The quest for bases of the intersection cohomology of Schubert varieties

Leonardo Patimo

University of Freiburg, Germany

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Roma Tor Vergata

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Part 1:
Schubert varieties in complex Grassmannians

Complex Grassmannian

$$\mathrm{Gr}(k,N) = \{k\text{-dimensional subspaces of } \mathbb{C}^N\}.$$

It is a smooth complex projective variety of dim $k(N-k)$.
What is $H^\bullet(\mathrm{Gr}(k,N),\mathbb{Q})$?

To any tableau $\lambda \subseteq \overbrace{\begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \; & \; \\ \hline \; & \; & \; & \; & \; \\ \hline \; & \; & \; & \; & \; \\ \hline \end{array}}^{k}\Big\}(N-k)$ it corresponds a **Schubert cell**.

Example

$$\begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \; & \; \\ \hline \; & \; \\ \cline{1-2} \; & \; \\ \cline{1-2} \end{array} \mapsto C_\lambda := \mathrm{Im}\begin{pmatrix} * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cong \mathbb{C}^{\ell(\lambda)}$$

$\ell(\lambda) = \#\{\text{boxes in } \lambda\}$

Cohomology of Schubert varieties

Schubert cells give a cell decomposition:

$$\mathrm{Gr}(k, N) = \coprod_{\lambda} C_\lambda$$

Let $X_\mu = \overline{C_\mu}$ be a Schubert variety $\rightsquigarrow [X_\mu] \in H_{2\ell(\mu)}(\mathrm{Gr}(k, N))$.

$$H_\bullet(\mathrm{Gr}(k, N), \mathbb{Q}) = \bigoplus_{\lambda} \mathbb{Q}[X_\lambda] \overset{\text{dualizing}}{\rightsquigarrow} H^\bullet(\mathrm{Gr}(k, N), \mathbb{Q}) = \bigoplus_{\lambda} \mathbb{Q}\mathcal{S}_\lambda$$

Taking dual basis of $\{[X_\lambda]\}$ we obtain a basis $\{\mathcal{S}_\lambda\}$,
with $\mathcal{S}_\lambda \in H^{2\ell(\lambda)}(\mathrm{Gr}(k, N))$, called **Schubert basis**

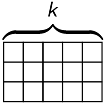
Let $i_\mu : X_\mu \hookrightarrow \mathrm{Gr}(k, N)$ be the inclusion.
Via the pullback $i_\mu^* : H^\bullet(\mathrm{Gr}(k, N), \mathbb{Q}) \to H^\bullet(X_\mu, \mathbb{Q})$ we get

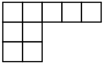
$$H^\bullet(X_\mu, \mathbb{Q}) = \bigoplus_{\lambda \subseteq \mu} \mathbb{Q} i_\mu^* \mathcal{S}_\lambda = \bigoplus_{\lambda \subseteq \mu} \mathbb{Q}\mathcal{S}_\lambda$$

The T-equivariant story is analogous

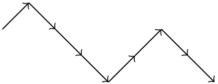
Many equivalent combinatorial descriptions

The following sets are in bijections:

- Subtableaux of  $\}(N-k)$.



- Piece-wise linear paths from $(0, k)$ to $(N, N-k)$ with steps $(1, \pm 1)$.



- elements of $\{0,1\}^N$ with k 0's.

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- Subsets of $\{1, \ldots, N\}$ with $N-k$ elements

$\{1, 5, 6\}$

- cosets $S_N / S_k \times S_{N-k}$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 6 & 2 & 3 & 4 & 7 & 8 \end{pmatrix}$$

- ...

Pieri's formula

The Schubert basis is an important tool to study $H^\bullet(\mathrm{Gr}(k,N),\mathbb{Q})$ and $H^\bullet(X_\mu,\mathbb{Q})$

Let $\mathcal{S}_\square \in H^2(\mathrm{Gr}(k,N),\mathbb{Q})$.

Pieri's formula

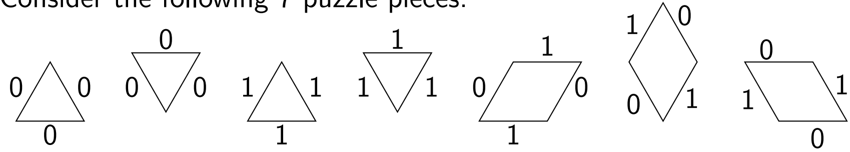
$$\mathcal{S}_\square \cdot \mathcal{S}_\lambda = \sum_{\substack{\mu \text{ tableau which can be} \\ \text{obtained by adding a box to } \lambda}} S_\mu$$

Example

$$\mathcal{S}_\square \cdot \mathcal{S}_{\ydiagram{2,2}} = \mathcal{S}_{\ydiagram{3,2}} + \mathcal{S}_{\ydiagram{2,2,1}}$$

Littlewood-Richardson Rule

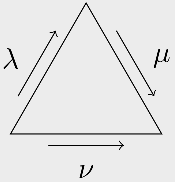
What about $\mathcal{S}_\lambda \cdot \mathcal{S}_\mu$ for arbitrary λ, μ?
Consider the following 7 puzzle pieces.



Theorem (Knutson-Tao-Woodward '04)

$$\mathcal{S}_\lambda \cdot \mathcal{S}_\mu = \sum_\nu c_{\lambda,\mu}^{\nu} \mathcal{S}_\nu$$

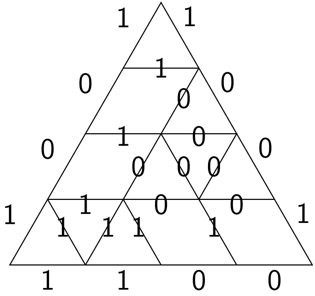
where $c_{\lambda,\mu}^{\nu}$ is the number of puzzles with boundary

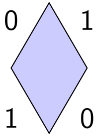


Here we regard λ, μ, ν as elements of $\{0,1\}^N$.

Example of the puzzle rule

We have $\mathcal{S}_{1001} \cdot \mathcal{S}_{1001} = c^{1100}_{1001,1001} \mathcal{S}_{1100} \in H^8(\mathrm{Gr}(2,4))$
and $c^{1100}_{1001,1001} = 1$ because the only possible puzzle is:





There is more! Adding a new "equivariant piece" one can compute $\mathcal{S}_\lambda \cdot \mathcal{S}_\mu \in H^\bullet_T(\mathrm{Gr}(k,N))$.
(Knutson-Tao '05)

Intersection Cohomology of Schubert varieties

If X_μ is singular, Poincaré duality fails.

$$H^{\ell(\mu)-d}(X_\mu, \mathbb{Q}) \not\cong H^{\ell(\mu)+d}(X_\mu, \mathbb{Q})$$

We can embed the cohomology into the **intersection cohomology**:

$$H^\bullet(X_\mu, \mathbb{Q}) \subseteq IH^{\bullet-\ell(\mu)}(X_\mu, \mathbb{Q}).$$

$IH^\bullet(X_\mu, \mathbb{Q})$ can be thought of as a replacement of $H^\bullet$ where Poincaré duality holds.

Motivation

$IH(X_\mu)$ is related to the representation theory of $\mathfrak{sl}_N(\mathbb{C})$ (more precisely to the maximally parabolic category $\mathcal{O}$)

Question

Is there some sort of "Schubert basis" for $IH^\bullet(X_\mu, \mathbb{Q})$?
Can we extend in a natural way $\{\mathcal{S}_\lambda\}$ to a basis of $IH^\bullet(X_\mu, \mathbb{Q})$?

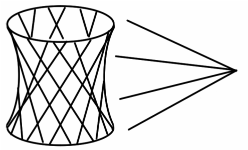
An example in $\mathrm{Gr}(2,4)$

Let $\mu = \ydiagram{2,1} \subseteq \ydiagram{2,2}$.

$X_\mu = \{V \subseteq \mathbb{C}^4 \mid \dim V = 2 \text{ and } \dim(V \cap \mathbb{C}^2) \geq 1\} \subseteq \mathrm{Gr}(2,4)$

X_μ is a singular variety of dim 3:

It is the projective cone over a non-degenerate quadric $Y \subseteq \mathbb{P}^3(\mathbb{C})$.



$$H^\bullet(X_\mu) = \begin{array}{cccc} 6 & & \ydiagram{2,1} & \\ 4 & \ydiagram{2} & & \ydiagram{1,1} \\ 2 & & \ydiagram{1} & \\ 0 & & \emptyset & \end{array} \qquad IH^\bullet(X_\mu) = \begin{array}{cccc} 3 & & \ydiagram{2,1} & \\ 1 & \ydiagram{2} & & \ydiagram{1,1} \\ -1 & \ydiagram{1} & & ? \\ -3 & & \emptyset & \end{array}$$

Kazhdan-Lusztig polynomial

The (Grassmannian) **Kazhdan-Lusztig polynomial** can be defined as

$$h_{\lambda,\mu}(v) = \sum_i \dim IC_{C_\lambda}^{-i-\ell(\mu)}(X_\mu, \mathbb{Q}) v^i.$$

- **Kazhdan-Lusztig '80**:
  $h_{\lambda,\mu}(v)$ can be computed via a recursive formula.
- **Lascoux-Schützenberger '80**:
  explicit not-recursive formula (in terms of LS binary trees) for $h_{\lambda,\mu}(v)$.

A quote by **Bernstein**: "...I would say that if you can compute a polynomial P for intersection cohomologies in some case without a computer, then probably there is a **small resolution** which gives it..."

Small resolutions

A resolution of singularities $p : \widetilde{X} \to X$ is said **small** if

$$\forall r > 0: \quad \mathrm{codim}\{x \in X \mid \dim p^{-1}(x) = r\} > 2r.$$

Schubert varieties in Grassmannians are "very special" Schubert varieties:

Theorem (Zelevinsky '83)

All the Schubert varieties in Grassmannians admit a small resolution of singularities.

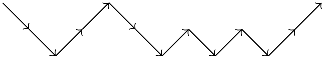
$$\begin{aligned} p \text{ small} \implies & p_* \mathbb{Q}_{\widetilde{X}}[\dim X] \cong IC^\bullet(X, \mathbb{Q}) \\ & \implies H^\bullet(\widetilde{X}, \mathbb{Q}) = IH^{\bullet - \dim X}(X, \mathbb{Q}). \end{aligned}$$

Dyck paths and strips

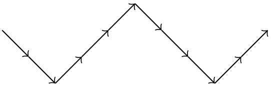
Definition

A **Dyck path** is a piecewise linear path consisting of the same number of segments $\searrow$ and $\nearrow$, such that it remains below the horizontal line.

Example

. , , 

This is **not** a Dyck path:



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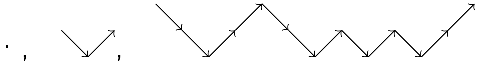
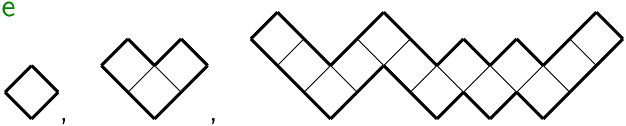
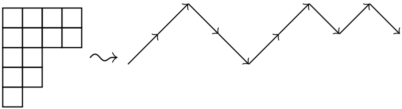


A **Dyck strip** is obtained by taking the unitary boxes, tilted by 45°, with center on the integral coordinates of a Dyck path

Example

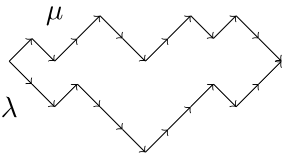


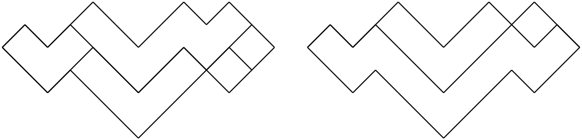
We reinterpret a tableau as a path:



Let λ, μ be paths with $\lambda \leq \mu$. A **Dyck partition** is a partition of the region between λ and μ into Dyck strips.

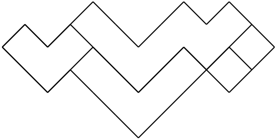
Example

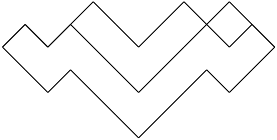




Dyck partitions of Type 1

A Dyck partition **P** is said of **type 1** if:
whenever a strip D contains a box just below a box in a strip C, then every box just above a box in D is in C.


Type 1


Not Type 1

We can reformulate LS results using **Dyck partitions**
$|\mathbf{P}|$ = number of strips in **P**.

Theorem (Shigechi - Zinn-Justin '12)

Dyck partitions of Type 1 describes KL polynomials.

$$\sum_{\substack{\mathbf{P} \text{ of type } 1 \\ \text{between } \lambda \text{ and } \mu}} v^{|\mathbf{P}|} = h_{\lambda,\mu}(v)$$

Singular Soergel bimodules

$R = \mathbb{Q}[x_1, x_2, \ldots, x_N] \curvearrowleft S_N$ acts by permutations.
R^{S_N} is the subring of invariants. Then:

$$H_T^\bullet(\mathrm{Gr}(k,N)) = R \otimes_{R^{S_N}} R^{S_k \times S_{N-k}}.$$

where $T = (\mathbb{C}^*)^N$ acts on $\mathrm{Gr}(k,N)$.
We regard $IH_\lambda := IH_T^\bullet(X_\lambda, \mathbb{Q})$ via the inclusion $X_\lambda \hookrightarrow \mathrm{Gr}(k,N)$ as a $H_T^\bullet(\mathrm{Gr}(k,N))$-module (or as a $(R, R^{S_k \times S_{N-k}})$-bimodule).
These bimodules are called **singular Soergel bimodules**.

Let $\lambda \subset \mu$. If we quotient out all morphisms factoring through

$$IH_\lambda \to IH_\nu \to IH_\mu \text{ for some } \nu \subset \lambda$$

we have

$$\mathrm{Hom}^\bullet_{\not\subset\lambda}(IH_\lambda, IH_\mu) \cong \bigoplus_i R(-i)^{m_i} \text{ with } \sum_i m_i v^i = h_{\lambda,\mu}(v)$$

Morphisms of degree one

$$\mathrm{Hom}^1(IH_\lambda, IH_\mu) \cong \begin{cases} \mathbb{Q} & \text{if } \lambda \text{ and } \mu \text{ differ by a Dyck strip} \\ 0 & \text{otherwise} \end{cases}$$

$$\{\text{Dyck strips}\} \to \{\text{morphisms of degree } 1\}$$
$$D \mapsto f_D$$

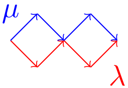
The map f_D can be explicitly constructed.

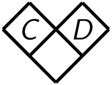
Naive idea:

$$\left\{ \begin{array}{c} \text{Dyck partition} \\ \text{with } m \text{ elements} \end{array} \right\} \to \{\text{morphisms of degree } m\}$$

$$\mathbf{P} = \{D_1, D_2, \ldots, D_m\} \mapsto f_{\mathbf{P}} := f_{D_1} \circ f_{D_2} \circ \ldots f_{D_m}$$

Dyck strips do not commute



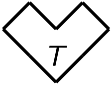


Then

$$f_C \circ f_D \neq f_D \circ f_C \in \mathrm{Hom}^2(IH_\lambda, IH_\mu).$$

and $f_C \circ f_D - f_D \circ f_C$ is a non-zero map factoring through

$$IH_\lambda \to IH_\emptyset \xrightarrow{f_T} IH_\mu \qquad \text{where}$$



In general, $f_{\mathbf{P}} := f_{D_1} \circ f_{D_2} \circ \ldots f_{D_k}$ **not** well defined.

A partial order on Dyck partitions

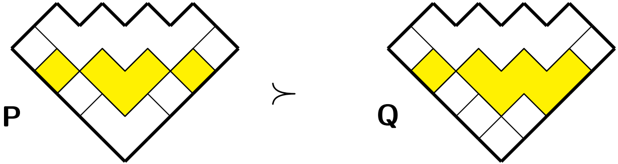
Let $\mathbf{P}$ and $\mathbf{Q}$ be Dyck partition, $\mathbf{P} \neq \mathbf{Q}$.

$\mathbf{P}(h) :=$ set of strips of height h in $\mathbf{P}$.

$h_0 :=$ largest index such that $\mathbf{P}(h_0) \neq \mathbf{Q}(h_0)$.

Then $\mathbf{P} \succ \mathbf{Q}$ if $\mathbf{P}(h_0)$ is finer than $\mathbf{Q}(h_0)$,
i.e. if every strip of $\mathbf{P}(h_0)$ is contained in a strip of $\mathbf{Q}(h_0)$.

Example



Construction of bases on morphisms spaces

Theorem (P. '19)

If $\mathbf{P} = \{D_1, D_2, \ldots, D_m\}$ is a Dyck partition between λ and μ, then the map

$$f_{\mathbf{P}} = f_{D_1} \circ f_{D_2} \circ \ldots \circ f_{D_m} \in \mathrm{Hom}^m_{\not<\mu}(IH_\mu, IH_\lambda).$$

does not depend on the order up to smaller terms wrt $\prec$, i.e. up to something contained in $span\langle f_{\mathbf{Q}} \mid \mathbf{Q} \prec \mathbf{P}\rangle$.

The set

$$\left\{ f_{\mathbf{P}} \mid \begin{array}{c} \mathbf{P} \text{ Dyck partition of type 1} \\ \text{between } \lambda \text{ and } \mu \end{array} \right\}$$

is a basis of $\mathrm{Hom}_{\not<\lambda}(IH_\lambda, IH_\mu)$ over R, for any choice of the order of the strips in $\mathbf{P}$.

Construction of the bases of Intersection Cohomology

$\mathcal{S}_{id}$ is the unity of the cohomology ring $H_T(X_\lambda)$.
Let $F_{\mathbf{P}} := f_{\mathbf{P}}(\mathcal{S}_{id})$.

Corollary

The element $F_{\mathbf{P}} \in IH_\mu^{-\ell(\mu)+2|\mathbf{P}|}$ does not depend on the order chosen, up to smaller elements wrt $\prec$.

The set

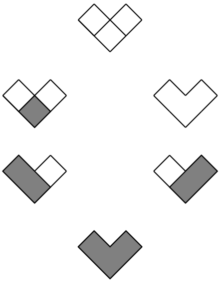
$$\left\{ F_{\mathbf{P}} \mid \begin{array}{c} \mathbf{P} \text{ Dyck partition of type } 1 \\ \text{between } \lambda \text{ and } \mu, \text{ for some } \lambda \leq \mu \end{array} \right\}$$

is a basis of IH_μ over R, for any choice of the order of the strips in $\mathbf{P}$.

Back to the example in $\mathrm{Gr}(2,4)$

$$\mu = \ydiagram{2,1} \subseteq \ydiagram{2,2}.$$

$$\begin{array}{cccc} 3 & & \ydiagram{2,1} & \\ 1 & \ydiagram{2} & & \ydiagram{1,1} \\ -1 & \ydiagram{1} & & ? \\ -3 & & \emptyset & \end{array} = IH^\bullet(X_\mu, \mathbb{Q}) =$$



Comparison with the Schubert basis

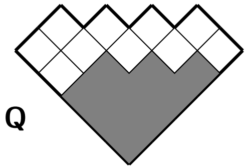
$\langle -, - \rangle_\mu$ Poincaré pairing on IH_μ.

P Dyck partition of type 1 between ν and μ. For any $\lambda \leq \mu$:

$$\langle \mathcal{S}_\lambda, F_{\mathbf{P}} \rangle = \begin{cases} 1 & \text{if } \nu = \lambda \text{ and } \mathbf{P} \text{ only consists of single boxes} \\ 0 & \text{otherwise} \end{cases}$$

Hence: if $\{F^*_{\mathbf{P}}\}$ is the dual basis to $\{F_{\mathbf{P}}\}$ then

$$\mathcal{S}_\lambda = F^*_{\mathbf{Q}} \text{ where } \mathbf{Q} \text{ consists only of single boxes.}$$

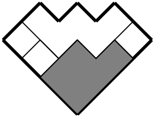


Pieri's formula in intersection cohomology

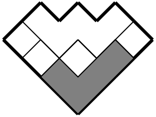
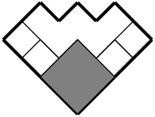
Proposition
$$\mathcal{S}_{\square} \cdot F_{\mathbf{P}} = \sum_{\substack{C \text{ box that can} \\ \text{be added to } \mathbf{P}}} F_{\mathbf{P} \cup \{C\}}$$

where the order on $\mathbf{P} \cup \{C\}$ is the same order on $\mathbf{P}$ plus C at the beginning.

Example

If $\mathbf{P} =$  then

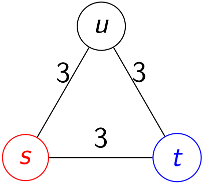
$$\mathcal{S}_{\square} \cdot F_{\mathbf{P}} = F_{\mathbf{Q}_1} + F_{\mathbf{Q}_2}$$

where $\mathbf{Q}_1 =$  and $\mathbf{Q}_2 =$ 

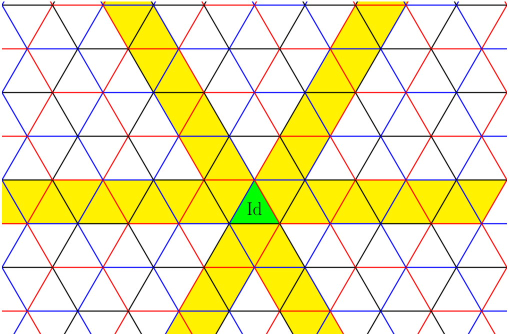
Part 2: the group $\widetilde{A}_2$

joint with Nicolas Libedinsky

W affine Weyl group of type $\widetilde{A}_2$.

Coxeter diagram: 

$$W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^3 = Id \rangle.$$



Schubert varieties for $\widetilde{A}_2$

Let $\mathcal{F}l = SL_3(\mathbb{C}(\!(t)\!))/\mathrm{Iw}$ be the *affine flag variety* where

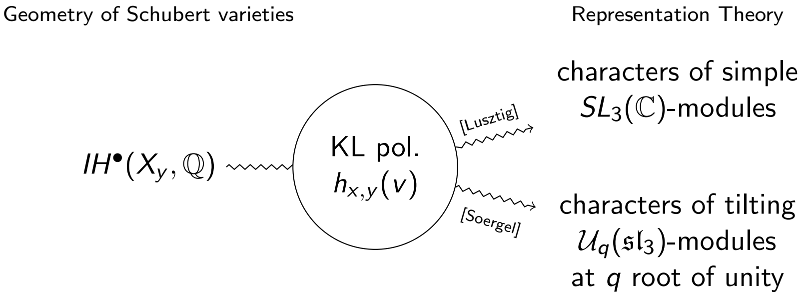
$$\mathrm{Iw} = \left\{ \begin{pmatrix} * & * & * \\ a & * & * \\ b & c & * \end{pmatrix} \in SL_3(\mathbb{C}[[t]]) \mid a, b, c \in t\mathbb{C}[[t]] \right\}$$

Bruhat decomposition:

$$\mathcal{F}l^\circ = \coprod_{y \in W} \mathrm{Iw} \cdot y\, \mathrm{Iw}$$

The Schubert variety $X_y = \overline{\mathrm{Iw} \cdot y \mathrm{Iw}}$ is a projective (usually singular) algebraic variety of dim $\ell(y)$.
As in the first part, we are interested in $IH^\bullet(X_y)$.

Connections with representation theory



Also in this case we have

$$H^\bullet(X_y, \mathbb{Q}) \subseteq IH^{\bullet - \ell(y)}(X_y, \mathbb{Q})$$

We want to extend the Schubert basis from $H^\bullet(X_y, \mathbb{Q})$ to $IH^\bullet(X_y, \mathbb{Q})$.

Soergel bimodules

Let $R = \mathbb{Q}[\alpha_s, \alpha_t, \alpha_u]$ (it is a polynomial ring with $\deg(\alpha_i) = 2$).

There is an action of W on R associated to the Cartan matrix

$$(\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Example: $s(\alpha_u) = \alpha_u - (\alpha_s, \alpha_u)\alpha_s = \alpha_u + \alpha_s$

For every reduced expression $y = st \dots u$ we can realize $IH_y := IH^\bullet_T(X_y, \mathbb{Q})$ as a direct summand of the Bott-Samelson bimodule

$$BS(\underline{st \dots u}) = R \otimes_{R^s} R \otimes_{R^t} R \otimes \dots \otimes_{R^u} R(\ell(y))$$

Example: IH_{sts} is a summand (as a R-bimodule) of

$$BS(\underline{sts}) = R \otimes_{R^s} R \otimes_{R^t} R \otimes_{R^s} R(3) \cong IH_{sts} \oplus IH_s$$

Light leaves morphisms

Morphisms between Bott-Samelson bimodules are well understood.
Let $y \leq w$. Let $\underline{w} = s_1 s_2 \dots s_k$ reduced expression.
For $e \in \{0,1\}^k$ we write $\underline{w}^e := s_1^{e_1} s_2^{e_2} \dots s_k^{e_k}$
Example: $\underline{sts}^{100} = s^1 t^0 s^0 = s$.

Theorem (Libedinsky '07)

$$Hom_{\not< y}(BS(\underline{y}), BS(\underline{w})) = \bigoplus_{\underline{w}^e = y} \mathbb{Q} LL_{\underline{w},e}$$

$LL_{\underline{w},e}$ is a map that can be constructed algorithmically out of $\underline{w}$ and e.

Question

Can we find a subset of $\{LL_{\underline{w},e}\}$ that gives a basis when restricted to the summand $IH_w \subset BS(\underline{w})$?

Morphisms diagrammatically

To depict morphisms between Bott-Samelson, it is very convenient to use diagrams.

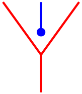
$$\begin{array}{c} BS(\underline{s}) = R \otimes_{R^s} R \\ \uparrow \\ BS(\emptyset) = R \end{array}$$

$$\begin{array}{c} \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \\ \mapsto \\ 1 \end{array}$$

$$\begin{array}{c} BS(\underline{ss}) = R \otimes_{R^s} R \otimes_{R^s} R \\ \uparrow \\ BS(\underline{s}) = R \otimes_{R^s} R \end{array}$$

$$\begin{array}{c} f \otimes 1 \otimes g \\ \mapsto \\ f \otimes g \end{array}$$

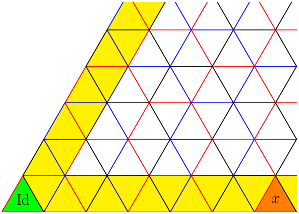
Example



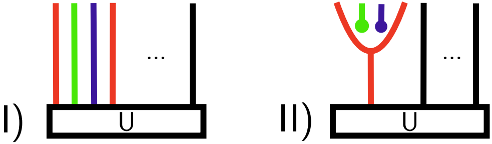
depicts a morphism $BS(\underline{s}) \to BS(\underline{sts})$

(which is actually $LL_{\underline{sts},100}$).

Basis in type $\widetilde{A}_2$ (on the walls)

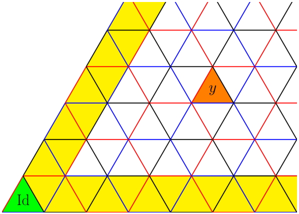


A basis of IH_x (x on the wall, $\ell(x)$ odd) is given by the following type of light leaves.

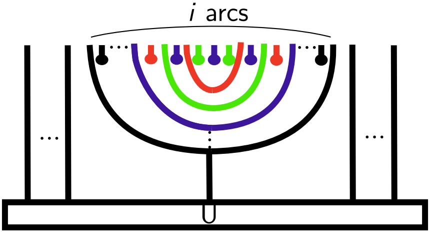


where the box with U is a light leaf containing only dots (no trivalent vertices allowed).

Basis in type $\widetilde{A}_2$ (outside the walls)



A basis of IH_y (y out of the walls, y spherical) is given by the following type of light leaves.



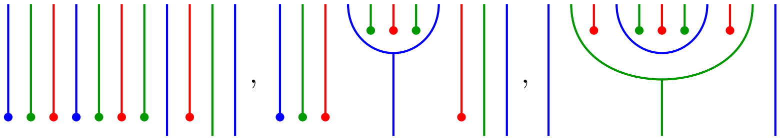
A visualization for KL polynomials

The basis that we produced give also a nice visualization of KL polynomials.

Example

We have $h_{tustusutsut,tut}(v) =$

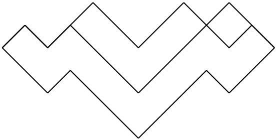
$$v^8 \qquad + \qquad v^6 \qquad + \qquad v^4$$



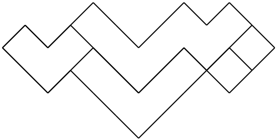
Thanks for your attention!

Dyck partitions of Type 2

A Dyck partition **P** is said of **type 2** if:
for any strip D that contains a box just below, SW or SE o a box in a strip C, then every box just below, SW or SE a box in C belongs either to D or C.



Type 2



Not Type 2

Remark

Between any two paths λ, μ there exists **at most one** Dyck partition of type 2.

Inverse KL polynomials

Inverse KL polynomials are related to the ordinary KL polynomials by the **inversion formula**:

$$\sum_{\mu}(-1)^{\ell(\mu)-\ell(\nu)}h_{\lambda,\mu}(v)g_{\mu,\nu}(v) = \delta_{\lambda,\nu}$$

Theorem (Brenti '02)

Dyck partitions of Type 2 describe **inverse** KL polynomials

$$\sum_{\substack{\mathbf{P}\text{ of type 2}\\ \text{between }\lambda\text{ and }\mu}} v^{|\mathbf{P}|} = g_{\lambda,\mu}(v)$$

Singular Rouquier Complexes

We can construct a complex of singular Soergel bimodules E_μ :

$$0 \to \bigoplus_{\lambda \in \Lambda_i} IH_\lambda(-i) \to \ldots \to \bigoplus_{\lambda \in \Lambda_1} IH_\lambda(-1) \to IH_\mu \to 0$$

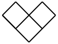
which is **exact** everywhere but in the term IH_μ.

Here $\Lambda_i = \left\{ \mu \leq \lambda \mid \begin{array}{c} \text{exists Dyck partition } \mathbf{P} \\ \text{of type 2 between } \lambda \text{ and } \mu \text{ with } |\mathbf{P}| = i \end{array} \right\}.$

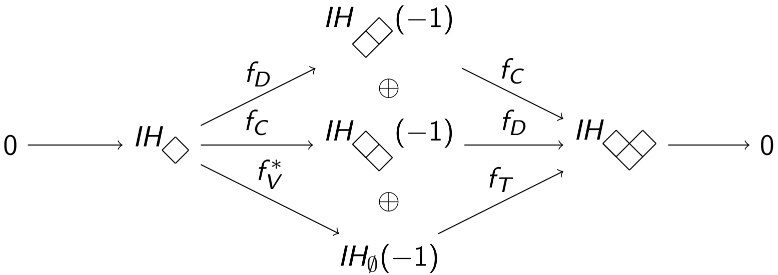
$$IH_\lambda(-i) \text{ occurs in } E_\mu \quad \Longleftrightarrow \quad g_{\lambda,\mu}(v) = v^i$$

E_μ is called **singular Rouquier complex**.

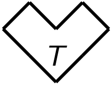
Example of a singular Rouquier complex

Let $\mu =$ . The singular Rouquier complex E_μ is:

${}^{-3}E_\mu$ ${}^{-2}E_\mu$ ${}^{-1}E_\mu$ ${}^{0}E_\mu$



where 

and 

.

From $d^2 = 0$ follows $f_D \circ f_C - f_C \circ f_D = f_T \circ f_V^*$ (up to scalar).