Quantity vs. Size in Representation Theory

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Tor Vergata, 22/05/2020

Preliminaries

§1) Representation-finiteness

§2) Torsion pairs

Based on joint work with Lidia Angeleri Hügel and Fredrik Marks ("A characterisation of τ-tilting finite algebras")

§3) Torsion-finiteness

Based on ongoing joint work with Lidia Angeleri Hügel and David Pauksztello

§4) t-discreteness

Notation:

$\Bbbk$: algebraically closed field

Λ : finite-dimensional $\Bbbk$-algebra

$\mathrm{Mod}(\Lambda)$: Category of left Λ-modules

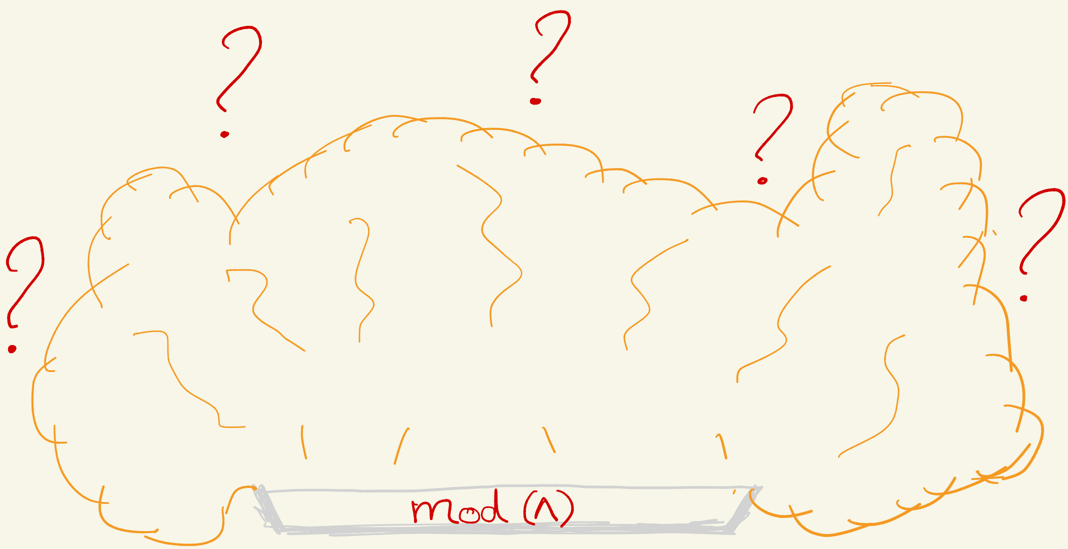
$\mathrm{mod}(\Lambda)$: Category of finite-dimensional Λ-modules

Guiding Question

To which extent does $\mathrm{mod}(\Lambda)$ control $\mathrm{Mod}(\Lambda)$?

mod(Λ) Controls Mod(Λ)

or, sometimes, less so:



§1) Representation-finiteness

<u>Thm</u> (Krull-Remak-Schmidt)

Every finite-dimensional Λ-module is a direct sum (in an essentially unique way) of indecomposable Λ-modules.

<u>Questions</u>: a) Is every Λ-module a direct sum of indecomposables?

b) Are all indecomposables finite-dimensional?

Example: (Ringel)

$\Lambda = \mathbb{k}Q$ $\qquad Q:$
$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left=40, "\alpha"] \arrow[r, "\beta"] \arrow[r, bend right=40, "\gamma"'] & \underset{2}{\bullet} \end{tikzcd}$$

(5-dimensional $\mathbb{k}$-algebra)

Recall: $\mathrm{Mod}(\Lambda) \cong \mathrm{Rep}(Q)$

- An indecomposable infinite-dimensional Λ-module

$$\begin{tikzcd} \mathbb{k}(x) \arrow[r, bend left=40, "x"] \arrow[r, "1"] \arrow[r, bend right=40, "0"'] & \mathbb{k}(x) \end{tikzcd}$$

- A Λ-module with no indecomposable summands (called superdecomposable)

$\mathbb{k}\langle x,y\rangle \hookrightarrow E(\mathbb{k}\langle x,y\rangle)$ injective envelope in $\mathrm{Mod}\ \mathbb{k}\langle x,y\rangle$

$$F : \mathrm{Mod}\ \mathbb{k}\langle x,y\rangle \longrightarrow \mathrm{Mod}\ \Lambda$$
$$M \longmapsto \begin{tikzcd} M \arrow[r, bend left=40, "x"] \arrow[r, "y"] \arrow[r, bend right=40, "1"'] & M \end{tikzcd}$$
full & exact

$F(E(\mathbb{k}\langle x,y\rangle))$ is a superdecomposable Λ-module.

Thm (Auslander, Ringel-Tachikawa, Fuller-Reiten)

The following are equivalent

Size		Quantity
(RF1) Every Λ-module is a direct sum of indecomposable Λ-modules		(RF3) There are only finitely many indec. Λ-modules
$\Updownarrow$	$\Longleftrightarrow$	$\Updownarrow$
(RF2) Every indecomp. is finite-dimensional		(RF4) There are only finitely many indec. finite-dimensional Λ-modules

"pure semisimple" $\Longleftrightarrow$ "representation-finite"

Pure Semisimplicity Conjecture: Does this hold for any ring? OPEN!

Example (finite representation type):

$$\Lambda = \mathbb{k}Q \qquad Q: \underset{1}{\cdot} \xrightarrow{\alpha} \underset{2}{\cdot} \xrightarrow{\beta} \underset{3}{\cdot}$$

Indecomposable Λ-modules and irreducible morphisms between them:

$$\begin{tikzcd}[column sep=tiny]
 & & P_1 \quad \mathbb{k} \xrightarrow{1} \mathbb{k} \xrightarrow{1} \mathbb{k} \quad I_3 \arrow[dr] & & \\
 & P_2 \quad 0 \to \mathbb{k} \xrightarrow{1} \mathbb{k} \arrow[ur] \arrow[dr] & & \mathbb{k} \xrightarrow{1} \mathbb{k} \to 0 \quad I_2 \arrow[dr] & \\
\underset{P_3 = S_3}{0 \to 0 \to \mathbb{k}} \arrow[ur] & & \underset{S_2}{0 \to \mathbb{k} \to 0} \arrow[ur] & & \underset{I_1 = S_1}{\mathbb{k} \to 0 \to 0}
\end{tikzcd}$$

Auslander - Reiten quiver of mod Λ

§2) Torsion pairs

Def.: $\mathcal{A}$ abelian cat. ($\mathcal{A} = \text{mod}\,\Lambda$ or $\mathcal{A} = \text{Mod}\,\Lambda$)

A pair $(\mathcal{T}, \mathcal{F})$ of subcategories is said to be torsion pair if

a) $\text{Hom}(T, F) = 0 \ \forall T \in \mathcal{T}, F \in \mathcal{F}$

b) $\forall X \in \mathcal{A} \ \exists T \in \mathcal{T}, F \in \mathcal{F}$ such that
$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$

($\mathcal{T}$ called torsion class, $\mathcal{F}$ torsionfree class)

Prop.: $\mathcal{T}$ is a torsion class in $\text{Mod}\,\Lambda$ $\iff$ $\mathcal{T}$ is closed under extensions, quotients, coproducts

$\mathcal{F}$ is a torsionfree class in $\text{Mod}\,\Lambda$ $\iff$ $\mathcal{F}$ is closed under submodules, extensions and products.

Rmk: Given a torsion class $\mathcal{T}$, the corresponding torsionfree class is $\mathcal{F} = \ker \operatorname{Hom}(\mathcal{T}, -)$
$= \mathcal{T}^{\perp}$

Dually $\mathcal{T} = {}^{\perp}\mathcal{F}$

Examples: $M \in \operatorname{Mod}\Lambda$

$M^{\perp}$ is a torsionfree class

$\rightsquigarrow$ torsion pair

$$\left(\underbrace{{}^{\perp}(M^{\perp})}_{?}, M^{\perp} \right)$$

$$\cup\!\!| \quad (*)$$

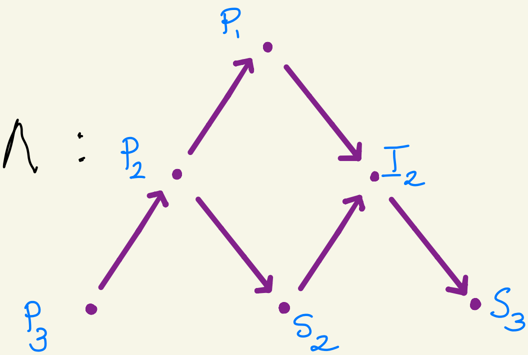
$$\operatorname{Gen} M = \{ X \in \operatorname{Mod}\Lambda : \exists\, M^{(I)} \twoheadrightarrow X \}$$

Equality in $(*)$ is sometimes attained, for example if M is projective.

Example (a torsion pair):

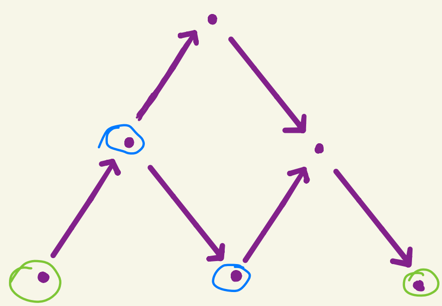
$\Lambda = kQ \qquad Q: \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{3}{\bullet}$

$\operatorname{mod} \Lambda$:



An example of a torsion pair:

$^{\perp}(P_2{}^{\perp}) = \operatorname{Gen} P_2$



$P_2{}^{\perp}$

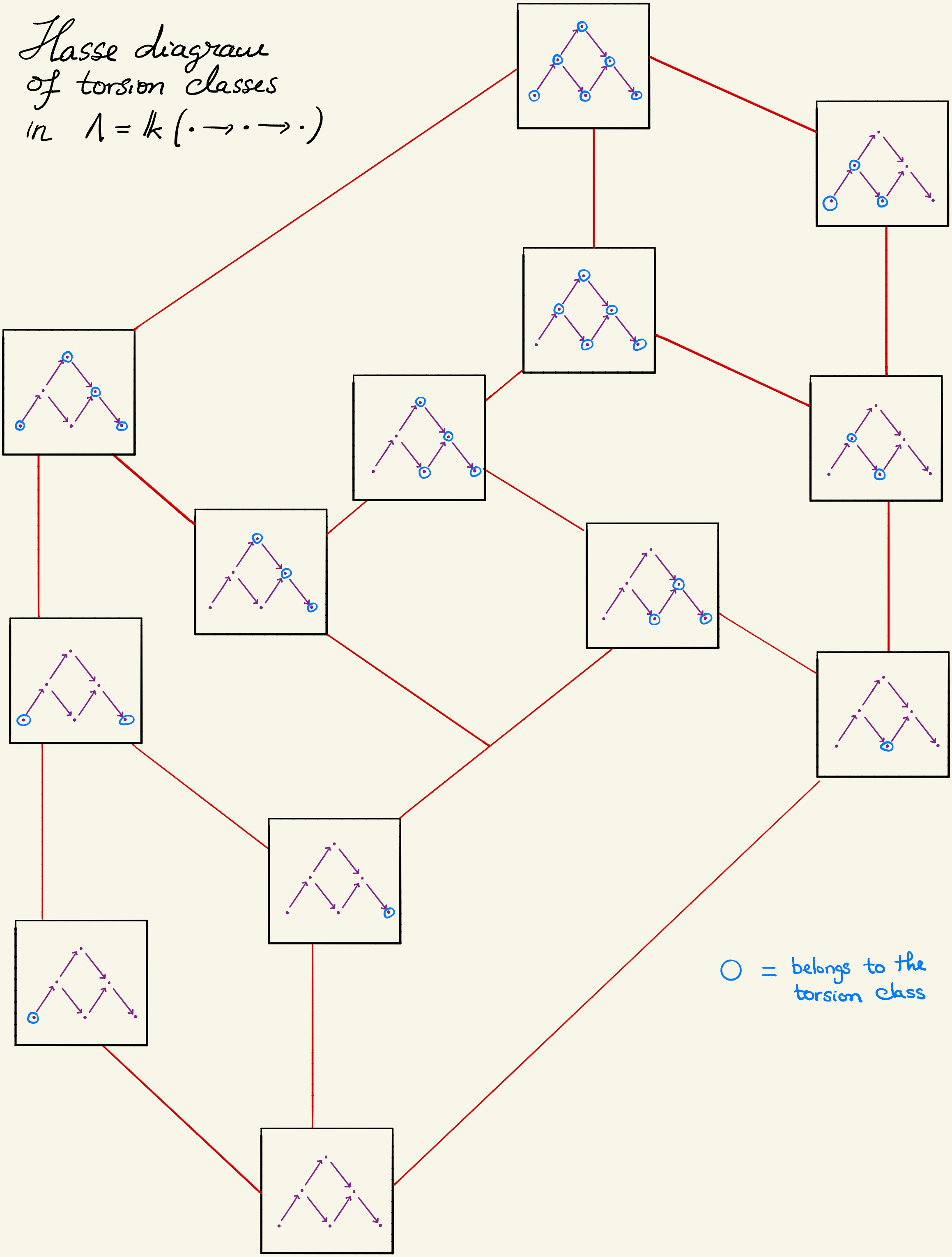
$\mathcal{T}$ torsion class

$\mathcal{F}$ torsionfree class

Λ representation finite $\Longrightarrow$ $\# \underbrace{\operatorname{Tors}(\operatorname{mod}\Lambda)}_{\text{poset}} < \infty$

$\rightsquigarrow$ Hasse diagram?

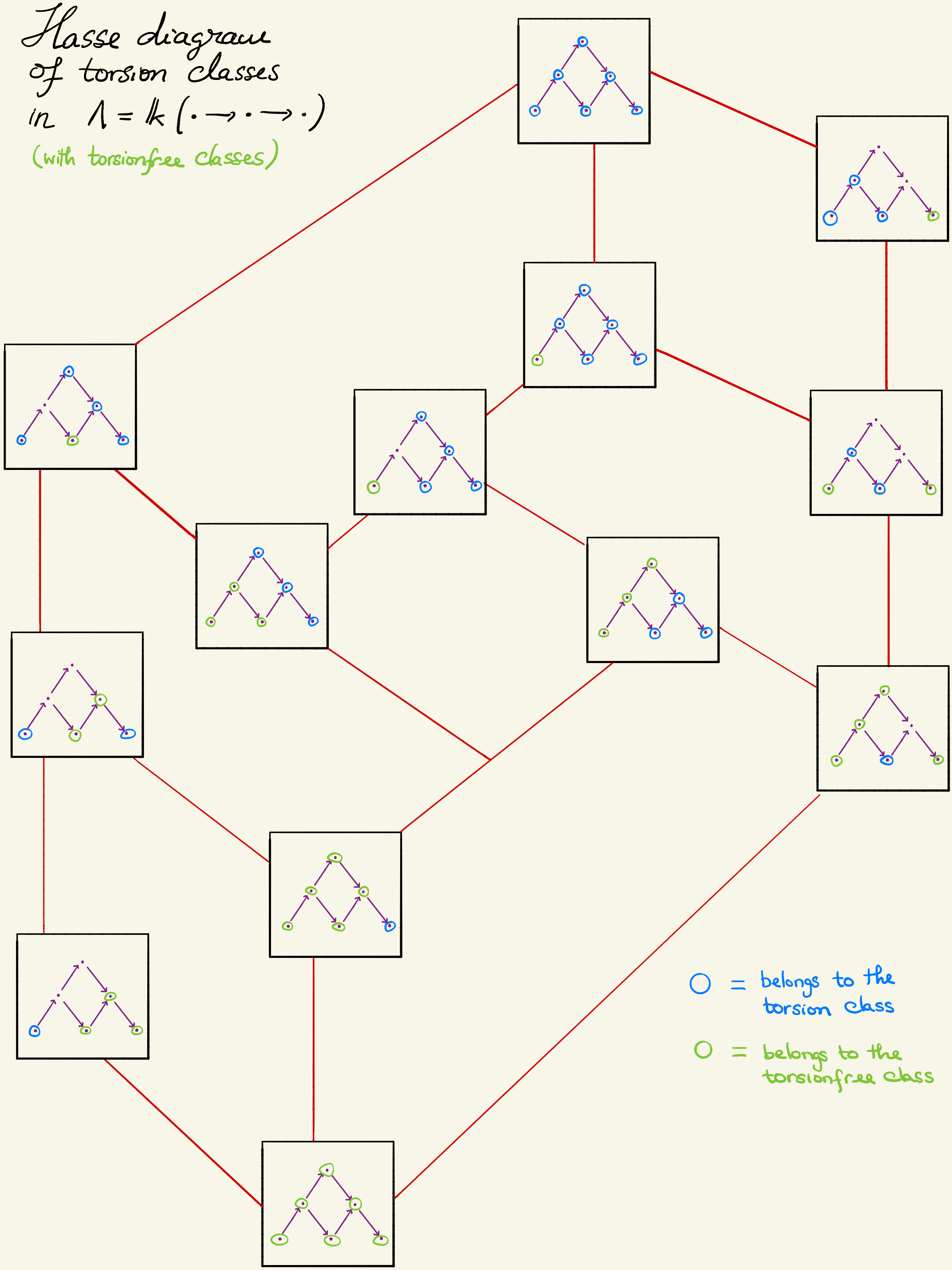
Hasse diagram of torsion classes in $\Lambda = \Bbbk(\cdot \to \cdot \to \cdot)$



○ = belongs to the torsion class

Hasse diagram of torsion classes in $\Lambda = \mathbb{k}(\cdot \to \cdot \to \cdot)$

(with torsionfree classes)



○ = belongs to the torsion class

○ = belongs to the torsionfree class

§3) Torsion-finiteness

$\mathcal{X} \subseteq \mathrm{Mod}\,\Lambda$

$$\varinjlim \mathcal{X} = \{ Z \in \mathrm{Mod}\,\Lambda : \exists \text{ directed system } (X_i)_{i \in I} \text{ in } \mathcal{X} \text{ such that } Z = \varinjlim X_i \}$$

<u>Thm</u> (Crawley-Boevey '94) there is an injection

$$\begin{aligned} \theta : \mathrm{Tors}(\mathrm{mod}\,\Lambda) &\longrightarrow \text{"}\mathrm{Tors}(\mathrm{Mod}\,\Lambda)\text{"} \\ (\mathcal{U}, \mathcal{V}) &\longmapsto (\varinjlim \mathcal{U}, \varinjlim \mathcal{V}) \end{aligned}$$

In fact

$$(\mathcal{U}, \mathcal{V}) = (\varinjlim \mathcal{U}, \varinjlim \mathcal{V}) \cap \mathrm{mod}\,\Lambda$$

and $\mathrm{Im}\,\theta = \{ (\mathcal{T}, \mathcal{F}) : \mathcal{F} = \varinjlim \mathcal{F} \}$

Thm (AMV 2019) The following statements are equivalent

Size		Quantity
(TF1) Every torsion pair in $\operatorname{Mod}(\Lambda)$ is in the image of Θ	$\Longleftrightarrow$	(TF3) There are only finitely many torsion pairs in $\operatorname{mod}(\Lambda)$
$\Updownarrow$		$\Updownarrow$
(TF2) Every torsion class in $\operatorname{Mod}\Lambda$ is of the form $\operatorname{Gen}(M)$, $M \in \operatorname{mod}\Lambda$ $\operatorname{Gen}(M) = \{X \in \operatorname{Mod}\Lambda : M^{(I)} \twoheadrightarrow X\}$		(TF4) There are only finitely many torsion pairs in $\operatorname{Mod}(\Lambda)$

Example: [The theorem does not hold for arbitrary rings Λ, with $\operatorname{mod}\Lambda$ denoting finitely generated Λ-modules]

$$\Lambda = \Bbbk[x]_{(x)} \qquad \text{local PID}$$

$$\#\operatorname{Tors}(\underbrace{\operatorname{mod}\Lambda}_{\text{finitely generated}}) = 3 \qquad (\Lambda \text{ satisfies (TF3)})$$

$\operatorname{Gen} \Bbbk(x)$ is a torsion class

$\qquad\|$

Injective Λ-modules $\quad \downarrow$ does not satisfy (TF2).

$$\text{Rep finite} \underset{\not\Longleftarrow}{\Longrightarrow} \text{Torsion-finite}$$

Ex. (Ringel)

$$\Lambda = \Bbbk\left(\begin{tikzcd} & & & & & 8 \arrow[dr, "\beta"] & \\ 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & 7 \arrow[urr, "\alpha"] \arrow[rrr] & & & 9 \end{tikzcd}\right) \Big/ \langle \beta\alpha \rangle$$

is wild, but it is torsion-finite

Remarks on the proof

The key tools are:

(1) In $\mathrm{mod}(\Lambda)$:

τ-tilting theory : mutation , torsion-finiteness
[Adachi - Iyama - Reiten]

Combinatorics of torsion classes
[Demonet - Iyama - Jasso]

(2) In $\mathrm{Mod}(\Lambda)$:

$\forall L_\Lambda,\ \ker(L \otimes_\Lambda \Phi) = 0$

$\Updownarrow$

Purity : (a) $\forall M \in \mathrm{Mod}(\Lambda)\ \exists\, \Phi : M \xhookrightarrow{\text{pure}} \prod_{\substack{N \leq M \\ M/N \in \mathrm{mod}(\Lambda)}} M/N$

[Crawley - Boevey '98]

(b) $\forall \mathcal{U} \in \mathrm{Tors}(\mathrm{mod}(\Lambda))$

$\mathcal{U} = \mathrm{gen}(M) \iff \varinjlim \mathcal{U}$ is closed under products

[Crawley - Boevey '94]

§4) There is an analogous result if we replace

$\mathrm{mod}\,\Lambda \rightsquigarrow D^b(\mathrm{mod}\,\Lambda)$

$\mathrm{Mod}\,\Lambda \rightsquigarrow D(\mathrm{Mod}\,\Lambda)$

torsion pairs $\rightsquigarrow$ intermediate t-structures

<u>t-structure in $D(\mathrm{Mod}\,\Lambda)$:</u>

pair of full subcategories $(\mathcal{U},\mathcal{V})$:

(1) $\mathrm{Hom}(U,V)=0 \quad \forall U\in\mathcal{U},\ V\in\mathcal{V}$

(2) $\mathcal{U}[1]\subseteq\mathcal{U}$

(3) $\forall X\in D(\mathrm{Mod}\,\Lambda)\ \exists U_X\in\mathcal{U},\ V_X\in\mathcal{V}$ & triangle $U_X\longrightarrow X\longrightarrow V_X\longrightarrow U_X[1]$

<u>Example</u>:

$(D^{\leq 0},D^{\geq 1})$ standard t-structure:

$D^{\leq 0}=\{X\in D(\mathrm{Mod}\,\Lambda): H^k(X)=0\ \forall k>0\}$

$D^{\geq 1}=\{X\in D(\mathrm{Mod}\,\Lambda): H^k(X)=0\ \forall k<1\}$

<u>Intermediate</u>: A t-structure $(\mathcal{U},\mathcal{V})$ is said to be $[k,\ell]$-intermediate if $D^{\leq k}\subseteq\mathcal{U}\subseteq D^{\leq \ell}$

Summary

Setting	Size	Quantity
"Micro" $\mathrm{Mod}(\Lambda)$	(RF1) $\mathrm{Mod}\,\Lambda = \mathrm{Add}(\mathrm{ind}(\mathrm{Mod}\,\Lambda))$ $\Updownarrow$ (RF2) $\mathrm{ind}(\mathrm{Mod}\,\Lambda) \subseteq \mathrm{mod}\,\Lambda$	(RF3) $\#\,\mathrm{ind}(\mathrm{mod}(\Lambda)) < \infty$ $\Updownarrow$ (RF4) $\#\,\mathrm{ind}(\mathrm{Mod}(\Lambda)) < \infty$
	[Auslander, Ringel–Tachikawa, Fuller–Reiten]	
"Macro" $\mathrm{Mod}(\Lambda)$	(TF1) $\forall\,(\mathcal{T},\mathcal{F}) \in \mathrm{Tors}(\mathrm{Mod}\,\Lambda)$ $\exists\,(\mathcal{U},\mathcal{V}) \in \mathrm{Tors}(\mathrm{mod}\,\Lambda)$: $(\mathcal{T},\mathcal{F}) = (\varinjlim \mathcal{U}, \varinjlim \mathcal{V})$ $\Updownarrow$ (TF2) $\forall\,(\mathcal{T},\mathcal{F}) \in \mathrm{Tors}(\mathrm{Mod}\,\Lambda)$ $\exists\, M \in \mathrm{mod}\,\Lambda$: $\mathcal{T} = \mathrm{Gen}(M)$	(TF3) $\#\,\mathrm{Tors}(\mathrm{mod}\,\Lambda) < \infty$ $\Updownarrow$ (TF4) $\#\,\mathrm{Tors}(\mathrm{Mod}\,\Lambda) < \infty$
	[Angeleri Hügel–Marks–V,	Demonet–Iyama–Jasso]
"Macro" $D(\Lambda)$	(tD1) $\forall\,(\mathcal{T},\mathcal{F}) \in t\text{-}st^{int}(D(\Lambda))$ $\exists\,(\mathcal{U},\mathcal{V}) \in t\text{-}st^{int}(D^b(\Lambda))$: $(\mathcal{T},\mathcal{F}) = (\mathrm{aisle}(\mathcal{U}), \mathcal{U}^{\perp})$ $\Updownarrow$ (tD2) $\forall\,(\mathcal{T},\mathcal{F}) \in t\text{-}st^{int}(D(\Lambda))$ $\exists\, M \in K^b(\mathrm{proj}\,\Lambda)$: $\mathcal{T} = \mathrm{aisle}(M)$	(tD3) $\#\, t\text{-}st^{[k,l]}(D^b(\mathrm{mod}(\Lambda))) < \infty$ $\forall\, k < l \in \mathbb{Z}$ $\Updownarrow$ (tD4) $\#\, t\text{-}st^{[k,l]}(D(\Lambda)) < \infty$ $\forall\, k < l \in \mathbb{Z}$
	[Angeleri Hügel–Pauksztello–V.,	Adachi–Mizuno–Yang]

(Size $\Longleftrightarrow$ Quantity in each setting.)