# Reeder's Conjecture for Classical Lie Algebras

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# Main Points of Presentation

1. Small representations and Zero weight spaces.
2. Reeder's Conjecture.
3. Stembridge Recurrences.
4. Strategies for Classical Lie Algebras.
    - Type $B$
    - Type $C$: The case of weights $\omega_{2k}$.
    - Type $C$: The case of weights $\omega_1 + \omega_{2k+1}$.
    - Type $D$: an Overview.

# Setting

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank $n$, $\mathfrak{h}$ a maximal toral subalgebra.

$\Phi$ is the associated root system with Weyl group $W$.

$\Delta$ is a simple system for $\Phi$ and $\Phi^+$ is the associated set of positive roots.

$\Pi$ and $\Pi^+$ are respectively the set of weights and the set of dominant weights. $\omega_1, \dots, \omega_n$ are the fundamental weights.

We will denote with $\theta$ the highest dominant root and with and $\theta_s$ the short dominant root.

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{n} \omega_i$$

# Small Weights and Zero Weight Space Representations

**Remark (Dominance Order)**

$$\lambda \geq \mu \iff \lambda - \mu = \sum_{\alpha \in \Phi^+,\ m_\alpha \geq 0} m_\alpha \alpha$$

**Definition**

- *$\lambda \in \Pi^+ \cap \mathbb{Z}[\Phi]$ is small if $\lambda \not\geq 2\alpha$ for all dominant roots $\alpha$.*
- *An irreducible representation $V_\lambda$ is small if $\lambda$ is small.*

Consider $V_\lambda \supset V_\lambda^0$ eigenspace of eigenvalue 0 for the action of $\mathfrak{h}$

**Remark**

$V_\lambda^0$ is a $W$ representation.

We are interested in zero weight representations when $\lambda$ is small.

Let $V_\lambda$ be an irreducible representation.

**Definition**

$P(V_\lambda, \Lambda\mathfrak{g}, t) = \sum_{i\geq 0} dim\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, \Lambda^i\mathfrak{g})t^i$

**Definition**

$P_W(V_\lambda^0, x, y) = \sum_{i\geq 0} dim\mathrm{Hom}_W(V_\lambda^0, \Lambda^i\mathfrak{h} \otimes \mathcal{H}^j_{(2)})x^i y^j$

**Conjecture (Reeder)**

*If $\lambda$ is small*

$$P(V_\lambda, \Lambda\mathfrak{g}, q) = P_W(V_\lambda^0, q, q^2)$$

Motivation: Cohomology of compact Lie groups

$$\Lambda\mathfrak{g}^{\mathfrak{g}} \simeq \left[\Lambda\mathfrak{h} \otimes \mathcal{H}_{(2)}\right]^W$$

# Reeder's Conjecture: The $A$ case.

Small representations: partitions of $n$ of the form $2^k 1^{n-2k}$.

**Theorem (Stembridge)**

*Let $\lambda$ be a partition of $n$*

$$P(V_\lambda, \Lambda\mathfrak{g}, q) = \prod_{i=1}^{n}(1-q^{2i}) \prod_{(ij)\in\lambda} \left(\frac{q^{2j-2}+q^{2i-1}}{1-q^{2h(ij)}}\right)$$

Zero weight representation: $V_\lambda^0 = \chi_{\lambda'}$

**Theorem**

$$P(\chi_\alpha, \Lambda\mathfrak{h}\otimes\mathcal{H}, y, x) = \prod_{i=1}^{n}(1-x^{i})\cdot x^{n(\alpha)} \prod_{(ij)\in\alpha} \left(\frac{1+yx^{c(ij)}}{1-x^{h(ij)}}\right)$$

where $n(\alpha) = \sum(i-1)\alpha_i$ and $c(ij) = j - i$

# Our Strategy

Lie Algebras Part: Polynomials $P(V_\lambda, \Lambda\mathfrak{g}, q)$ can be computed solving a triangular system of linear equations with polynomial coefficients (Stembridge, '05).

Weyl Group Part: There exist some general closed formulae for $P_W(V_\lambda^0, x, y)$ (Gyoja, Nishiyama, Shimura, '99).

Idea: Prove that $P_W(V_\lambda^0, q, q^2)$ satisfies Stembridge's recurrences.

Hard part:

- Deal with the coefficients appearing in the system of equations.
- Find transition formulae (w.r.t. dominance order) between $P_W(V_\lambda^0, q, q^2)$ to apply iterative arguments.

# Macdonald Kernels

## Definition (Macdonald Kernel)

$$MDK(q,t) := \prod_{i\geq 0}\left(\frac{1-q^{i+1}}{1-tq^i}\right)^n \cdot \prod_{i\geq 0}\prod_{\alpha\in\Phi}\frac{1-q^{i+1}e^{\alpha}}{1-tq^ie^{\alpha}}.$$

$$MDK(q,t) = \sum_{\mu\in Pi^+} C_\mu(q,t)\chi_\mu$$

## Remark

$$C_\mu(-q,q^2) = P(V_\mu, \Lambda\mathfrak{g}, q)$$

If $\mu$ is not dominant, consider ${O^\bullet}_\mu = \{\sigma\circ\mu = \sigma(\mu+\rho)-\rho\,|\sigma\in W\}$

$$C_\mu(q,t) = \begin{cases} 0 & \text{if } \mu+\rho \text{ is not regular,} \\ (-1)^{l(\sigma)}C_\lambda(q,t) & \text{if } \lambda\in {O^\bullet}_\mu\,, \lambda\in\Pi^+. \end{cases}$$

Fix a dominant weight $\lambda$. We want to compute $C_\lambda(q,t)$.

**Definition**

*A coweight $\omega$ is minuscule iff $(\omega,\alpha) \in \{\pm 1, 0\}$ for all $\alpha \in \Phi$.*

Stembridge proves that: (Minuscule Recurrence)

$$\sum_{i=1}^{k} C_{w_i\lambda}(q,t) \left( \sum_{\psi \in O_\omega} \left( t^{-(\rho, w_i\psi)} - q^{(\lambda,\omega)} t^{(\rho, w_i\psi)} \right) \right) = 0.$$

Here $w_1, \ldots, w_k$ are minimal coset representatives of $W/W_\lambda$, $W_\lambda$ is the stabilizer of $\lambda$ and $O_\omega$ is the orbit $W_\lambda \cdot \omega$.

**Example**

- $\omega_1$ is a minuscule coweight in type $B$ and $D$.
- $\frac{1}{2}\omega_n$ is a minuscule coweight for $C_n$.

# Stembridge's Work

Quasi-minuscule Recurrence:

$$\sum_{(\nu,\beta)} \sum_{i \geq 0} \left[ f_i^{\beta}(q,t) - q^{(\lambda,\theta^\vee)} f_i^{\beta}(q^{-1},t^{-1}) \right] C_{\nu - i\beta}(q,t) = 0.$$

Here $(\nu, \beta) \in \{(w\lambda, w\theta) | w \in W,\ w\theta \geq 0\}$ and the $f_i^{\beta}(q,t)$ are defined by

$$(1-tz)(1-qtz)\frac{((t^2 z)^{(\rho,\beta^\vee)} - 1)}{t^2 z - 1} = \sum_{i \geq 0} t^{(\rho,\beta^\vee)} f_i^{\beta}(q,t) z^i.$$

**Remark**

Stembridge uses the recurrences to compute explicitely $C_\mu(q,t)$ when $\mu = \theta,\ \theta_s$.

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**Remark**

Stembridge uses the recurrences to compute explicitely $C_\mu(q,t)$ when $\mu = \theta,\ \theta_s$.

# Reduced Recurrences

**Remark**

If $\mu$ is not dominant we can reduce to the dominant case by

$$C_\mu(q,t) = \begin{cases} 0 & \text{if } \mu+\rho \text{ is not regular,} \\ (-1)^{l(\sigma)} C_\lambda(q,t) & \text{if } \lambda \in O^{\bullet}{}_{\mu}. \end{cases}$$

$$R_\lambda := \Lambda_\lambda C_\lambda + \sum_{\mu \in \Pi^+} \Lambda_\mu C_\mu = 0$$

**Proposition**

*In the reduced recurrence* $\Lambda_\mu \neq 0 \Longrightarrow \mu \leq \lambda$

$$R_\lambda := \Lambda_\lambda C_\lambda + \sum_{\mu \leq \lambda} \Lambda_\mu C_\mu = 0$$

# A Concrete Example: Recurrences for $\lambda = \omega_k$ in $B_n$

**Remark**

$\lambda \in \Pi^+$ is smaller than $\omega_k$ iff $\lambda = 0$ or $\lambda = \omega_h$ with $h \leq k$.

$$
\begin{cases}
\Lambda_k^{k,n} C_{\omega_k} + \Lambda_{k-1}^{k,n} C_{\omega_{k-1}} + \dots + \Lambda_1^{k,n} C_{\omega_1} + \Lambda_0^{k,n} C_0 = 0 \\
\Lambda_{k-1}^{k-1,n} C_{\omega_{k-1}} + \dots + \Lambda_1^{k-1,n} C_{\omega_1} + \Lambda_0^{k-1,n} C_0 = 0 \\
\ddots \qquad \vdots \\
\Lambda_1^{1,n} C_{\omega_1} + \Lambda_0^{1,n} C_0 = 0
\end{cases}
$$

**Remark**

In general $\lambda$ small and $\mu \leq \lambda \implies \mu$ small.

# Combinatorial description of Zero Weight Representations.

**Remark**

The IrrReps of $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ are indexed by pair of partitions $(\alpha, \beta)$ such that $|\alpha| + |\beta| = n$

$$P_W(\pi_{\alpha,\beta}; x, y) = \prod_{(i,j)\in\alpha} \frac{x^{2(i-1)} + yx^{2j-1}}{1 - x^{2h(ij)}} \prod_{(i,j)\in\beta} \frac{x^{2i-1} + yx^{2(j-1)}}{1 - x^{2h(ij)}} \prod_{i=1}^{n} \left(1 - x^{m_i+1}\right).$$

**Example ($\lambda = \omega_{2k}$ in $C_n$)**

$$V_\lambda^0 \longleftrightarrow ((n-k, k), \emptyset)$$

$$P_W(V^0_{\omega_{2k}}, q, q^2) = q^{4k-1}(q+1)\binom{n}{k}_{q^4} \frac{\left(q^{4(n-2k+1)} - 1\right)}{\left(q^{4(n-k+1)} - 1\right)} \prod_{i=1}^{n-k}\left(q^{4i-1}+1\right)\prod_{i=1}^{k-1}\left(q^{4i-1}+1\right)$$

# Reeder's Conjecture in type $B$.

## Remark

$$\Phi = \{\pm e_i \pm e_j\}_{i<j} \cup \{\pm e_1, \dots, \pm e_n\}$$

$$\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n\}$$

$$\Phi^+ = \{e_i \pm e_j\}_{i<j} \cup \{e_1, \dots, e_n\} \quad W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$$

$$\omega_i = e_1 + \cdots + e_i \quad \omega_n = \frac{e_1 + \cdots + e_n}{2}$$

| **Small Representation** Highest weight | **Zero Weight Space** $(\alpha, \beta)$ description |
|---|---|
| $\omega_i,\ i < n,\ i = 2k$ | $((n-k),(k))$ |
| $\omega_i,\ i < n,\ i = 2k+1$ | $((k),(n-k))$ |
| $2\omega_n,\ n = 2k$ | $((k),(k))$ |
| $2\omega_n,\ n = 2k+1$ | $((k),(k+1))$ |

# Reeder's Conjecture in type $B$.

Minuscule cowieght: $\omega_1 = e_1$.

Stabilizer of $\omega_k$: $S_k \times B_{n-k}$.

$$\sum_{i=1}^{l} C_{w_i\lambda} \sum_{j=1}^{k} (q^{-2(\rho, w_i e_j)} + q^{1+2(\rho, w_i e_j)}) = 0.$$

## Proposition

$$C_{\omega_m} c_m = \sum_{i=1}^{[\frac{m+1}{2}]} C_{\omega_{m-2i+1}} b_i + \sum_{i=1}^{[\frac{m}{2}]} C_{\omega_{m-2i}} b_{n-m+i+1}.$$

$$c_m = \frac{1-q^{2m}}{1-q^2} q^{-2n+1}(1+q^{4n-2m+1}), \quad b_m = (q+1)q^{-2m+2}\frac{1-q^{4m-2}}{1-q^2}.$$

# Reeder's Conjecture in type $C$.

## Remark

$$\Phi = \{\pm e_i \pm e_j\}_{i<j} \cup \{\pm 2e_1, \dots, \pm 2e_n\}$$

$$\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n, 2e_n\}$$

$$\Phi^+ = \{e_i \pm e_j\}_{i<j} \cup \{2e_1, \dots, 2e_n\} \quad W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$$

$$\theta = 2e_1 \quad \theta_s = e_1 + e_2 \quad \omega_i = e_1 + \cdots + e_i$$

$$\rho = ne_1 + (n-1)e_2 + \cdots + 2e_{n-1} + e_n$$

| **Small Representation** Highest weight | **Zero Weight Space** $(\alpha, \beta)$ description |
|---|---|
| $2\omega_1$ | $((n-1),(1))$ |
| $\omega_{2i}$ | $((n-i,i),\emptyset)$ |
| $\omega_1 + \omega_{2i+1}$ | $((n-i-1,i),(1)) \oplus ((n-i-1,i,1),\emptyset)$ |

Using minuscule recurrence is harder (on the computational point of view)!

$$\sum_{i=1}^{l} C_{w_i\lambda} \sum_{\mu} (q^{-(\rho, w_i \omega_{2k} + \mu)} + q^{k + (\rho, w_i \omega_{2k} + \mu)}) = 0.$$

Quasi minuscule recurrence:

$$\sum_{(\nu,\beta)} \sum_{i \geq 0} \left[ f_i^{\beta}(q,t) - q^{(\lambda, e_1)} f_i^{\beta}(q^{-1}, t^{-1}) \right] C_{\nu - 2i\beta}(q,t) = 0.$$

Here $(\nu, \beta) \in \{(w\lambda, we_1) | w \in W,\ w\theta \geq 0\}$.

$$F_i^j = f_i^{e_j}(q,t) - q f_i^{e_j}(q^{-1}, t^{-1})$$

For each $\mu \leq \omega_{2k}$ we have to find pairs $(i, j)$ such that $F_i^j$ contributes to coefficient $\Lambda_\mu$.

# The case $\omega_{2k}$: Symmetries.

Table: Contributions of pairs $(w\omega_2, e_j)$ in $C_4$

| j=4 | j=3 | j=2 | j=1 | |
|---|---|---|---|---|
| | | $\omega_2+$ | $\omega_2+$ | i=0 |
| | $0-$ | $0-$ | $0-$ | i=1 |
| | $0+$ | | | i=2 |
| | | $0+$ | | i=3 |
| | | $\omega_2-$ | $0+$ | i=4 |
| | | | $\omega_2-$ | i=5 |

$$(i,j) \leftrightarrow (n-i-j-2, j)$$

Wlog $2i-1 \leq n-j+1 \Rightarrow$ the coordinates of $\rho + w\omega_{2k} - 2ie_j$ are positive $\Rightarrow$ we can reduce to considere ${O_\mu}^{\bullet S_n}$ instead of ${O_\mu}^{\bullet}$

# The case $\omega_{2k}$: Reducing the System

## Example

If $\lambda = \omega_{2k}$

$$\begin{cases} \Lambda_k^{k,n} C_{\omega_{2k}} + & \Lambda_{k-1}^{k,n} C_{\omega_{2(k-1)}} & + & \dots & \Lambda_1^{k,n} C_{\omega_2} & + & \Lambda_0^{k,n} C_0 & = 0 \\ & \Lambda_{k-1}^{k-1,n} C_{\omega_{2(k-1)}} & + & \dots & \Lambda_1^{k-1,n} C_{\omega_2} & + & \Lambda_0^{k-1,n} C_0 & = 0 \\ & & & \ddots & & \vdots & & \\ & & & & \Lambda_1^{1,n} C_{\omega_2} & + & \Lambda_0^{1,n} C_0 & = 0 \end{cases}$$

$$\omega_{2k} \in O_{w\omega_{2k}}{}^{\bullet S_n} \iff w \in W_{\omega_{2k}};\ i = 0$$

$$\Lambda_k^{k,\,n} = \frac{(t^{2k-1} - qt^{2n})(t^{2k} - 1)}{t^{n+2k-1}(t-1)}$$

# The case $\omega_{2k}$: Reducing the System

Using the combinatorics of weights it can be showed that:

$$\Lambda_h^{k,\,n} = \Lambda_0^{k-h,\,n-2h} + (-1)^{k-h}\Lambda_h^{h,\,n}\binom{n-h-k}{k-h}$$

$$\omega_{2k} \in O_{w\omega_{2k}}{}^{\bullet S_n} \iff w\omega_{2k} = \omega_{2h} + \mu$$

where $\mu \in O_0^{\bullet S_{n-2h}}$ when contracted to $C_{n-2h}$.

$$\Lambda_0^{k,\,n} = \Lambda_0^{k,\,n-1} - \Lambda_0^{k-1,n-2} + (-1)^k\binom{n-k-1}{k-1}F_0^{2,n} + \sum_{i=1}^{k}(-1)^{k-i+1}F_i^{1,n}\binom{n-i-k}{k-i}$$

$0 \in O_{w\omega_{2k}}{}^{\bullet S_n}$ only if $w\omega_{2k}$ is of the form:

$$(0,\dots), \qquad (-1,1,\dots), \qquad (2-2i,1,\dots,1,0\dots,0)+\mu$$

# The case $\omega_{2k}$: Reducing the System

## Proposition

*There exists a family of integers* $\{A_i^{k,n}\}_{i\leq k}$ *such that*

$$\sum_{i=1}^{k} A_i^{k,n} R_{\omega_{2i}} = \Lambda_k^{k,n} C_{\omega_{2k}}(q,t) + \Lambda_0^{1,n-2k+2}\left(C_{\omega_{2(k-1)}}(q,t) + \cdots + C_0(q,t)\right)$$

$$C_{\omega_{2(k+1)}}(q,t) = \frac{(t^{2(n-2k-1)}-1)(t^{2(n-k+1)}-1)(1-qt^{2k-1})t^2}{(t^{2(n-2k+1)}-1)(t^{2(k+1)}-1)(1-qt^{2(n-k)-1})} C_{\omega_{2k}}(q,t).$$

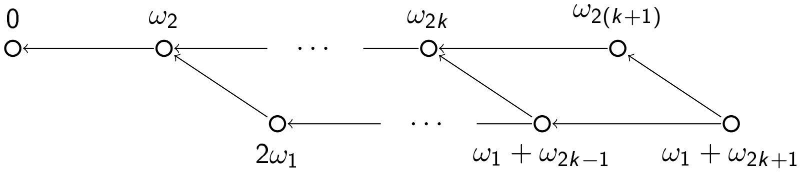
## Remark

$$P_W(V^0_{\omega_{2(k+1)}}, q, q^2) = P_W(V^0_{\omega_{2k}}, q, q^2)\frac{q^4(q^{4(n-k+1)}-1)(q^{4(n-2k-1)}-1)(q^{4k-1}+1)}{(q^{4(k+1)}-1)(q^{4(n-2k+1)}-1)(q^{4(n-k)-1}+1)}$$

# The case $\omega_1 + \omega_{2k+1}$

It is more complex because:

- The zero weight representations are not irreducible.
- There are less symmetries in the coefficients of reduced recurrence
- The dominance order on the weights smaller or equal to $\omega_1 + \omega_{2k+1}$ is not a total order.




- In case $n = 2k + 1$ the poset of weights smaller than $\omega_1 + \omega_{2k+1}$ is different from the general case.

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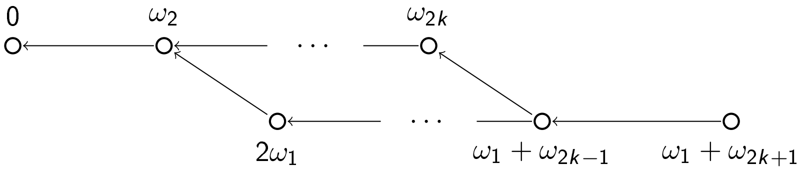


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# The case $\omega_1 + \omega_{2k+1}$

## Proposition

*If $\lambda = \omega_1 + \omega_{2k+1}$ the system can be reduced to*

$$\Lambda_\lambda^{\lambda,n} C_\lambda + \Gamma_{k+1}^{k,n} C_{\omega_{2(k+1)}} + \Gamma_k^{k,n} C_{\omega_{2k}} + \Gamma_0^{k,n} \left( C_{\omega_{2(k-1)}} + \cdots + C_0 \right) = 0$$

*for generic $n$ and to*

$$\Lambda_\lambda^{\lambda,2k+1} C_\lambda + \Gamma_k^{k,2k+1} C_{\omega_{2k}} + \Gamma_0^{k,2k+1} \left( C_{\omega_{2(k-1)}} + \cdots + C_0 \right) = 0$$

*in the special case $n = 2k+1$.*

$$\Gamma_k^{kn} = -\frac{(t-q)(1+qt^{2(n-2k)-1})}{t^{n-2k}} - \frac{(1-q^2t^{2n-2k-1})}{t^{n-1}} \frac{(t^{2k}-1)}{(t-1)}$$

$$\Gamma_{k+1}^{kn} = -\frac{(1-q^2t^{2(n-k-1)})}{t^{n-1}} \frac{(t^{2k+1}-1)}{(t-1)} \qquad \Gamma_0^{kn} = \frac{(t^2+q)(t-q)(t^{2(n-2k)}-1)}{t^{n-2k+1}(t-1)}.$$

## Remark

$$\Phi = \{\pm e_i \pm e_j\}_{i<j} \quad \Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$$

$$\Phi^+ = \{e_i \pm e_j\}_{i<j} \quad W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$$

$$\omega_i = e_1 + \cdots + e_i \quad \omega_{n-1} = \frac{e_1 + \cdots - e_n}{2} \quad \omega_n = \frac{e_1 + \cdots + e_n}{2}$$

$$\rho = (n-1, \ldots, 1, 0)$$

Table: Zero weights space of small representation: 1st Family

| **Small Representation** Highest weight | **Zero Weight Space** $(\alpha, \beta)$ description |
|---|---|
| $\omega_{2i}$, $i < \frac{n-1}{2}$ | $((n-i),(i))$ |
| $2\omega_{n-1}$, $2\omega_n$ $(n$ even$)$ | $((\frac{n}{2}),(\frac{n}{2}))_{I/II}$ |
| $\omega_{n-1} + \omega_n$ $(n$ odd$)$ | $((\frac{n+1}{2}),(\frac{n-1}{2}))$ |

# Representations of $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$

**Remark ( $\alpha \neq \beta$)**

$\pi_{\alpha,\beta} \in \mathrm{Irr}_{S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n}$ restricts to an Irr.Rep. $\widetilde{\pi_{\alpha,\beta}}$ of $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$.

$$\pi_{\alpha,\beta} \otimes \mathrm{sg}_{|(\mathbb{Z}/2\mathbb{Z})^n} \simeq \pi_{\beta,\alpha} \Longrightarrow P_{D_n}(\widetilde{\pi_{\alpha,\beta}}) = P_{B_n}(\pi_{\alpha,\beta}) + P_{B_n}(\pi_{\beta,\alpha})$$

**Remark**

The restriction of $\pi_{\alpha,\alpha}$ splits into two non isomorphic IrrReps $\widetilde{\pi_{\alpha,\alpha}}_I$ and $\widetilde{\pi_{\alpha,\alpha}}_{II}$.

$$P_{D_n}(\widetilde{\pi_{\alpha,\alpha_I}}) = P_{D_n}(\widetilde{\pi_{\alpha,\alpha_{II}}}) = P_{B_n}(\pi_{\alpha,\alpha})$$

# Reeder's Conjecture in type $D$

Idea: mixed strategy!

$e_1$ is a minuscule coweight $\Rightarrow$ use minuscule recurrence.

It is possible to find recurrences between the coefficeints.

## Proposition

*If $\lambda$ is a small weight of the 1st family, it is possible to reduce the recurrence as:*

$$a_k C_{\omega_{2k}} = \sum_{i=1}^{k} b_{i,n-2(k-i)} C_{\omega_{2(k-i)}}$$

*where* $b_{k,n} = \frac{(q+1)(q^{2n}-1)(q^{n-2k}+1)}{(q^2-1)q^{2(n-k)-1}}$

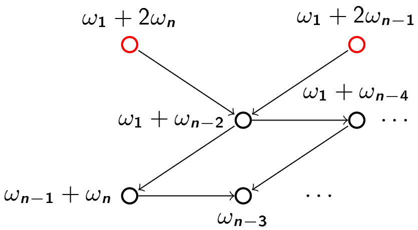
## Remark ($n$ is even)

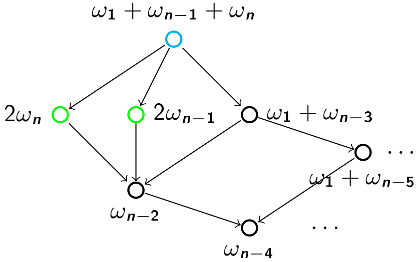
$$O_{w2\omega_n}{}^{\bullet} \overset{\epsilon_n}{\longleftrightarrow} O_{w2\omega_{n-1}}{}^{\bullet}$$

# Reeder's Conjecture in type $D$

Table: Zero weights space of small representation: 2nd Family

| **Small Representation**<br>Highest weight | **Zero Weight Space**<br>$(\alpha, \beta)$ description |
|---|---|
| $2\omega_1$ | $((n-1,1),\emptyset)$ |
| $\omega_1+\omega_{2i+1}$ , $i<\frac{n-1}{2}$ | $((n-i-1,1),(i))\oplus((n-i-1),(i,1))$ |
| $\omega_1+\omega_{n-1}+\omega_n$ $\quad$ ($n$ even) | $((\frac{n}{2},1),(\frac{n-2}{2}))\oplus((\frac{n}{2}),(\frac{n-2}{2},1))$ |
| $\omega_1+2\omega_{n-1}$, $\omega_1+2\omega_{n-1}$ $\quad$ ($n$ odd) | $((\frac{n-1}{2},1),(\frac{n-1}{2}))$ |



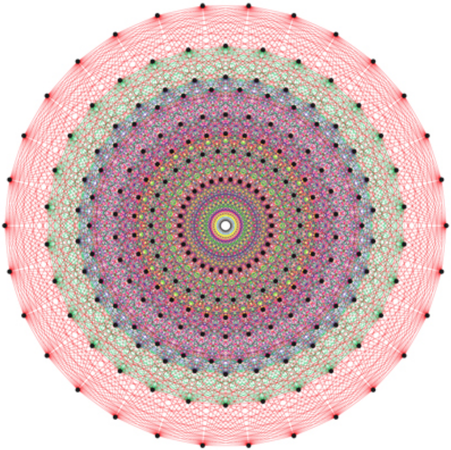



# Conclusions and Future Developments

- Complete type $D$ and prove the conjecture for exceptional cases.
- In a recent paper De Concini and Papi propose a "uniform" approach to the Reeder's conjecture:

$$\Phi_V : \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, \bigwedge \mathfrak{g}) \to \mathrm{Hom}_W(V_\lambda^0, \bigwedge \mathfrak{h} \otimes \mathcal{H})$$

  They conjecture that this map is injective for all f.d. irreducible $\mathfrak{g}$-representations. A Reeder result then implies the Conjecture.
- The small representations and their zero weight representations appeared recently in geometric context (see Achar, Henderson, Riche - " Geometric Satake, Springer correspondence, and small representations").



Thank you for the attention!