Parabolic K-matrices for quantum groups

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ARTS 8 May 2020

Yang–Baxter

$R \in \mathrm{End}(V\otimes V)$ (R-matrix) $\qquad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$

origin: QISM on the line
quantum integrable syst.
braided mon. categ.

source: Hopf algebras $\quad R\in H\otimes H$

$$R_{12}(R_{13}R_{23})R_{12}^{-1} = (R_{13}R_{23})_{21}$$

quasi-triang. Hopf alg.

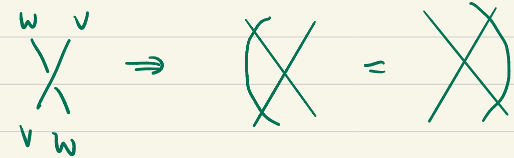
- $R\,\Delta(x) = \Delta^{21}(x)R$
- $\Delta\otimes 1(R) = R_{13}R_{23}$
- $1\otimes\Delta(R) = R_{12}R_{13}$

$V\in \mathrm{Rep}(H) \Rightarrow \mathrm{YBE}(R|_{V\otimes V}) = 0$

however $R: V\otimes V \to V\otimes V$ not an intertw.

$\Longrightarrow R^\vee := (12)\circ R : V\otimes W \xrightarrow{\sim} W\otimes$

$R^\vee$:



Examples: Drinfeld–Jimbo quantum gp

$$R = \sum \mathcal{X}_\mu \otimes \mathcal{X}_{-\mu} \in U_\hbar b_+ \hat{\otimes} U_\hbar b_-$$

- ss Lie alg. $\Longrightarrow$ action on f.d. reps
- sym. KM $\Longrightarrow$ action on $\mathcal{O}^{int}_+$
- affine Lie alg. $\Longrightarrow$ f.d. reps over $U_\hbar L\mathfrak{g}$

↳ $R(z): V(z)\otimes W \xrightarrow{\sim} V(z)\otimes W$

"spectral" YBE

Reflection Equation

$k \in \mathrm{End}(V)$ $\qquad R_{21} \cdot 1\otimes k \cdot R \cdot k\otimes 1 =$

(k-matrix) $\qquad = k\otimes 1 \cdot R_{21} \cdot 1\otimes k \cdot R$

origin: QISM on half-line — Cherednik

qint syst. with bound. — Sklyanin

braided MODULE cat. — 80s

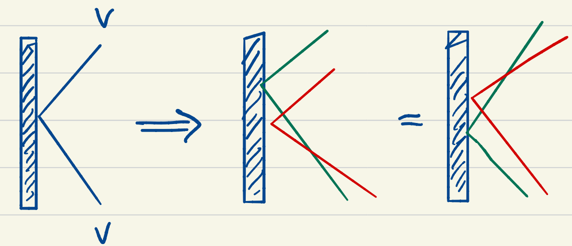
[...]

source: quasi triangular HA (H,R)

$$\underbrace{k\otimes 1 \cdot R_{21} \cdot 1\otimes k \cdot R}_{\Delta(k)} = \Big(R\cdot\big(k\otimes 1 \cdot R_{21} \cdot 1\otimes k \cdot R\big)\cdot R^{-1}\Big)_{21}$$

$$\Delta(k) = \big(R\,\Delta(k)\,R^{-1}\big)_{21}$$

$k \in H$

k : 

$k \in H$ s.t. $\Delta(k) = k\otimes 1 \cdot R_{21} \cdot 1\otimes k \cdot R$

Q: no invariance condition?

$\rightsquigarrow$ $\mathrm{Rep}(H) \longrightarrow \mathrm{Vect}$

$\rightsquigarrow$ "cylindrical HA" by tom Dieck & Häring-Oldenburg '90s

$\rightsquigarrow$ braid groups of type B

RE $\Longrightarrow$ $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$

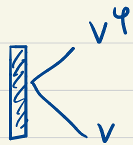
Balanced Hopf algebra (+ ribbon Hopf)

(H,R) + $v \in Z(H)$ s.t.

$$\Delta(v) = v\otimes v \cdot R_{21} R$$
$$= v\otimes 1 \cdot R_{21} \cdot 1\otimes v \cdot R$$

First modification

$\varphi : (H,R) \to (H,R)$ s.t. $\varphi^2 = 1$

k : 

$$\Longrightarrow \Delta(k) = k\otimes 1 \cdot R_{2,\varphi(1)} \cdot 1\otimes k \cdot R$$

$$\Longrightarrow \underbrace{R_{21}}_{\parallel} \cdot 1\otimes k \cdot R_{1,\varphi(2)} \cdot 1\otimes k = $$
$$= k\otimes 1 \cdot R_{2,\varphi(1)} \cdot 1 \otimes k \cdot R$$

$R_{\varphi(2)\varphi(1)}$

$$\begin{tikzcd} \mathrm{Rep}(H) \arrow[rr, "{(\varphi^*, R)}"] \arrow[dr] \arrow[rr, phantom, shift right=4, "\overset{k}{\Longrightarrow}"] & & \mathrm{Rep}(H^{cop}) \arrow[dl] \\ & \mathrm{Vect} & \end{tikzcd}$$

$$\Delta(k) = k\otimes k \cdot \tilde{R}_{21} \cdot R$$

$\parallel$

$1\otimes k^{-1} \cdot R_{2,\varphi(1)} \cdot 1\otimes k$

Source : quantum symmetric pairs

$\mathfrak{g}$ = ss Lie alg / sym KM

$\theta : \mathfrak{g} \to \mathfrak{g}$ s.t. $\theta^2 = 1$ $\big(\dim(\theta(b_+)\cap b_+)\big)$ $< \infty$

Kac–Wang (92) : $\theta = \tilde{w}_X \circ \tau \circ \omega$ ← Chevalley

[...]

longest element of W_X $X \subseteq I$

diagr. aut. of the Dynkin

& $\tau|_X = op_X$ ← opposition involution.

$\Longrightarrow \quad \mathfrak{g}_X \subseteq \mathfrak{g}^\vartheta \subseteq \mathfrak{g}$

$\hookrightarrow$ coideal Lie subalg. wrt std LBA on $\mathfrak{g}$

$\delta(\mathfrak{g}^\vartheta) \subseteq \mathfrak{g}^\vartheta \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{g}^\vartheta$

Quantization is given by

$$U_\hbar \mathfrak{g}_X \subset B_{c,s} \subset U_\hbar \mathfrak{g}$$

$\uparrow$ (left) coideal

[...]

Letzter '99, '00 $\quad \nearrow$ ss Lie alg

Kolb '15 $\quad \searrow$ sym KM

$\rightsquigarrow B_{c,s} \longrightarrow U(\mathfrak{g}^\vartheta)$

<u>Thm</u> (Balagovic-Kolb '15)

$\mathfrak{g}$ = ss Lie alg $/\mathbb{C}$ $\circlearrowright \vartheta$ $\qquad \varphi = \mathrm{op}_I \circ \tau$

$\exists\, k \in \widehat{U_\hbar \mathfrak{g}}^{fd}$ s.t.

$$\forall u \in B_{c,s} \quad k \cdot u = \varphi(u) \cdot k$$

<u>idea</u>: adapt Lusztig proof of the existence of R-matrix

$$\mathfrak{X} \cdot \overline{\Delta(u)} = \Delta(\bar{u})\, \mathfrak{X}$$

<u>Alternative construction</u> (A. Jordan)

<u>Half-balance</u>: $t \in H$

$$\Delta(t) = t \otimes t \cdot R \quad \& \quad t^2 \in Z(H)$$

[Kamnitzer-Tingley '08]

$\dim \mathfrak{g} < \infty$, $\quad t = \xi_0 \cdot S_{w_0} \in \widehat{U_\hbar \mathfrak{g}}^{\mathcal{O}^{int}_+}$

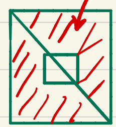
$$\Delta(k) = k\otimes 1 \cdot R_{2\varphi(1)} \cdot 1\otimes k \cdot R$$

$$\Updownarrow$$

$k = a\cdot t$ with $\Delta(a) = a\otimes 1 \cdot R_{2,t\varphi(1)} \cdot 1\otimes a$

Rmk for quantum groups $X \subseteq I$

$$U_\hbar\mathfrak{g}_X \subseteq U_\hbar\mathfrak{g}$$



$\mathfrak{h}_+$

$$\rightsquigarrow \quad R = \bar{R}\cdot R_X$$

Second reduction:

$$k = a\cdot t_X \cdot t$$

$\in L_+ \otimes L_+$

$$\Delta(a) = a\otimes 1 \cdot \bar{R}_{2,t_X t\varphi(1)} \cdot 1\otimes a$$

$$a = 1 + \sum_{\mu>0} \mathcal{X}_\mu$$

Thm (A-Jordan)

$\exists!$ solution for $a = 1 + \sum \mathcal{X}_\mu$ with prescribed "elementary" terms and $k_c = a_c \cdot t_X \cdot t$ is a k-matrix and

$\forall u$ $\quad k_c\cdot u = \varphi(u)\cdot k \rightsquigarrow \psi(u) = k^{-1}\varphi(u)k$

$$\mathcal{B}_k := \{x\in U_\hbar\mathfrak{g} \mid \psi\otimes 1(\Delta(x)) = \Delta(x)\}$$

Moreover, $\mathcal{B}_k \subseteq U_\hbar\mathfrak{g}$ is coideal and quantize $\mathfrak{g}^\theta$.

$$U(\mathfrak{g}^\theta) \subseteq (U\mathfrak{g})^\theta$$

$$\parallel$$

$$\{x\in U\mathfrak{g} \mid \theta\otimes 1\,\Delta(x) = \Delta(x)\}$$

$$\varphi = \varphi_I \circ \tau \rightsquigarrow (\varpi \circ \tau)^* : \mathcal{O}_+^{int} \longrightarrow \mathcal{O}_-^{int}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "{(G,J)}"] \arrow[dr] \arrow[rr, phantom, shift right=6, "\overset{k}{\Rightarrow}"] & & \mathcal{D} \arrow[dl] \\
& \mathrm{Vect} &
\end{tikzcd}$$

$+$ "$A \in H^{\otimes 2}$"

Thm (A-Vlaar)

(X,τ) = Satake diagr., $Y \subseteq I$ finite type s.t. $\tau(Y) = Y$.

Set
$$\psi := \mathrm{Ad}(t_Y^{-1}) \circ \varpi \circ \tau \in \mathrm{Aut}(U_\hbar\mathfrak{g})$$

Let $B_{c,s} \subseteq U_\hbar\mathfrak{g}$ be the coideal sub.

$\exists!$ $a = 1 + \sum_{\mu > 0} \mathcal{X}_\mu$ (supported on $\mathfrak{h} \backslash \mathfrak{h}_X$)

s.t.
$$k = a \cdot t_X \cdot t_Y^{-1}$$

satisfies

(a) $k u = \psi(u) k \quad \forall u \in B_{c,s}$

(b) $\Delta(k^{-1}) = R_Y^{-1} \cdot 1 \otimes k^{-1} \cdot R_{\psi(1)2} \cdot k^{-1} \otimes 1$

and therefore

$$k \otimes 1 \cdot R_{\psi(2)1} \cdot 1 \otimes k \cdot R =$$
$$= R_{\psi(2)\psi(1)} \cdot 1 \otimes k \cdot R_{\psi(1)2} \cdot k \otimes 1$$

$$\Delta^\psi(x) = R_Y^{-1}\, \Delta^{21}(x)\, R_Y$$
$$R^{\psi\psi} = R_{Y,21}^{-1} \cdot R_{21} \cdot R_Y$$