Parabolic K-matrices for quantum groups

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Yang-Baxter

$R \in \mathrm{End}(V\otimes V)$ (R-matrix) $\qquad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$

origin: QISM on the line
quantum integrable syst.
braided mon. categ.

source: Hopf algebras $R \in H\otimes H$

$$R_{12}(R_{13}R_{23})R_{12}^{-1} = (R_{13}R_{23})_{21}$$

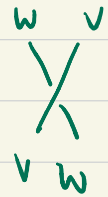
quasi-triang. Hopf alg.

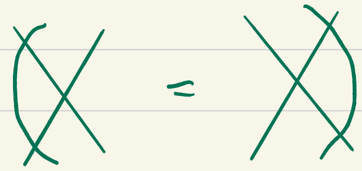
- $R\,\Delta(x) = \Delta^{21}(x)\,R$
- $\Delta\otimes 1(R) = R_{13}R_{23}$
- $1\otimes\Delta(R) = R_{12}R_{13}$

$V \in Rep(H) \Rightarrow YBE(R|_{V\otimes V}) = 0$

however $R : V\otimes V \to V\otimes V$ not an intertw.

$\Rightarrow R^\vee := (12)\circ R : V\otimes W \xrightarrow{\sim} W\otimes$

$R^\vee$: 

$\Rightarrow$ 

Examples: Drinfeld-Jimbo quantum gp

$$R = \sum \mathfrak{X}_\mu \otimes \mathfrak{X}_{-\mu} \in U_\hbar b_+ \hat{\otimes} U_\hbar b_-$$

- ss Lie alg. $\Rightarrow$ action on f.d. reps
- sym. KM $\Rightarrow$ action on $\mathcal{O}^{int}_+$
- affine Lie alg. $\Rightarrow$ f.d. reps over $U_\hbar L\mathfrak{g}$

↳ $R(z) : V(z)\otimes W \xrightarrow{\sim} V(z)\otimes W$

"spectral" YBE

Reflection Equation

$k \in \mathrm{End}(V)$ $\quad R_{21}\cdot 1\otimes k \cdot R \cdot k\otimes 1 =$

(k-matrix)

$$= k\otimes 1 \cdot R_{21}\cdot 1\otimes k \cdot R$$

origin: QISM on half-line — Cherednik

qint syst. with bound. — Sklyanin

braided MODULE cat. — 80s

[...]

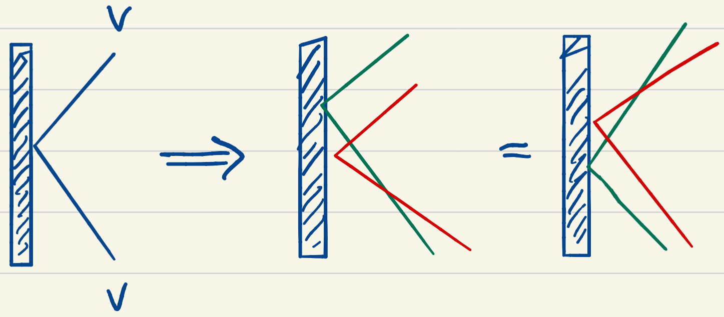
source: quasi triangular HA (H, R)

$$\underbrace{k\otimes 1\cdot R_{21}\cdot 1\otimes k\cdot R}_{\Delta(k)} = \underbrace{\left(R\cdot\left(k\otimes 1\cdot R_{21}\cdot 1\otimes k\cdot R\right)\cdot R^{-1}\right)_{21}}_{(R\,\Delta(k)\,R^{-1})_{21}}$$

$$\Delta(k) = (R\,\Delta(k)\,R^{-1})_{21}$$

$k \in H$

k :



$k \in H$ s.t. $\Delta(k) = k\otimes 1\cdot R_{21}\cdot 1\otimes k\cdot R$

Q: no invariance condition ?

$\rightsquigarrow$ $\mathrm{Rep}(H) \longrightarrow \mathrm{Vect}$

$\rightsquigarrow$ "cylindrical HA" by tom Dieck & Häring-Oldenburg '90s

$\rightsquigarrow$ braid groups of type B

RE $\Rightarrow$ $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$

Balanced Hopf algebra (+ ribbon Hopf)

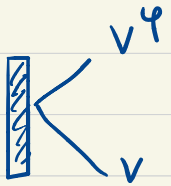
$(H, R) + v \in Z(H)$ s.t.

$$\Delta(v) = v \otimes v \cdot R_{21} R$$

$$= v \otimes 1 \cdot R_{21} \cdot 1 \otimes v \cdot R$$

First modification

$\varphi : (H,R) \longrightarrow (H,R)$ s.t. $\varphi^2 = 1$

k : 

$$\Longrightarrow \Delta(k) = k \otimes 1 \cdot R_{2,\varphi(1)} \cdot 1 \otimes k \cdot R$$

$$\Longrightarrow R_{21} \cdot 1 \otimes k \cdot R_{1,\varphi(2)} \cdot 1 \otimes k = $$

$$= k \otimes 1 \cdot R_{2,\varphi(1)} \cdot 1 \otimes k \cdot R$$

$\|$ $R_{\varphi(2)\varphi(1)}$

$$\begin{tikzcd}
\mathrm{Rep}(H) \arrow[rr, "{(\varphi^*, R)}"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "\overset{k}{\Longrightarrow}"] & & \mathrm{Rep}(H^{cop}) \arrow[dl] \\
& \mathrm{Vect} &
\end{tikzcd}$$

$$\Delta(k) = k \otimes k \cdot \tilde{R}_{21} \cdot R$$

$\|$

$1 \otimes k^{-1} \cdot R_{2,\varphi(1)} \cdot 1 \otimes k$

Source: quantum symmetric pairs

$\mathfrak{g}$ = ss Lie alg / symm KM

$\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$ s.t. $\theta^2 = 1$ $\left(\dim(\theta(b_+) \cap b_+)\right)$ $< \infty$

Kac–Wang (92) : $\theta = \tilde{w}_X \circ \tau \circ \varpi$ ← Chevalley

↖ diagr. aut. of the Dynkin

longest element of

W_X $X \subseteq I$ & $\tau|_X = op_X$ ← opposition involution.

[...]

$$\Longrightarrow \quad \mathfrak{g}_X \subseteq \mathfrak{g}^\vartheta \subseteq \mathfrak{g}$$

↳ coideal Lie subalg. wrt std LBA on $\mathfrak{g}$

$$\delta(\mathfrak{g}^\vartheta) \subseteq \mathfrak{g}^\vartheta \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{g}^\vartheta$$

Quantization is given by

$$U_\hbar \mathfrak{g}_X \subset B_{c,s} \subset U_\hbar \mathfrak{g}$$

↑ (left) coideal

[...]

ss Lie alg

Letzter '99, '00 $\rightsquigarrow B_{c,s} \longrightarrow U(\mathfrak{g}^\vartheta)$

Kolb '15

sym KM

<u>Thm</u> (Balagović-Kolb '15)

$\mathfrak{g}$ = ss Lie alg / $\mathbb{C}$ ⟲ ϑ $\qquad \varphi = \varpi_I \circ \tau$

$\exists\, k \in \widehat{U_\hbar \mathfrak{g}}^{fd}$ s.t.

$$\forall u \in B_{c,s} \quad k \cdot u = \varphi(u) \cdot k$$

<u>idea</u>: adapt Lusztig's proof of the existence of R-matrix

$$\mathfrak{X} \cdot \overline{\Delta(u)} = \Delta(\bar{u})\, \mathfrak{X}$$

<u>Alternative construction</u> (A. Jordan)

<u>Half-balance</u>: $t \in H$

$$\Delta(t) = t \otimes t \cdot R \quad \& \quad t^2 \in Z(H)$$

[Kamnitzer-Tingley '08]

$\dim \mathfrak{g} < \infty$, $\quad t = \xi_0 \cdot S_{w_0} \in \widehat{U_\hbar \mathfrak{g}}^{\mathcal{O}^{int}_+}$

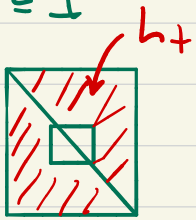
$\Delta(k) = k\otimes 1 \cdot R_{2\psi(1)} \cdot 1\otimes k \cdot R$

$\Updownarrow$

$k = a\cdot t$ with $\Delta(a) = a\otimes 1 \cdot R_{2,t\psi(1)} \cdot 1\otimes a$

<u>Rmk</u> for quantum groups $X \subseteq I$

$U_\hbar \mathfrak{g}_X \subseteq U_\hbar \mathfrak{g}$



$\rightsquigarrow R = \bar{R} \cdot R_X$

<u>Second reduction:</u>

$k = a\cdot t_X \cdot t$

$$\Delta(a) = a\otimes 1 \cdot \overbrace{\bar{R}_{2,t_X t\psi(1)}}^{\in L_+\otimes L_+} \cdot 1\otimes a$$

$$a = 1 + \sum_{\mu>0} \mathcal{X}_\mu$$

<u>Thm</u> (A-Jordan)

$\exists!$ solution for $a = 1 + \sum \mathcal{X}_\mu$ with prescribed "elementary" terms and

$k_c = a_c \cdot t_X \cdot t$ is a k-matrix

and

$\forall u \quad k_c \cdot u = \varphi(u)\cdot k \rightsquigarrow \psi(u) = k^{-1}\varphi(u) k$

$$B_k := \{x \in U_\hbar \mathfrak{g} \mid \psi\otimes 1(\Delta(x)) = \Delta(x)\}$$

Moreover, $B_k \subseteq U_\hbar\mathfrak{g}$ is coideal and quantize $\mathfrak{g}^\vartheta$.

$$U(\mathfrak{g}^\vartheta) \subseteq (U\mathfrak{g})^\vartheta$$

$$\shortparallel$$

$$\{x \in U\mathfrak{g} \mid \vartheta\otimes 1\, \Delta(x) = \Delta(x)\}$$

$\varphi = \varphi_I \circ \tau \rightsquigarrow (\varpi\circ\tau)^* : \mathcal{O}_+^{int} \longrightarrow \mathcal{O}_-^{int}$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "{(G,J)}"] \arrow[dr] & \arrow[d, phantom, "\overset{k}{\Rightarrow}"] & \mathcal{D} \arrow[dl] \\ & \mathrm{Vect} & \end{tikzcd} \quad + \text{"} A \in H^{\otimes 2} \text{"}$$

<u>Thm</u> (A-Vlaar)

(X,τ) = Satake diagr., $Y \subseteq I$ finite type s.t. $\tau(Y) = Y$.

Set
$$\psi := \mathrm{Ad}(t_Y^{-1})\circ\varpi\circ\tau \in \mathrm{Aut}(U_\hbar\mathfrak{g})$$

Let $B_{c,s} \subseteq U_\hbar\mathfrak{g}$ be the coideal sub.

$\exists!$ $a = 1 + \sum_{\mu>0} \mathcal{X}_\mu$ (supported on $\mathfrak{h}\backslash\mathfrak{h}_X$)

s.t.
$$k = a\cdot t_X\cdot t_Y^{-1}$$

satisfies

(a) $k u = \psi(u) k \quad \forall u \in B_{c,s}$

(b) $\Delta(k^{-1}) = R_Y^{-1}\cdot 1\otimes k^{-1}\cdot R_{\psi(1)2}\cdot k^{-1}\otimes 1$

and therefore

$$k\otimes 1\cdot R_{\psi(2)1}\cdot 1\otimes k\cdot R =$$
$$= R_{\psi(2)\psi(1)}\cdot 1\otimes k\cdot R_{\psi(1)2}\cdot k\otimes 1$$

$$\Delta^{\psi}(x) = R_Y^{-1}\,\Delta^{21}(x)\,R_Y$$
$$R^{\psi\psi} = R_{Y,21}^{-1}\cdot R_{21}\cdot R_Y$$