Variational approaches to PDE's

A short school in Rome

March 13–14, 2019

	Wed 13	Thu 14
9:30-11:00	Rivière – Lesson 1	Bartsch – Lesson 2
11:00-11:30	Coffee break	Coffee break
11:30-12:00	Mazzoleni	Pigati
12:00-12:30	Musina	Sicbaldi
12:30-14:00	Lunch break	Lunch break
14:00-15:30	Bartsch – Lesson 1	Rivière – Lesson 3
15:30-16:00	Break	Break
16:00-17:30	Rivière – Lesson 2	Bartsch – Lesson 3

Abstracts of the courses

Normalized solutions of nonlinear Schrödinger equations and systems

Thomas Bartsch
Universität Giessen

Nonlinear Schrödinger equations and systems have been in the focus of nonlinear analysis since decades. A model system that has been investigated in quite some detail in recent years is

$$\begin{cases} -i\partial_t \Phi_1 = \Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1 \\ -i\partial_t \Phi_2 = \Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2 \end{cases} \quad \text{in } \mathbb{R}^3. \tag{S}$$

An important type of solutions are solitary waves, i.e. solutions of the form $\Phi_k(t,x) = e^{-i\lambda_k t}u_k(x)$, with $\lambda_k \in \mathbb{R}$ and $u_k : \mathbb{R}^3 \to \mathbb{R}$. There is an enormous amount of literature on the existence and shape of solitary waves when the frequencies λ_1, λ_2 are prescribed. However, since the masses $\|\Phi_k(t,\cdot)\|_{L^2} = \|u_k\|_{L^2}$ are preserved during the time evolution it is natural to ask for solitary waves that satisfy the normalization constraint

$$\|\Phi_k(t,\cdot)\|_{L^2} = a_1 \quad \text{and} \quad \|\Phi_k(t,\cdot)\|_{L^2} = a_2\,. \tag{N}$$

The existence of normalized solutions is far from being understood, compared with the fixed frequency problem.

The goal of this lecture is to present recent results and methods covering problems like (S)-(N).

Minmax methods for the area of surfaces

Tristan Rivière
ETH Zürich

The study of the variations of curvature functionals takes its origins in the works of Euler and Bernouilli from the XVIIIth century on the Elastica. Since these very early times, special curves and surfaces such as geodesics, minimal surfaces, elastica, Willmore surfaces...etc have become central objects in mathematics much beyond the field of geometry stricto sensu with applications in analysis, in applied mathematics, in theoretical physics and natural sciences in general.

Despite its venerable age the calculus of variations of length, area or curvature functionals for curves and surfaces is still a very active field of research with important developments that took place in the last decades.

In the proposed mini-course we shall concentrate on the various minmax constructions of these critical surfaces for the area in closed manifolds.

We will start by recalling the origins of minmax methods for the length functional and present in particular the "curve shortening process" of Birkhoff. We will mention the generalization of Birkhoff's approach to surfaces and the "harmonic map replacement" method by Colding, Minicozzi and Zhou.

We will review the Geometric Measure Theory approach to the minmax construction of minimal surfaces in relation with the recent proof of the Yau Conjecture by Marques, Neves and Song.

We will then change minmax strategy completely and recall some fundamental notions of Palais Smale deformation theory in infinite dimensional spaces and apply it first to the construction of closed geodesics and Elastica.

In the last part of the mini-course we will present a new method based on smoothing arguments combined with Palais Smale deformation theory for performing successful minmax procedures for surfaces.

Abstracts of the talks

Asymptotic spherical shapes in some spectral optimization problems

Dario Mazzoleni

Università Cattolica – Brescia

We study the positive principal eigenvalue of a weighted problem associated with the Neumann-Laplacian settled in a box $\Omega \subset \mathbb{R}^N$, which arises from the investigation of the survival threshold in population dynamics. When trying to minimize such eigenvalue with respect to the weight, one is lead to consider a shape optimization problem, which is known to admit spherical optimal shapes only in trivial cases. We investigate whether spherical shapes can be recovered in some singular perturbation limit or not. More precisely, the shape optimization problem appearing in the limit is the so called *spectral drop* problem, which involves the minimization of the first eigenvalue of the mixed Dirichlet-Neumann Laplacian and whose optimal shapes are spherical for suitable choices of the box, the most interesting case being when Ω is a convex polytope.

These is a joint work with Benedetta Pellacci and Gianmaria Verzini.

Some advances on Arnold's problem about the existence of multiple K-magnetic geodesics

Roberta Musina

Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università di Udine

We report on some recent contributions to a problem proposed by V. I. Arnold, see [A, Problems 1988/30, 1994/14, 1996/18]. This talk is based on the papers [MZ1], [MZ2].

[A] V. I. Arnold, *Arnold's problems*, translated and revised edition of the 2000 Russian original, Springer-Verlag, Berlin, 2004.

[MZ2] R. Musina and F. Zuddas, Embedded loops in the hyperbolic plane with prescribed, almost constant curvature, Ann. Glob. Anal. Geom. (2018). doi.org/10.1007/s10455-018-9638-9.

[MZ2] R. Musina and F. Zuddas, Multiple closed K-magnetic geodesics on $\mathbb{S}^2$, preprint arXiv:1803.00856 (2018).

Parametrized 2D stationary varifolds and the multiplicity one conjecture

Alessandro Pigati
ETH Zürich

We complement the mini-course by Rivière on his existence theory for min-max minimal surfaces, with arbitrary genus and codimension, based on a viscous relaxation of the area functional. The regularity theory requires the study of a 2D stationary varifold V, which comes enriched with an additional parametrized structure. We discuss how to obtain axiomatically the full regularity of this "parametrized stationary varifold" V, without any extra hypotheses (such as stability or codimension one). We also show that the natural counterpart of Marques-Neves multiplicity one conjecture holds in this setting, deducing an upper bound on the Morse index of V. This is joint work with Tristan Rivière.

Overdetermined elliptic problems in exterior domains

Pieralberto Sicbaldi
Universidad de Granada

Overdetermined elliptic systems of the form

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega \subset \mathbb{R}^n,\ n \geq 2 \\ u = 0 & \text{on} \quad \partial\Omega \\ \frac{\partial u}{\partial \vec{n}} = \text{constant} & \text{on} \quad \partial\Omega \end{cases}$$

appear in many problems in Physics and Applied Mathematics. In this talk, I will consider overdetermined elliptic systems in exterior domains, i.e. domains that are the complement of a compact region. I will present a symmetry results with classification of solution for the case where the PDE is a Allen-Cahn type equation, and a perturbation results with the construction of new solutions in the case where the PDE is the Nonlinear Schrödinger equation. This is based on joint works with A. Ros and D. Ruiz.
