Analisi Matematica II
Integrali curvilinei (svolgimenti)

Svolgimento esercizio 1

(1) Si ha, successivamente,

$$\begin{aligned}\gamma'(t) &= (3t^2, 2t),\\ \|\gamma'(t)\|^2 &= 9t^4 + 4t^2 = t^2(9t^2+4),\\ \ell(\gamma) &= \int_0^1 \|\gamma'(t)\|\, dt = \int_0^1 t\sqrt{9t^2+4}\, dt \overset{(a)}{=} \frac{1}{18}\int_4^{13} \sqrt{x}\, dx = \frac{1}{27}\Big[x^{2/3}\Big]_4^{13} = \frac{1}{27}(13\sqrt{13}-8),\end{aligned}$$

dove in (a) si è usata la sostituzione $9t^2+4 = x \implies 18t\, dt = dx$.

(2) Si ha, successivamente,

$$\begin{aligned}\gamma'(t) &= \Big(-\frac{1}{\sqrt{1-t^2}}, \frac{1}{t}\Big),\\ \|\gamma'(t)\|^2 &= \frac{1}{1-t^2} + \frac{1}{t^2} = \frac{1}{t^2(1-t^2)},\\ \ell(\gamma) &= \int_{1/2}^1 \|\gamma'(t)\|\, dt = \int_{1/2}^1 \frac{dt}{t\sqrt{1-t^2}} \overset{(a)}{=} \int_{\pi/6}^{\pi/2} \frac{dx}{\sin x} \overset{(b)}{=} \int_{\operatorname{tg}(\pi/12)}^1 \frac{y^2+1}{2y}\frac{2}{y^2+1}\, dy\\ &= \Big[\log y\Big]_{\operatorname{tg}(\pi/12)}^1 = -\log \operatorname{tg}\frac{\pi}{12},\end{aligned}$$

dove si è usato, in (a) la sostituzione $t = \sin x \implies dt = \cos x\, dx$, in (b) la sostituzione $x = 2\operatorname{arctg} y$, per cui $\sin x = \frac{2y}{1+y^2}$, $dx = \frac{2\,dy}{1+y^2}$.

(3) Si ha, successivamente,

$$\begin{aligned}\gamma'(t) &= (e^t, 2e^{2t}),\\ \|\gamma'(t)\|^2 &= e^{2t} + 4e^{4t} = e^{2t}(1+4e^{2t}),\\ \ell(\gamma) &= \int_0^1 \|\gamma'(t)\|\, dt = \int_0^1 e^t\sqrt{1+4e^{2t}}\, dt \overset{(a)}{=} \frac{1}{2}\int_2^{2e} \sqrt{x^2+1}\, dx \overset{(b)}{=} \frac{1}{2}\int_{2+\sqrt{5}}^{2e+\sqrt{4e^2+1}} \frac{y^2+1}{2y}\frac{y^2+1}{2y^2}\, dy\\ &= \frac{1}{8}\int_{2+\sqrt{5}}^{2e+\sqrt{4e^2+1}} \Big(y + \frac{2}{y} + \frac{1}{y^3}\Big)\, dy = \frac{1}{8}\Big[\frac{y^2}{2} + 2\log y - \frac{1}{2y^2}\Big]_{2+\sqrt{5}}^{2e+\sqrt{4e^2+1}}\\ &= \frac{1}{2}(e\sqrt{4e^2+1} - \sqrt{5}\,) + \frac{1}{4}\log\frac{2e+\sqrt{4e^2+1}}{2+\sqrt{5}},\end{aligned}$$

dove si è usato, in (a) la sostituzione $2e^t = x \implies 2e^t\, dt = dx$, in (b) la sostituzione $\sqrt{x^2+1} = y - x$, per cui $x = \frac{y^2-1}{2y}$, $\sqrt{x^2+1} = \frac{y^2+1}{2y}$, $dx = \frac{y^2+1}{2y^2}\, dy$.

(4) Si ha, successivamente,

$$\begin{aligned}
\gamma'(t) &= (6t+10, 8t+5),\\
\|\gamma'(t)\|^2 &= (6t+10)^2 + (8t+5)^2 = 25(4t^2+8t+5),\\
\ell(\gamma) &= \int_{-1}^{1} \|\gamma'(t)\|\, dt = 5\int_{-1}^{1}\sqrt{4t^2+8t+5}\, dt \overset{(a)}{=} \frac{5}{4}\int_0^8 \sqrt{x^2+1}\, dx \overset{(b)}{=} \frac{5}{4}\int_1^{8+\sqrt{65}} \frac{y^2+1}{2y}\frac{y^2+1}{2y^2}\, dy\\
&= \frac{5}{16}\int_1^{8+\sqrt{65}} \Big(y + \frac{2}{y} + \frac{1}{y^3}\Big)\, dy = \frac{5}{16}\Big[\frac{y^2}{2} + 2\log y - \frac{1}{2y^2}\Big]_1^{8+\sqrt{65}} = 5\sqrt{65} + \frac{5}{8}\log(8+\sqrt{65}),
\end{aligned}$$

dove si è usato, in (a) la sostituzione $4(t+1) = x$, in (b) la sostituzione $\sqrt{x^2+1} = y - x$, per cui $x = \frac{y^2-1}{2y}$, $\sqrt{x^2+1} = \frac{y^2+1}{2y}$, $dx = \frac{y^2+1}{2y^2}\, dy$.

(5) Si ha, successivamente,

$$\begin{aligned}
\gamma'(t) &= \Big(1, \sqrt{\frac{2}{t}}, \frac{1}{t}\Big),\\
\|\gamma'(t)\|^2 &= 1 + \frac{2}{t} + \frac{1}{t^2} = \frac{(t+1)^2}{t^2},\\
\ell(\gamma) &= \int_1^2 \|\gamma'(t)\|\, dt = \int_1^2 \frac{t+1}{t}\, dt = \int_1^2 \Big(1 + \frac{1}{t}\Big)\, dt = \Big[t + \log t\Big]_1^2 = 1 + \log 2.
\end{aligned}$$

(6) Si ha, successivamente,

$$\begin{aligned}
\gamma'(t) &= \Big(-\frac{1}{2}\sin(2t), -3\cos^2 t\sin t, 3\sin^2 t\cos t\Big),\\
\|\gamma'(t)\|^2 &= \frac{1}{4}\sin^2(2t) + 9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t = \frac{5}{2}\sin^2(2t),\\
\ell(\gamma) &= \int_{-\pi/2}^{\pi/2} \|\gamma'(t)\|\, dt = \sqrt{\frac{5}{2}}\int_{-\pi/2}^{\pi/2} |\sin(2t)|\, dt = \sqrt{10}\int_0^{\pi/2} \sin(2t)\, dt = -\frac{\sqrt{10}}{2}\Big[\cos(2t)\Big]_0^{\pi/2} = \sqrt{10}.
\end{aligned}$$

(7) Si ha, successivamente,

$$\begin{aligned}
\gamma'(t) &= (-4\sin t + 4\sin 4t, 4\cos t - 4\cos 4t),\\
\|\gamma'(t)\|^2 &= (-4\sin t + 4\sin 4t)^2 + (4\cos t - 4\cos 4t)^2 = 32(1 - \cos 3t) = 64\sin^2\Big(\frac{3}{2}t\Big),\\
\ell(\gamma) &= \int_0^{2\pi} \|\gamma'(t)\|\, dt = 16\int_0^{2\pi} \Big|\sin\Big(\frac{3}{2}t\Big)\Big|\, dt = 24\int_0^{2\pi/3} \sin\Big(\frac{3}{2}t\Big)\, dt \overset{(a)}{=} 16\int_0^{\pi} \sin x\, dx\\
&= 16\Big[-\cos x\Big]_0^{\pi} = 32,
\end{aligned}$$

dove in (a) si è usata la sostituzione $\frac{3}{2}t = x \implies \frac{3}{2}\, dt = dx$.

(8) Si ha, successivamente,

$$\begin{aligned}
\gamma'(t) &= (\sinh t\cos t - \cosh t\sin t, \sinh t\sin t - \cosh t\cos t, 1),\\
\|\gamma'(t)\|^2 &= (\sinh t\cos t - \cosh t\sin t)^2 + (\sinh t\sin t - \cosh t\cos t)^2 + 1 = 2(\cosh t)^2,\\
\ell(\gamma) &= \int_0^1 \|\gamma'(t)\|\, dt = \sqrt{2}\int_0^1 \cosh t\, dt = \sqrt{2}\Big[\sinh t\Big]_0^1 = \sqrt{2}\sinh 1 = \frac{e^2-1}{e\sqrt{2}}.
\end{aligned}$$

(9) Si ha

$$\ell(\gamma)=\int_0^2 \sqrt{1+9(3+2x)}\,dx \overset{(a)}{=} \frac{1}{9}\int_{2\sqrt7}^{8} y^2\,dy=\frac{1}{27}\Big[y^3\Big]_{2\sqrt7}^{8}=\frac{512-56\sqrt7}{27},$$

dove in (a) si è usata la sostituzione $\sqrt{18x+28}=y$, per cui $x=\frac{y^2-28}{18}$, $dx=\frac19 y\,dy$.

(10) Si ha

$$\begin{aligned}\ell(\gamma)&=\int_0^{\pi/3}\sqrt{1+\operatorname{tg}^2 x}\,dx=\int_0^{\pi/3}\frac{dx}{\cos x}\overset{(a)}{=}\int_0^{\sqrt3/3}\frac{1+y^2}{1-y^2}\,\frac{2}{1+y^2}\,dy\\ &=\int_0^{\sqrt3/3}\Big(\frac{1}{y+1}-\frac{1}{y-1}\Big)\,dy=\Big[\log\Big|\frac{y+1}{y-1}\Big|\Big]_0^{\sqrt3/3}=\log\frac{3+\sqrt3}{3-\sqrt3}=\log(2+\sqrt3\,),\end{aligned}$$

dove in (a) si è usata la sostituzione $x=2\operatorname{arctg} y$, per cui $\cos x=\frac{1-y^2}{1+y^2}$, $dx=\frac{2}{y^2+1}\,dy$.

(11) Si ha

$$\begin{aligned}\ell(\gamma)&=\int_0^{3/2}\sqrt{1+\frac{1}{1+2x}}\,dx=\int_0^{3/2}\sqrt{\frac{2x+2}{2x+1}}\,dx\overset{(a)}{=}\int_{\sqrt{5/2}}^{\sqrt2}\frac{y^2}{(y^2-1)^2}\,dy\\ &\overset{(b)}{=}\frac14\Big[\log\Big|\frac{y-1}{y+1}\Big|-\frac{1}{y-1}-\frac{1}{y+1}\Big]_{\sqrt{5/2}}^{\sqrt2}=\frac14\log\frac{\sqrt2-1}{\sqrt2+1}-\frac14\log\frac{\sqrt5-2}{\sqrt5+2}-\frac{\sqrt2}{2}+\sqrt5,\end{aligned}$$

dove si è usato in (a) la sostituzione $\sqrt{\frac{2x+2}{2x+1}}=y$, per cui $x=-\frac{y^2-2}{2(y^2-1)}$, $dx=-\frac{y}{(y^2-1)^2}\,dy$, in (b) la decomposizione $\frac{y^2}{(y^2-1)^2}=\frac14\big(\frac{1}{y-1}+\frac{1}{(y-1)^2}-\frac{1}{y+1}+\frac{1}{(y+1)^2}\big)$.

(12) Si ha

$$\begin{aligned}\ell(\gamma)&=\int_0^5\sqrt{1+\frac{x(9-x)^2}{(6-x)^3}}\,dx=3\sqrt3\int_0^5\frac{1}{6-x}\sqrt{\frac{8-x}{6-x}}\,dx\overset{(a)}{=}3\sqrt3\int_{\sqrt{4/3}}^{\sqrt3}\frac{y^2-1}{2}\,y\,\frac{4y}{(y^2-1)^2}\,dy\\ &=6\sqrt3\int_{\sqrt{4/3}}^{\sqrt3}\frac{y^2}{(y^2-1)^2}\,dy=6\sqrt3\int_{\sqrt{4/3}}^{\sqrt3}\Big(1+\frac12\frac{1}{y-1}-\frac12\frac{1}{y+1}\Big)\,dy\\ &=6\sqrt3\Big[y+\frac12\log\Big|\frac{y-1}{y+1}\Big|\Big]_{\sqrt{4/3}}^{\sqrt3}=6+3\sqrt3\log\frac{\sqrt3-1}{\sqrt3+1}-3\sqrt3\log\frac{2-\sqrt3}{2+\sqrt3}=6+3\sqrt3\,\log(2+\sqrt3\,),\end{aligned}$$

dove in (a) si è usata la sostituzione $\sqrt{\frac{8-x}{6-x}}=y$, per cui $x=\frac{6y^2-8}{y^2-1}$, $dx=\frac{4y}{(y^2-1)^2}\,dy$.

(13) Si ha

$$\begin{aligned}\ell(\gamma)&=\int_1^2\sqrt{1+(2x)^2}\,dx\overset{(a)}{=}\int_{2+\sqrt5}^{4+\sqrt{17}}\frac{y^2+1}{2y}\,\frac{y^2+1}{4y^2}\,dy=\frac18\int_{2+\sqrt5}^{4+\sqrt{17}}\Big(y+\frac{2}{y}+\frac{1}{y^3}\Big)\,dy\\ &=\frac18\Big[\frac{y^2}{2}+2\log|y|-\frac{1}{2y^2}\Big]_{2+\sqrt5}^{4+\sqrt{17}}=\sqrt{17}-\frac{\sqrt5}{2}+\frac14\log\frac{4+\sqrt{17}}{2+\sqrt5},\end{aligned}$$

dove in (a) si è usata la sostituzione $\sqrt{1+4x^2}=y-2x$, per cui $x=\frac{y^2-1}{4y}$, $dx=\frac{y^2+1}{4y^2}\,dy$, $\sqrt{1+4x^2}=\frac{y^2+1}{2y}$.

Alternativamente, si ha

$$\ell(\gamma) = \int_1^2 \sqrt{1+4x^2}\,dx \overset{(a)}{=} \frac{1}{2}\int_{\operatorname{arsinh} 2}^{\operatorname{arsinh} 4} \cosh^2 y\,dy = \frac{1}{4}\int_{\operatorname{arsinh} 2}^{\operatorname{arsinh} 4} (\cosh 2y + 1)\,dy = \frac{1}{4}\Big[\frac{1}{2}\sinh 2y + y\Big]_{\operatorname{arsinh} 2}^{\operatorname{arsinh} 4}$$
$$= \frac{1}{4}\sinh \operatorname{arsinh} 4 \cosh \operatorname{arsinh} 4 - \frac{1}{4}\sinh \operatorname{arsinh} 2 \cosh \operatorname{arsinh} 2 + \frac{1}{4}\log\frac{4+\sqrt{17}}{2+\sqrt{5}}$$
$$= \sqrt{17} - \frac{\sqrt{5}}{2} + \frac{1}{4}\log\frac{4+\sqrt{17}}{2+\sqrt{5}},$$

dove in (a) si è usata la sostituzione $2x = \sinh y$, $dx = \frac{1}{2}\cosh y\,dy$.

(14) Si ha

$$\ell(\gamma) = \int_0^1 \sqrt{1+(1+\sqrt{x})^2}\,dx \overset{(a)}{=} \int_1^2 2(y-1)\sqrt{y^2+1}\,dy \overset{(b)}{=} \int_2^5 \sqrt{z}\,dz - 2\int_{1+\sqrt{2}}^{2+\sqrt{5}} \frac{z^2+1}{2z}\,\frac{z^2+1}{2z^2}\,dz$$
$$= \frac{2}{3}\Big[z^{3/2}\Big]_2^5 - \frac{1}{2}\int_{1+\sqrt{2}}^{2+\sqrt{5}}\Big(z + \frac{2}{z} + \frac{1}{z^3}\Big)\,dz = \frac{2}{3}\big(5\sqrt{5} - 2\sqrt{2}\big) - \frac{1}{2}\Big[\frac{z^2}{2} + 2\log z - \frac{1}{2z^2}\Big]_{1+\sqrt{2}}^{2+\sqrt{5}}$$
$$= \frac{10}{3}\sqrt{5} - \frac{4}{3}\sqrt{2} - 2\sqrt{5} + \sqrt{2} - \log\frac{2+\sqrt{5}}{1+\sqrt{2}} = \frac{4}{3}\sqrt{5} - \frac{1}{3}\sqrt{2} - \log\frac{2+\sqrt{5}}{1+\sqrt{2}},$$

dove si è usato in (a) la sostituzione $1+\sqrt{x} = y$, per cui $x = (y-1)^2$, $dx = 2(y-1)\,dy$, in (b) la sostituzione $z = y^2+1$, per cui $dz = 2y\,dy$, nel primo integrale, e $\sqrt{1+y^2} = z - y$, per cui $y = \frac{z^2-1}{2z}$, $dy = \frac{z^2+1}{2z^2}\,dz$, $\sqrt{1+y^2} = \frac{z^2+1}{2z}$, nel secondo integrale.

(15) Si ha

$$\ell(\gamma) = \int_0^1 \sqrt{9e^{6\vartheta} + e^{6\vartheta}}\,d\vartheta = \sqrt{10}\int_0^1 e^{3\vartheta}\,d\vartheta = \frac{\sqrt{10}}{3}\Big[e^{3\vartheta}\Big]_0^1 = \frac{\sqrt{10}}{3}(e^3 - 1).$$

(16) Si ha

$$\ell(\gamma) = \int_0^1 \sqrt{4+4\vartheta^2}\,d\vartheta = 2\int_0^1 \sqrt{1+\vartheta^2}\,d\vartheta \overset{(a)}{=} 2\int_1^{1+\sqrt{2}} \frac{z^2+1}{2z}\,\frac{z^2+1}{2z^2}\,dz$$
$$= \frac{1}{2}\int_1^{1+\sqrt{2}}\Big(z + \frac{2}{z} + \frac{1}{z^3}\Big)\,dz = \frac{1}{2}\Big[\frac{z^2}{2} + 2\log z - \frac{1}{2z^2}\Big]_1^{1+\sqrt{2}} = \sqrt{2} + \log(1+\sqrt{2}),$$

dove in (a) si è usata la sostituzione $\sqrt{1+\vartheta^2} = z - \vartheta$, per cui $\vartheta = \frac{z^2-1}{2z}$, $d\vartheta = \frac{z^2+1}{2z^2}\,dz$, $\sqrt{1+\vartheta^2} = \frac{z^2+1}{2z}$.

□

Svolgimento esercizio 2

(1) Poiché $\int_\gamma f\,ds = \int_0^1 f(\gamma(t))\,\|\gamma'(t)\|\,dt$, calcoliamo $f(\gamma(t)) = \sin(\pi t) + \cos(2\pi t)$, e $\|\gamma'(t)\| = \pi\sqrt{5}$. Quindi

$$\int_\gamma f\,ds = \pi\sqrt{5}\int_0^1 \big(\sin(\pi t) + \cos(2\pi t)\big)\,dt = \pi\sqrt{5}\,\Big[\frac{-\cos(\pi t)}{\pi} + \frac{\sin(2\pi t)}{2\pi}\Big]_0^1 = 2\sqrt{5}.$$

(2) Poiché $\int_\gamma f\,ds = \int_0^\pi f(\gamma(t))\,\|\gamma'(t)\|\,dt$, calcoliamo $f(\gamma(t)) = \sqrt{1-\sin^2 t} = |\cos t|$, e $\|\gamma'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$. Quindi

$$\int_\gamma f\,ds = \int_0^\pi |\cos t|\,dt = 2\int_0^{\pi/2} \cos t\,dt = 2\Big[\sin t\Big]_0^{\pi/2} = 2.$$

(3) Poiché $\int_\gamma f\,ds = \int_{\pi/2}^\pi f(\gamma(t))\,\|\gamma'(t)\|\,dt$, calcoliamo $f(\gamma(t)) = (2\cos t)^2 \cdot 2\sin t = 8\cos^2 t \sin t$, e $\|\gamma'(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$. Quindi

$$\int_\gamma f\,ds = \int_{\pi/2}^\pi 16\cos^2 t \sin t\,dt \overset{(a)}{=} 16\int_{-1}^0 x^2\,dx = \frac{16}{3}\Big[x^3\Big]_{-1}^0 = \frac{16}{3},$$

dove in (a) si è usata la sostituzione $\cos t = x$, per cui $dx = -\sin t\,dt$.

(4) Poiché $\int_\gamma f\,ds = \int_0^{\pi/2} f(\gamma(t))\,\|\gamma'(t)\|\,dt$, calcoliamo $f(\gamma(t)) = \frac{\cos t}{1+\sin^2 t}$, e $\|\gamma'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$. Quindi

$$\int_\gamma f\,ds = \int_0^{\pi/2} \frac{\cos t}{1+\sin^2 t}\,dt \overset{(a)}{=} \int_0^1 \frac{dx}{1+x^2} = \Big[\operatorname{arctg} x\Big]_0^1 = \operatorname{arctg}\frac{\pi}{4} = 1,$$

dove in (a) si è usata la sostituzione $\sin t = x$, per cui $dx = \cos t\,dt$.

(5) Poiché $\int_\gamma f\,ds = \int_0^1 f(\gamma(t))\,\|\gamma'(t)\|\,dt$, calcoliamo $f(\gamma(t)) = (2t)^3 + t^3 = 9t^3$, e $\|\gamma'(t)\| = \sqrt{2^2 + (3t^2)^2} = \sqrt{4+9t^4}$. Quindi

$$\int_\gamma f\,ds = \int_0^1 9t^3\sqrt{4+9t^4}\,dt \overset{(a)}{=} \frac{1}{4}\int_4^{13} \sqrt{x}\,dx = \frac{1}{6}\Big[x^{3/2}\Big]_4^{13} = \frac{1}{6}\big(13\sqrt{13} - 8\big),$$

dove in (a) si è usata la sostituzione $4 + 9t^4 = x$, per cui $dx = 36t^3\,dt$.

(6) Poiché $\int_\gamma f\,ds = \int_0^{2\pi} f(\gamma(t))\,\|\gamma'(t)\|\,dt$, calcoliamo $f(\gamma(t)) = 4t$, e $\|\gamma'(t)\| = 5$. Quindi

$$\int_\gamma f\,ds = \int_0^{2\pi} 5\cdot 4t\,dt = \Big[10t^2\Big]_0^{2\pi} = 40\pi^2.$$

(7) Poiché $\int_\gamma f\,ds = \int_0^1 f(\gamma(t))\,\|\gamma'(t)\|\,dt$, calcoliamo $f(\gamma(t)) = t$, e $\|\gamma'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4t^2} = \sqrt{1+4t^2}$. Quindi

$$\int_\gamma f\,ds = \int_0^1 t\sqrt{1+4t^2}\,dt \overset{(a)}{=} \frac{1}{8}\int_1^5 \sqrt{x}\,dx = \frac{1}{12}\Big[x^{3/2}\Big]_1^5 = \frac{1}{12}\big(5\sqrt{5} - 1\big),$$

dove in (a) si è usata la sostituzione $1 + 4t^2 = x$, per cui $dx = 8t\,dt$.

(8) Intanto le equazioni parametriche della curva γ sono $\gamma(t) = (1-t)\big(1,2\big) + t\big(3,6\big) = (1+2t, 2+4t)$, $t \in [0,1]$. Poiché $\int_\gamma f\,ds = \int_0^1 f(\gamma(t))\,\|\gamma'(t)\|\,dt$, calcoliamo $f(\gamma(t)) = \sqrt{(1+2t) + 2(2+4t)} = \sqrt{5+10t}$, e $\|\gamma'(t)\| = \sqrt{4+16} = 2\sqrt{5}$. Quindi

$$\int_\gamma f\,ds = 10\int_0^1 \sqrt{1+2t}\,dt \overset{(a)}{=} 5\int_1^3 \sqrt{x}\,dx = \frac{10}{3}\Big[x^{3/2}\Big]_1^3 = \frac{10}{3}\big(3\sqrt{3} - 1\big),$$

dove in (a) si è usata la sostituzione $1 + 2t = x$, per cui $dx = 2\,dt$.

(9) Intanto le equazioni parametriche della curva γ sono $\gamma(t) = (t, \log t)$, $t \in [1, 2]$. Poiché $\int_\gamma f\, ds = \int_1^2 f(\gamma(t))\, \|\gamma'(t)\|\, dt$, calcoliamo $f(\gamma(t)) = t^2$, e $\|\gamma'(t)\| = \sqrt{1 + \frac{1}{t^2}}$. Quindi

$$\int_\gamma f\, ds = \int_1^2 t^2 \sqrt{1 + \frac{1}{t^2}}\, dt = \int_1^2 t\sqrt{t^2+1}\, dt \overset{(a)}{=} \frac{1}{2}\int_2^5 \sqrt{x}\, dx = \frac{1}{3}\Big[x^{3/2}\Big]_2^5 = \frac{1}{3}\big(5\sqrt{5} - 2\sqrt{2}\big),$$

dove in (a) si è usata la sostituzione $t^2 + 1 = x$, per cui $dx = 2t\, dt$.

(10) Intanto le equazioni parametriche della curva γ sono $\gamma(t) = (t, e^t)$, $t \in [0, \log 2]$. Poiché $\int_\gamma f\, ds = \int_0^{\log 2} f(\gamma(t))\, \|\gamma'(t)\|\, dt$, calcoliamo $f(\gamma(t)) = e^{2t}$, e $\|\gamma'(t)\| = \sqrt{1 + e^{2t}}$. Quindi

$$\int_\gamma f\, ds = \int_0^{\log 2} e^{2t}\sqrt{1 + e^{2t}}\, dt \overset{(a)}{=} \frac{1}{2}\int_2^5 \sqrt{x}\, dx = \frac{1}{3}\Big[x^{3/2}\Big]_2^5 = \frac{1}{3}\big(5\sqrt{5} - 2\sqrt{2}\big),$$

dove in (a) si è usata la sostituzione $1 + e^{2t} = x$, per cui $dx = 2e^{2t}\, dt$.

(11) Intanto le equazioni parametriche della curva γ sono $\gamma(t) = (t, \sqrt{1+t^2})$, $t \in [0, 1]$. Poiché $\int_\gamma f\, ds = \int_0^1 f(\gamma(t))\, \|\gamma'(t)\|\, dt$, calcoliamo $f(\gamma(t)) = \sqrt{1+t^2}$, e $\|\gamma'(t)\| = \sqrt{1 + \frac{t^2}{1+t^2}} = \sqrt{\frac{2t^2+1}{t^2+1}}$. Quindi

$$\begin{aligned}\int_\gamma f\, ds &= \int_0^1 \sqrt{2t^2+1}\, dt \overset{(a)}{=} \int_1^{\sqrt{2}+\sqrt{3}} \frac{x^2+1}{2x}\,\frac{x^2+1}{2\sqrt{2}x^2}\, dx = \frac{1}{4\sqrt{2}}\int_1^{\sqrt{2}+\sqrt{3}} \Big(x + \frac{2}{x} + \frac{1}{x^3}\Big)\, dx \\ &= \frac{1}{4\sqrt{2}}\Big[\frac{x^2}{2} + 2\log x - \frac{1}{2x^2}\Big]_1^{\sqrt{2}+\sqrt{3}} = \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{2}}\log(\sqrt{2}+\sqrt{3}\,),\end{aligned}$$

dove in (a) si è usata la sostituzione $\sqrt{2t^2+1} = x - \sqrt{2}t$, per cui $t = \frac{x^2-1}{2\sqrt{2}x}$, $dt = \frac{x^2+1}{2\sqrt{2}x^2}\, dx$, $\sqrt{1+2t^2} = \frac{x^2+1}{2x}$.

(12) Intanto le equazioni parametriche della curva γ sono $\gamma(t) = (t\cos t, t\sin t)$, $t \in [0, 1]$. Poiché $\int_\gamma f\, ds = \int_0^1 f(\gamma(t))\, \|\gamma'(t)\|\, dt$, calcoliamo $f(\gamma(t)) = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = t$, e $\|\gamma'(t)\| = \sqrt{1+t^2}$. Quindi

$$\int_\gamma f\, ds = \int_0^1 t\sqrt{1+t^2}\, dt \overset{(a)}{=} \frac{1}{2}\int_1^2 \sqrt{x}\, dx = \frac{1}{3}\Big[x^{3/2}\Big]_1^2 = \frac{1}{3}\big(2\sqrt{2} - 1\big),$$

dove in (a) si è usata la sostituzione $1 + t^2 = x$, per cui $dx = 2t\, dt$.

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