On (special versions of) Hartshorne Conjecture

- I will report on joint work/thoughts with Paltin Ionescu.

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4. $$d := \sum_{i=1}^{c} (d_i - 1) \geq c.$$

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X^n **is a complete intersection** $\Longleftrightarrow -K_X = \mathcal{O}_X(n + 1 - d)$.

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Equations for $\mathcal{L}_{x,X} \subset \mathbb{P}((t_x X)^*)$

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1. $X \subset \mathbb{P}^N$ quadratic $\iff d = c$.
2. $X^n \subset \mathbb{P}^{n+c}$ quadratic $\implies I = I^* = J$.

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  with $\widetilde{I} \subset S$ generated by $r \leq c$ quadratic equations.

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- Recall that $\mathcal{L}_{x,Y}$ can be scheme-theoretically defined by $d = \sum_{i=1}^{c}(d_i - 1)$ equations.

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- Recall that $\mathcal{L}_{x,Y}$ can be scheme-theoretically defined by $d = \sum_{i=1}^{c}(d_i - 1)$ equations.
- Also remark that

$$\text{supp}(\mathcal{L}_{x,X}) = \text{supp}(\mathcal{L}_{x,Y}).$$

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Proposition

Let $X^n \subset \mathbb{P}^{n+c}$ manifold, $x \in U$ be a general point. Then :

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so that $\mathcal{L}_{x,X} \subset \mathbb{P}((t_xX)^*)$ is a quadratic variety (in fact a manifold !) **scheme theoretically** *defined by $r \leq c$ linearly independent quadratic equations.*

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- **index of** $X^n \subset \mathbb{P}^N$ defined by

$$-K_X = i(X)H,$$

with $H \subset X$ a hyperplane section of $X^n \subset \mathbb{P}^N$.

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5. [Hwang] *If $i(X) \geq \frac{n+3}{2}$, then $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a non-degenerate manifold of dimension $i(X) - 2$.*

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- $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a smooth complete intersection of type

$$(2, \dots, d_1; 2, \dots, d_2; \dots; 2, \dots d_{c-1}; 2, \dots, d_c).$$

Theorem (Bertram–Ein–Lazarsfeld)

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Beyond [BEL] Main Application

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2. $X^n \subset \mathbb{P}^{n+c}$ is arithmetically Cohen-Macaulay if $d \leq n$.

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$X^n \subset \mathbb{P}^{n+c}$ manifold.
Assume $d \leq n-1$ (and also $n \geq c+2$ if $X^n \subset \mathbb{P}^N$ is quadratic).
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$$i(X) = \dim(\mathcal{L}_x) + 2 \geq n + 1 - d \geq 2;$$

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$X^n \subset \mathbb{P}^{n+c}$ manifold.
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Hartshorne Conjecture on Complete Intersections

Conjecture

(**Complete Intersection Conjecture**, Hartshorne 1974) *Let* $X^n \subset \mathbb{P}^{n+c}$ *smooth manifold*

$$\text{If } 2c < n (\text{i.e. if } c \leq \frac{n-1}{2}), \implies X \text{ is a complete intersection.}$$

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In particular $\mathbb{G}(1,4) \subset \mathbb{P}^9$ and $S^{10} \subset \mathbb{P}^{15}$ are the unique Hartshorne varieties defined by quadratic equations, modulo projective equivalence.

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Theorem (Faltings)

Let $X^n = V(f_1, \ldots, f_m) \subset \mathbb{P}^{n+c}$ *manifold. If* $m \leq \frac{N}{2}$, *then*

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Proof

$$c \leq \frac{n-1}{2}$$

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6. $-K_X = \mathcal{O}(n+1-c) \Longrightarrow X^n \subset \mathbb{P}^{n+c}$ *complete intersection.*

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Assume that $c \leq \frac{n-1}{2}$ *and* $X^n \subset \mathbb{P}^{n+c}$ *Fano.*

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Remarks

- When $n \geq 2c+1$, by the Barth–Larsen Theorem $\mathrm{Pic}(X) \cong \mathbb{Z}\langle H \rangle$. In particular $K_X = bH$ for some integer b. So, saying that X is Fano means exactly that $b < 0$; this happens, for instance, if X is covered by lines.

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- Dual defective and LQEL manifolds satisfy the (HCL) (Ionescu, –).
- Prime Fano manifolds of high index tend to be complete intersections. Note that for complete intersections $X^n \subset \mathbb{P}^{n+c}$, $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is also a complete intersection.

Conjecture

If $X \subset \mathbb{P}^N$ is covered by lines and $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a (say smooth irreducible non-degenerate) complete intersection, then X is a complete intersection too.

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Question

Let $X \subset \mathbb{P}^N$ be as above. If $d \leq \frac{n-1}{2}$, then $X^n \subset \mathbb{P}^{n+c}$ is a complete intersection ?

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Hartshorne was the first to realize in 1973 that there exists a function $f(\deg(X))$ such that, for smooth $X^n \subset \mathbb{P}^{n+c}$,

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- Other refinements for $c = 2$, due to Ran, Ballico–Chiantini, Holme–Schneider.

Open problems and Conjectures

Conjecture (Barth–Ionescu)

If $\deg(X) \leq n-1$, *then* $X^n \subset \mathbb{P}^{n+c}$ *is a complete intersection, unless it is projectively equivalent to* $\mathbb{G}(1,4) \subset \mathbb{P}^9$.

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- If $n \leq c + 1$, then

$$0 \leq \deg(X) - \operatorname{codim}(X) - 1 \leq n - 1 - c - 1 < 0.$$

Remarks

- If $n \geq c+2$ we have $\text{Pic}(X) \simeq \mathbb{Z}\langle\mathcal{O}(1)\rangle$. To prove that X is Fano it suffices to prove that $p_g(X) = h^0(K_X) = 0$.

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- Recall the Harris bound

$$p_g(X) \leq c\binom{M}{n+1} + \epsilon\binom{M}{n},$$

where $M = [\frac{\deg(X)-1}{c}]$ and $\epsilon = \deg(X) - 1 - Mc$.

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- From $\deg(X) \leq n - 1$ we deduce

$$M = [\frac{\deg(X) - 1}{c}] \leq 1\frac{n-2}{c} \leq n - 2 < n$$

and $p_g(X) = 0$.

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On Barth–Ionescu Conjecture

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In conclusion if $\deg(X) \leq n-1$ and $X \not\simeq \mathbb{G}(1,4) \subset \mathbb{P}^9$, then

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(HCF) implies Barth–Ionescu

In conclusion if $\deg(X) \leq n-1$ and $X \not\simeq \mathbb{G}(1,4) \subset \mathbb{P}^9$, then $n > 2c$ and $X^n \subset \mathbb{P}^{n+c}$ is a prime Fano manifold. Therefore

(HCF) $\Longrightarrow$ **Barth–Ionescu Conjecture**.