FINAL EXAM
after the lecture course

AFFINE ALGEBRAIC SURFACES
and
the ZARISKI CANCELLATION PROBLEM

by MIKHAIL ZAIDENBERG

1. Epimorphism Theorem

a) Let $\Gamma \subset \mathbb{A}^2 = \mathbb{A}^2_{\mathbb{C}}$ be a reduced, irreducible plane affine curve over the base field $k = \mathbb{C}$. Verify that the following conditions are equivalent:
   (i) Γ is isomorphic to the affine line $\mathbb{A}^1 = \mathbb{A}^1_{\mathbb{C}}$;
   (ii) Γ is a smooth rational curve with one place at infinity;
   (iii) Γ is smooth and $\pi_1(\Gamma) = 1$;
   (iv) Γ is smooth and $e(\Gamma) = 1$, where e stands for the Euler characteristic;
   (v) Γ admits a parameterization $\Gamma = \varphi(\mathbb{A}^1)$, where $\varphi = (p, q)$ with $p, q \in \mathbb{C}[t]$ separating points of $\mathbb{A}^1$ and separating directions (that is, $p'(t)$ and $q'(t)$ do not vanish simultaneously);
   (vi) $\Gamma = \varphi(\mathbb{A}^1)$, where $\varphi = (p, q) \in (\mathbb{C}[t])^2$ with $\mathbb{C}[p, q] = \mathbb{C}[t]$;
   (vii) $\Gamma = \varphi(\mathbb{A}^1)$, where $\varphi = (p, q) \in (\mathbb{C}[t])^2$ are such that $t = f(p(t), q(t))$ for some $f \in \mathbb{C}[x, y]$.

b) Show that the Epimorphism Theorem of Abhyankar-Moh and Suzuki is equivalent to the following assertion. Let k be a field of characteristic π, and let $\gamma\colon k[X, Y] \to k[Z]$ be the k-epimorphism with $\gamma(X) = 0$ and $\gamma(Y) = Z$. Let $\alpha\colon k[X, Y] \to k[Z]$ be any k-epimorphism such that at least one of $\deg_Z \alpha(X)$ and $\deg_Z \alpha(Y)$ is not divisible by π. Then there exists an automorphism $\beta\colon k[X, Y] \to k[X, Y]$ such that $\gamma = \alpha \circ \beta$.

2. Simply connected affine curves

a) Let $\Gamma \subset \mathbb{A}^2 = \mathbb{A}^2_{\mathbb{C}}$ be a reduced, irreducible plane affine curve over $k = \mathbb{C}$. Verify that the following conditions are equivalent:
   (i) $\pi_1(\Gamma) = 1$;
   (ii) Γ is homeomorphic to the complex affine line $\mathbb{A}^1 = \mathbb{A}^1_{\mathbb{C}}$;
   (iii) Γ is a rational curve with one place at infinity and only unibranch singular points;
   (iv) $e(\Gamma) = 1$, where e stands for the Euler characteristic;
   (v) Γ admits a parameterization $\Gamma = \varphi(\mathbb{A}^1)$, where $\varphi = (p, q)$ with $p, q \in \mathbb{C}[t]$ separating points of $\mathbb{A}^1$.

b) Verify that the conditions (i)-(v) are fulfilled for the curve $\Gamma = \Gamma_{k,l}$ given in $\mathbb{A}^2$ by equation $x^k - y^l = 0$, where $k, l \in \mathbb{N}$, $\gcd(k, l) = 1$.

c) Show that any reduced, irreducible, quasihomogeneous affine plane curve Γ different from the coordinate axes coincides with $\Gamma_{k,l}$ for some $k, l \in \mathbb{N}$ with $\gcd(k, l) = 1$.

d) Let $C \subset \mathbb{A}^3 = \mathbb{A}^3_{\mathbb{C}}$ be an affine space curve isomorphic to the affine line $\mathbb{A}^1$. Show that if C admits two parallel tangent lines $T_P C$, $T_Q C$, where $P, Q \in C$, $P \neq Q$, then there is a secant line (S, R), where $S, R \in C$, $S \neq R$, parallel to $T_P C$ and $T_Q C$.

3. Locally nilpotent derivations

Let A be an integral domain over an algebraically closed field k of characteristic zero, and let $\partial\colon A \to A$ be a locally nilpotent derivation of A.

a) Show that
   (i) $A^\partial = \ker \partial$ is an inert domain, that is, $a \cdot b \in A^\partial \Rightarrow a, b \in A^\partial$;
   (ii) A is an affine domain provided $A[t]$ is;
   (iii) $(\mathrm{Frac}A)^\partial = \mathrm{Frac}(A^\partial)$;
   (iv) if A is integrally closed in $\mathrm{Frac}A$ then A^∂ is integrally closed in $\mathrm{Frac}(A^\partial)$.

b) Find an example of an affine domain A such that A^∂ is not an affine domain.

c) Let A be graded: $A = \bigoplus_{n\in\mathbb{Z}} A_n$. Show that any derivation $\partial \in \mathrm{Der}A$ is a sum of homogeneous derivations: $\partial = \sum_{i=k}^{l} \partial_i$, where $\deg \partial_i = i$, and that ∂_k, ∂_l are locally nilpotent provided ∂ is.

d) Find explicitly a locally nilpotent derivation of the algebra $\mathcal{O}(X)$, where X is the surface given in $\mathbb{A}^3$ by equation $x^n y - p(z) = 0$, where $p \in k[z]$.

e) Let $S_n = \{x^n y - z_2 + 1 = 0\} \subset \mathbb{A}^2$. Show that any locally nilpotent derivation of $\mathcal{O}(S_2)$ is a replica of a fixed such derivation that you will make explicit. Is this true for S_1?

4. Cancellation

a) Prove the cancellation for cylinders over smooth affine curves $/\mathbb{C}$.

b) Show that the strong cancellation does not hold for the affine line $\mathbb{A}^1$ and holds for any smooth affine curve non-isomorphic to $\mathbb{A}^1$.

c) Show that the strong cancellation does not hold for the cylinder over the surface X as in Exercise 3.d) above.

d) Prove the Danielewski-Fieseler Theorem in case of the cylinders over the Danielewski surfaces S_1 and S_2 (see Exercise 3.e).