqrt{z}}\varrho^3\, d\varrho = \lim_{a\to+\infty}\big[\frac{\vartheta}{2} + \frac{\sin 2\vartheta}{4}\big]_0^{2\pi} \int_0^a \big[\frac{\varrho^4}{4}\big]_0^{\sqrt{z}} \frac{dz}{1+z^6} = \lim_{a\to+\infty}\frac{\pi}{4}\int_0^a \frac{z^2\,dz}{1+z^6} = \frac{\pi}{4}\lim_{a\to+\infty}\big[\frac{1}{3}\operatorname{arctg}(z^3)\big]_0^a = \frac{\pi^2}{12}$.

(21) La funzione integranda è positiva, per cui si può calcolare l'integrale usando un'esaustione qualsiasi. Usando il cambiamento di coordinate sferiche $x = \varrho\cos\vartheta\sin\varphi, y = \varrho\sin\vartheta\sin\varphi, z = \varrho\cos\varphi$ si ottiene $D_{\varrho\vartheta\varphi} = \{(\varrho,\vartheta,\varphi) \in \mathbb{R}^3 : 0 \le \vartheta \le 2\pi, 0 \le \varphi \le \pi, \varrho \ge 1\}$. Posto, per ogni $a > 0$, $D^a_{\varrho\vartheta\varphi} = \{(\varrho,\vartheta,\varphi) \in \mathbb{R}^3 : 0 \le \vartheta \le 2\pi, 0 \le \varphi \le \pi, 1 \le \varrho \le a\}$, si ha

$$\iiint_D \frac{x^2\,dxdydz}{(x^2+y^2+z^2)^4} = \lim_{a\to+\infty}\iiint_{D^a_{\varrho\vartheta\varphi}} \frac{\varrho^2\cos^2\vartheta\sin^2\varphi}{\varrho^8}\varrho^2\sin\varphi\, d\varrho d\vartheta d\varphi$$
$$= \lim_{a\to+\infty}\int_0^{\pi}(1-\cos^2\varphi)\sin\varphi\, d\varphi \int_0^{2\pi}\frac{1+\cos 2\vartheta}{2}\, d\vartheta \int_1^a \frac{d\varrho}{\varrho^4}$$
$$= \Big[-\cos\varphi + \frac{\cos^3\varphi}{3}\Big]_0^{\pi} \cdot \Big[\frac{\vartheta}{2} + \frac{\sin 2\vartheta}{4}\Big]_0^{2\pi} \lim_{a\to+\infty}\Big[-\frac{1}{3\varrho^3}\Big]_1^a = \frac{4\pi}{9}\,.$$

□