Derived categories and cubic hypersurfaces

Paolo Stellari

UNIVERSITÀ DEGLI STUDI DI MILANO

Roma, February 2011

Outline

1 The geometric setting

Outline

1. **The geometric setting**
2. **3-folds**
   - Geometry
   - Derived categories
   - Bridgeland stability conditions
   - Irrationality

Outline

1. The geometric setting
2. 3-folds
   - Geometry
   - Derived categories
   - Bridgeland stability conditions
   - Irrationality
3. 4-folds
   - Geometry
   - Derived categories
   - Categorical Torelli theorem
   - Fano varieties of lines
   - Rationality

Aim

The aim of the talk is to propose a ‘categorical’ treatment for some fundamental (often unknown) geometric properties of smooth (complex) hypersurfaces of degree 3

$$Y \subseteq \mathbb{P}^{n+1}.$$

We will study **cubic 3-fold** ($n = 3$) and **cubic 4-fold** ($n = 4$).

For example:

- Rationality/irrationality of those varieties;
- Torelli type theorems;
- Geometric description of the Fano varieties of lines of those cubics.

The definition

Let **A** be an abelian category (e.g., **mod**-R, right R-modules, R an ass. ring with unity, and **Coh**(X)).

Define $C^b(\mathbf{A})$ to be the (abelian) **category of bounded complexes** of objects in **A**. In particular:

- Objects:

$$M^\bullet := \{\cdots \to M^{p-1} \xrightarrow{d^{p-1}} M^p \xrightarrow{d^p} M^{p+1} \to \cdots\}$$

- Morphisms: sets of arrows $f^\bullet := \{f^i\}_{i\in\mathbb{Z}}$ making commutative the following diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^{i-2}_{M^\bullet}} & M^{i-1} & \xrightarrow{d^{i-1}_{M^\bullet}} & M^i & \xrightarrow{d^{i}_{M^\bullet}} & M^{i+1} & \xrightarrow{d^{i+1}_{M^\bullet}} & \cdots \\
 & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^{i}} & & \downarrow{\scriptstyle f^{i+1}} & & \\
\cdots & \xrightarrow{d^{i-2}_{L^\bullet}} & L^{i-1} & \xrightarrow{d^{i-1}_{L^\bullet}} & L^i & \xrightarrow{d^{i}_{L^\bullet}} & L^{i+1} & \xrightarrow{d^{i+1}_{L^\bullet}} & \cdots
\end{array}$$

The definition

For a complex $M^\bullet \in C^b(\mathbf{A})$, its i-th cohomology is

$$H^i(M^\bullet) := \frac{\ker(d^i)}{\mathrm{im}(d^{i-1})} \in \mathbf{A}.$$

A morphism of complexes is a **quasi-isomorphism** (qis) if it induces isomorphisms on cohomology.

Definition

The **bounded derived category** $\mathrm{D}^b(\mathbf{A})$ of the abelian category $\mathbf{A}$ is such that:

- Objects: $\mathrm{Ob}(C^b(\mathbf{A})) = \mathrm{Ob}(\mathrm{D}^b(\mathbf{A}))$;
- Morphisms: (very) roughly speaking, obtained 'by inverting qis in $C^b(\mathbf{A})$'.

Semi-orthogonal decompositions

Suppose we have a sequence of full triangulated subcategories $\mathbf{T}_1, \ldots, \mathbf{T}_n \subseteq \mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$, where X is smooth projective, such that:

- $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathbf{T}_i, \mathbf{T}_j) = 0$, for $i > j$,
- For all $K \in \mathrm{D}^{\mathrm{b}}(X)$, there exists a chain of morphisms in $\mathrm{D}^{\mathrm{b}}(X)$

  $$0 = K_n \to K_{n-1} \to \ldots \to K_1 \to K_0 = K$$

  with $\mathrm{cone}(K_i \to K_{i-1}) \in \mathbf{T}_i$, for all $i = 1, \ldots, n$.

This is a **semi-orthogonal** decomposition of $\mathrm{D}^{\mathrm{b}}(X)$:

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathbf{T}_1, \ldots, \mathbf{T}_n \rangle.$$

Derived categories and Fano varieties

Theorem (Bondal–Orlov)

Let X be a smooth projective complex Fano variety and assume that Y is a smooth projective variety such that

$$\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y).$$

Then $X \cong Y$.

Thus, if Y is a cubic hypersurface as above, then $\mathrm{D}^{\mathrm{b}}(Y)$ is a too strong invariant.

Question

Does some 'piece' in a semi-orthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(Y)$ behave nicely?

Outline

1 The geometric setting

2 3-folds
- Geometry
- Derived categories
- Bridgeland stability conditions
- Irrationality

3 4-folds
- Geometry
- Derived categories
- Categorical Torelli theorem
- Fano varieties of lines
- Rationality

First properties

The geometric setting
3-folds
Geometry
Derived categories
Bridgeland stability conditions
Irrationality
4-folds
Geometry
Derived categories
Categorical Torelli theorem
Fano varieties of lines
Rationality

Let $Y \subseteq \mathbb{P}^4$ be a smooth cubic 3-fold. The following are classical results:

Torelli Theorem (Clemens–Griffiths, Tyurin)

Let Y_1 and Y_3 be cubic 3-folds. Then $Y_1 \cong Y_2$ if and only if the intermediate Jacobians $(J(Y_1), \Theta_1)$ and $(J(Y_2), \Theta_2)$ are isomorphic.

Theorem (Clemens–Griffiths)

Cubic 3-folds are not rational.

Use that $J(Y)$ does not decompose as direct sum of Jacobians of curves.

Outline

1. The geometric setting
2. 3-folds
   - Geometry
   - Derived categories
   - Bridgeland stability conditions
   - Irrationality
3. 4-folds
   - Geometry
   - Derived categories
   - Categorical Torelli theorem
   - Fano varieties of lines
   - Rationality

The decomposition

Let $Y \subseteq \mathbb{P}^4$ be a smooth cubic 3-fold.

Theorem (Kuznetsov)

The derived category $\mathrm{D}^{\mathrm{b}}(Y)$ has a semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathbf{T}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle.$$

The subcategory $\mathbf{T}_Y$ is highly non-trivial and cannot be the derived category of a smooth projective variety.

Indeed the Serre functor $S_{\mathbf{T}_Y}$ is such that $S_{\mathbf{T}_Y}^3 \cong [5]$. So $\mathbf{T}_Y$ is a so called **Calabi–Yau category of fractional dimension** $\frac{5}{3}$.

Categorical Torelli

Question (Kuznetsov)

Given two cubic 3-folds Y_1 and Y_2, is it true that $Y_1 \cong Y_2$ if and only if $\mathbf{T}_{Y_1} \cong \mathbf{T}_{Y_2}$?

Theorem (Bernardara–Macrì–Mehrotra–S.)

The answer to the above question is positive.

Idea: realize the Fano variety of lines of Y_i as moduli space of stable objects according to a Bridgeland stability condition on $\mathbf{T}_{Y_i}$.

Outline

1 The geometric setting

2 3-folds
- Geometry
- Derived categories
- Bridgeland stability conditions
- Irrationality

3 4-folds
- Geometry
- Derived categories
- Categorical Torelli theorem
- Fano varieties of lines
- Rationality

The definition

A **stability condition** on a triangulated category **T** is a pair $\sigma = (Z, \mathcal{P})$ where

- $Z : K(\mathbf{T}) \to \mathbb{C}$ is a linear map called **central charge** (similar to the slope for sheaves);
- $\mathcal{P}(\phi) \subset \mathbf{T}$ are full additive subcategories for each $\phi \in \mathbb{R}$ (**semistable objects** of phase ϕ)

satisfying some compatibilities.

The minimal objects in $\mathcal{P}(\phi)$ are called **stable objects**.

$\mathrm{Stab}(\mathbf{T})$ is the space parametrizing stability conditions on **T**.

Some questions

The geometric setting

3-folds
Geometry
Derived categories
Bridgeland stability conditions
Irrationality

4-folds
Geometry
Derived categories
Categorical Torelli theorem
Fano varieties of lines
Rationality

Let Y be a cubic 3-fold. As a consequence of the result of Bernardara–Macrì–Mehrotra–S. above, we have that

$$\mathrm{Stab}(\mathrm{D}^{\mathrm{b}}(Y)) \neq \emptyset \neq \mathrm{Stab}(\mathbf{T}_Y).$$

The category $\mathbf{T}_Y$ behaves almost as the derived category of a smooth complex curve C. The stability conditions on $\mathrm{D}^{\mathrm{b}}(C)$ are completely classified.

Problem

Classify completely all the stability conditions in $\mathrm{Stab}(\mathbf{T}_Y)$.

Outline

1 The geometric setting

2 3-folds
- Geometry
- Derived categories
- Bridgeland stability conditions
- Irrationality

3 4-folds
- Geometry
- Derived categories
- Categorical Torelli theorem
- Fano varieties of lines
- Rationality

Open question and new perspectives

Question

Does the category $\mathbf{T}_Y$ encode the irrationality of Y?

A new perspective in this direction is provided by the recent work of Ballard–Favero–Katzarkov:

1. **Idea:** the irrationality of Y should be related to the presence of gaps in the interval of integers corresponding to the 'generation time' of the objects in $\mathrm{D}^{\mathrm{b}}(Y)$.
2. This is related to a conjecture of Orlov. In this case: the dimension of the category $\mathrm{D}^{\mathrm{b}}(Y)$ is $3 = \dim(Y)$.

Outline

1. The geometric setting
2. 3-folds
   - Geometry
   - Derived categories
   - Bridgeland stability conditions
   - Irrationality
3. 4-folds
   - Geometry
   - Derived categories
   - Categorical Torelli theorem
   - Fano varieties of lines
   - Rationality

The basic definitions

Let $Y \subseteq \mathbb{P}^5$ be a smooth cubic 4-fold. Denote by H^2 the self-intersection of the hyperplane class of Y.

The moduli space $\mathcal{C}$ of smooth cubic 4-folds is a quasi-projective variety of dimension 20.

Voisin: Smooth cubic 4-folds Y containing a plane P form a divisor $\mathcal{C}_8$ in $\mathcal{C}$.

Denote by $T := \langle H^2, P \rangle$ the primitive sublattice (with respect to the intersection form) of $H^4(Y, \mathbb{Z})$ generated by H^2 and P. Then the intersection form is of type

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

The basic definitions

Projecting from P onto a disjoint $\mathbb{P}^2$, we get $\pi_P : Y \dashrightarrow \mathbb{P}^2$. Blowing up the plane inside Y gives a quadric fibration

$$\pi'_P : \tilde{Y} \longrightarrow \mathbb{P}^2$$

whose fibres degenerate along a plane sextic C.

The double cover of $\mathbb{P}^2$ ramified along C is a **K3 surface** S (i.e. a smooth complex projective simply connected surface with trivial canonical bundle).

The quadric fibration provides an element

$$\beta \in \mathrm{Br}(S) := H^2(S, \mathcal{O}_S^*)_{\mathrm{tor}}$$

in the **Brauer group** of S.

Hodge theory

Back to the case of Y any cubic 4-fold (not necessarily containing a plane). We have the following remarkable results:

- **Torelli theorem (Voisin):** Let Y_1 and Y_2 be two cubic 4-folds and assume that there exists a Hodge isometry

  $$\phi : H^4(Y_1, \mathbb{Z}) \to H^4(Y_2, \mathbb{Z})$$

  sending H_1^2 to H_2^2. Then there exists an isomorphism $f : Y_2 \cong Y_1$ such that $\phi = f^*$.

- **Surjectivity of the period map (Looijenga, Laza):** The period map surjects onto an explicitly described subset of the period domain.

Hassett: constructing divisors

Hassett proposed a very nice way to construct divisors in the moduli space $\mathcal{C}$.

For a positive integer d, define $\mathcal{C}_d$ to be the set of all $Y \in \mathcal{C}$ such that

- There is a rank-2 lattice K_d with $\det(K_d) = d$.
- There is a primitive embedding $K_d \hookrightarrow H^4(Y, \mathbb{Z})$.
- There is $h^2 \in K_d$ mapped to H^2.

Hassett: $\mathcal{C}_d$ is an irreducible divisor as soon as $d > 6$ and $d \equiv 0, 2 \pmod 6$.

Outline

1. The geometric setting
2. 3-folds
   - Geometry
   - Derived categories
   - Bridgeland stability conditions
   - Irrationality
3. 4-folds
   - Geometry
   - Derived categories
   - Categorical Torelli theorem
   - Fano varieties of lines
   - Rationality

The semi-orthogonal decomposition

Theorem (Kuznetsov)

The derived category $\mathrm{D}^{\mathrm{b}}(Y)$ has a semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathbf{T}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle.$$

Theorem (Kuznetsov)

The triangulated category $\mathbf{T}_Y$ is a 2-Calabi–Yau category.

Recall that a triangulated category $\mathbf{T}$ is a **2-Calabi–Yau category** if $\mathbf{T}$ has a Serre functor which is isomorphic to the shift by 2.

Which 2-Calabi–Yau category?

The geometric setting

3-folds
Geometry
Derived categories
Bridgeland stability conditions
Irrationality

4-folds
Geometry
Derived categories
Categorical Torelli theorem
Fano varieties of lines
Rationality

Theorem (Kuznetsov)

Let Y be a cubic 4-fold containing a plane and such that the plane sextic C is smooth. Then there exists an exact equivalence

$$\mathbf{T}_Y \cong \mathrm{D}^{\mathrm{b}}(S, \beta)$$

Remark

If Y is generic with the above properties (i.e. $H^4(Y, \mathbb{Z}) \cap H^{2,2}(Y) = \langle H^2, P \rangle$), then there is no smooth projective K3 surface S' such that

$$\mathbf{T}_Y \cong \mathrm{D}^{\mathrm{b}}(S').$$

Twisted sheaves

Represent $\beta \in \mathrm{Br}(S)$ as a Čech 2-cocycle

$$\{\beta_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$$

on an analytic open cover $S = \bigcup_{i \in I} U_i$.

A β-twisted coherent sheaf $\mathcal{E}$ is a collection of pairs $(\{\mathcal{E}_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j \in I})$ where

- $\mathcal{E}_i$ is a coherent sheaf on the open subset U_i;
- $\varphi_{ij} : \mathcal{E}_j|_{U_i \cap U_j} \to \mathcal{E}_i|_{U_i \cap U_j}$ is an isomorphism

such that

1. $\varphi_{ii} = \mathrm{id}$ and $\varphi_{ji} = \varphi_{ij}^{-1}$;
2. $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \beta_{ijk} \cdot \mathrm{id}$.

In this way we get the abelian category $\mathbf{Coh}(S, \beta)$.

Outline

1. The geometric setting
2. 3-folds
   - Geometry
   - Derived categories
   - Bridgeland stability conditions
   - Irrationality
3. 4-folds
   - Geometry
   - Derived categories
   - Categorical Torelli theorem
   - Fano varieties of lines
   - Rationality

Results and questions

Theorem (Bernardara–Macrì–Mehrotra–S.)

Given a cubic fourfold Y containing a plane P and such that C is smooth, there exist only finitely many isomorphism classes of cubic 4-folds $Y_1 = Y, Y_2, \ldots, Y_n$ containing a plane and with smooth plane sextics such that $\mathbf{T}_Y \cong \mathbf{T}_{Y_j}$, with $j \in \{1, \ldots, n\}$. Moreover, if Y is generic, then $n = 1$.

Questions

1. Can we prove a similar result for any possible cubic 4-fold (with a plane or not)?
2. Can the number n be arbitrarily large?

Outline

1 The geometric setting

2 3-folds
- Geometry
- Derived categories
- Bridgeland stability conditions
- Irrationality

3 4-folds
- Geometry
- Derived categories
- Categorical Torelli theorem
- Fano varieties of lines
- Rationality

Classical results

The geometric setting

3-folds
Geometry
Derived categories
Bridgeland stability conditions
Irrationality

4-folds
Geometry
Derived categories
Categorical Torelli theorem
Fano varieties of lines
Rationality

For a cubic 4-fold Y, we denote by $F(Y)$ the Fano variety of lines contained in Y.

Theorem (Beauville–Donagi)

1. $F(Y)$ is a irreducible holomorphic symplectic manifold of dimension 4 (i.e. a simply connected, Kähler manifold such that $H^{2,0}(F(Y))$ is generated by a non-degenerate 2-form).
2. $F(Y)$ is deformation equivalent to $\mathrm{Hilb}^2(S)$, the Hilbert scheme of length-2 0-dimensional subschemes on a K3 surface S.

Hassett's results

Theorem (Hassett)

Assume that $d = 2(n^2 + n + 1)$ for $n \geq 2$. Then the generic cubic 4-fold Y contained in $\mathcal{C}_d$ is such that $F(Y) \cong \mathrm{Hilb}^2(S)$ for some K3 surface S.

Question (Hassett)

Are there other d's such that the generic points in $\mathcal{C}_d$ have the same property for some K3 surface?

When there is a plane, the twist cannot be avoided...

The answer when there is a plane

Theorem (Macrì–S.)

If Y is a generic cubic fourfold containing a plane, then $F(Y)$ is isomorphic to a moduli space of stable objects in the derived category $\mathrm{D}^b(S, \beta)$ of bounded complexes of β-twisted coherent sheaves on S.

Theorem (Macrì–S.)

For all cubic fourfolds Y containing a plane, the Fano variety $F(Y)$ is birational to a smooth projective moduli space of twisted sheaves on a K3 surface. Moreover, if the plane sextic C is smooth, then such a birational map is either an isomorphism or a Mukai flop.

Outline

1. The geometric setting
2. 3-folds
   - Geometry
   - Derived categories
   - Bridgeland stability conditions
   - Irrationality
3. 4-folds
   - Geometry
   - Derived categories
   - Categorical Torelli theorem
   - Fano varieties of lines
   - Rationality

Hodge theoretical results

The geometric setting

3-folds
Geometry
Derived categories
Bridgeland stability conditions
Irrationality

4-folds
Geometry
Derived categories
Categorical Torelli theorem
Fano varieties of lines
Rationality

Beauville–Donagi: The provide examples of rational cubic 4-folds (Pfaffian cubic 4-folds).

Hassett: Using lattice and Hodge theory, he constructs countably many divisors in $\mathcal{C}_8$ consisting of rational cubic 4-folds.

The way he defines these families is by showing that the quadric fibration mentioned above has a section.

Notice that the presence of such a section implies that the Brauer class β in $\mathrm{Br}(S)$ is automatically trivial.

The categorical approach

The geometric setting

3-folds
Geometry
Derived categories
Bridgeland stability conditions
Irrationality

4-folds
Geometry
Derived categories
Categorical Torelli theorem
Fano varieties of lines
Rationality

Conjecture (Kuznetsov)

A cubic 4-fold Y is rational if and only if there exists a K3 surface S' and an exact equivalence $\mathbf{T}_Y \cong \mathrm{D}^{\mathrm{b}}(S')$.

The conjecture is verified by Beauville–Donagi's and Hassett's examples.

The generic cubic 4-fold with a plane is such that there are no K3 surfaces S' with the property above.

Problem

Use categorical methods to prove that the generic cubic 4-fold with a plane is not rational.